Trialmaths Enterprises

2012

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time –3 hours
- Write using black or blue pen
  Black pen is preferred
- Board- approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11-16

Total marks – 100

- Section I Pages 2-3
  10 Marks
- Attempt Questions 1-10
- Allow about 15 minutes for this section
- Section II Pages 4-7
  90 Marks
- Attempt Questions 11-16
- Allow about 2 hours 45 minutes for this section

Section I

10 marks
Attempt Questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

1. Find $\left(cis\frac{\pi}{3}\right)^9$

   (A) -1 (B) 1 (C) 0 (D) 2

2. Find $\int \frac{2x}{\sqrt{1-x^4}}\,dx$

   (A) $\cos^{-1}x^2+c$ (B) $\frac{1}{2}\sin^{-1}x^2+c$ (C) $\sin^{-1}x^2+c$ (D) $2\sin^{-1}x+c$

3. Find all pairs of integers a and b such that $(a+ib)^2 = 8+6i$

   (A) $a=\pm3, b=\pm1$ (B) $a=\pm1, b=\pm3$ (C) $a=3, b=1$ (D) $a=1, b=3$

4. Find the eccentricity of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$.

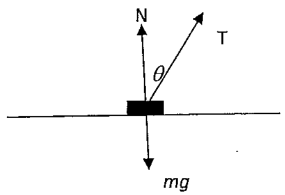
   (A) $\frac{3}{5}$ (B) $\frac{\sqrt{3}}{5}$ (C) $\frac{\sqrt{5}}{9}$ (D) $\frac{\sqrt{3}}{9}$

5. Find the parametric equations of the hyperbola $x^2-y^2=4$.

   (A) $x=2\tan\theta$ and $y=2\sec\theta$ (B) $x=2\sec\theta$ and $y=2\tan\theta$

   (C) $x=4\tan\theta$ and $y=4\sec\theta$ (D) $x=4\sec\theta$ and $y=4\tan\theta$

6. A body of mass m kg is being pulled along a smooth horizontal table by means of a string inclined at θ to the vertical. The diagram below indicates the forces acting on the body. Which one of the following is true?



   (A) $N-mg=0$

   (B) $N+T\sin\theta-mg=0$

   (C) $N-T\sin\theta-mg=0$

   (D) $N+T\cos\theta-mg=0$

7. The following letters CUFGUUT are arranged in a line. In how many ways can they be arranged if all the U's are separated?

(A) 10 (B) 240 (C) 35 (D) 24

8. The equation $x^3+2x-1=0$ has roots α,β,γ. Find the monic equations with roots $\alpha^2,\beta^2,\gamma^2$.

(A) $x^3+2x^2+2x-3=0$

(B) $x^3+4x^2+2x-1=0$

(C) $x^3+4x^2+4x-1=0$

(D) $4x^3+4x^2+4x-1=0$

9. In how many ways can a group of eight be divided into two groups of four to play a set of double table tennis with each other?

(A) 70 (B) 140 (C) 35 (D) 50

10. Find $\int \frac{1}{x^2+2x+2}dx$

(A) $\tan^{-1}(x+2)+c$ (B) $\tan^{-1}(x+1)+c$ (C) $\sin^{-1}(x+1)+c$ (D) $\cos^{-1}(x+1)+c$

Section II

90 marks

Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet.

All necessary working should be shown in every question.

Question 11 (15 Marks) Use a separate page/booklet **Marks**

(a) Use the substitution $u=x^2$ to calculate $\int_0^{\frac{1}{\sqrt{2}}} \frac{xdx}{\sqrt{1-x^4}}$ 3

(b) Find $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$ 3

(c) (i) Write $\frac{4}{x^2-1}$ as the sum of two fractions. 2

(ii) Hence, find $\int \frac{4}{x^2-1}dx$. 2

(d) Sketch (showing critical points) the graph of $y=x^2-|x|$. 3

(e) Show that $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$ 2

Question 12 (15 Marks) Use a separate page/booklet **Marks**

(a) Use integration by parts to evaluate $\int_0^{\frac{\pi}{2}} e^x \cos x dx$ 4

(b) (i) Express $-1+i\sqrt{3}$ in modulus-argument form. 2

(ii) Hence find $(-1+i\sqrt{3})^5$, giving your answer in the form $a+ib$ 2

(c) Given that $|z| = 1$, show that $z^{-1} = \bar{z}$ **2**

(d) Sketch in the Argand diagram the locus of a complex number z that satisfies $0 \le \arg(z - i) \le \frac{2\pi}{3}$ **2**

(e) Sketch the graph of $y = \sin|x|$ in the domain $-2\pi \le x \le 2\pi$ **3**

Question 13 (15 Marks) Use a separate page/booklet **Marks**

(a) An ellipse has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(i) Show that the equation of a tangent to the ellipse at the point (x_1, y_1) on the ellipse is $\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$ **3**

(ii) Two points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ lie on the ellipse such that $\theta + \phi = \frac{\pi}{2}$.

Find the coordinates of M the midpoint of P and Q in terms of θ **2**

(b) (i) Find all the solutions to the equation $z^6 = 1$ in the form $x + yi$ **2**

(ii) If ω is a non-real solution to the equation $z^6 = 1$, show that $\omega^4 + \omega^2 = -1$. **2**

(c) Sketch the graph of $f(x) = (x-4)(2-x)$ and hence draw separate sketches of the following graphs:

(i) $y = \left|\frac{1}{f(x)}\right|$ **1**

(ii) $y = [f(x)]^2$ **1**

(iii) $y^2 = f(x)$ **2**

(iv) $y = e^{f(x)}$ **2**

Question 14 (15 Marks) Use a separate page/booklet **Marks**

(a) The hyperbola has eccentricity $\frac{3}{2}$ and directrices $x = -4$ and $x = 4$.

Find the equation of this hyperbola. **2**

(b) The points $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$ lie on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

If PQ subtends a right angle at $(a, 0)$. Show that $\tan\frac{\theta}{2}\tan\frac{\phi}{2} = -\frac{b^2}{a^2}$ **4**

(c) Find $P(x)$, given that $P(x)$ is monic, of degree 4, with -1 as a single zero and 3 as a zero of multiplicity 3. **1**

(d) (i) Show that $\sin x + \sin 3x = 2\sin 2x \cos x$. **1**

(ii) Hence, or otherwise, solve $\sin x + \sin 2x + \sin 3x = 0$ for $0 \le x \le 2\pi$. **2**

(e) Evaluate $\int_0^3 \frac{x^2 + 4x + 5}{(x+1)(x+3)} dx$. **3**

(f) Find $\frac{dy}{dx}$ for $x^2 + xy + y^2 = 1$ **2**

Question 15 (15 Marks) Use a separate page/booklet **Marks**

(a) (i) Show that the recurrence (reduction) formula for

$$I_n = \int \tan^n x\,dx \text{ is } I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}.$$ **3**

(ii) Hence evaluate $\int_0^{\frac{\pi}{4}} \tan^3 x\,dx$ **2**

(b) A corner on a race track is an arc of a circle of radius 100m. The track is banked such that there is no tendency for a vehicle to move sideways when cornering at 100km/h.
Find the angle of banking. ($g = 10\,\text{ms}^{-2}$) **2**

(c) The base of a solid is the region in the first quadrant bounded by the curve $y = \sin x$, the x-axis and the line $x = \frac{\pi}{2}$. Find the volume of the solid if every cross-section perpendicular to the base and the $x-$axis is a square. **3**

(d) (i) Show that $\tan(\alpha+\beta+\gamma) = \dfrac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\alpha\tan\beta - \tan\beta\tan\gamma - \tan\gamma\tan\alpha}$. **3**

(ii) Hence or otherwise, evaluate $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{13}$. **2**

Question 16 (15 Marks) Use a separate page/booklet **Marks**

(b) A spherical planet of mass m_1 and radius R has a rocket launched vertically from its surface with an initial speed of V. The mass of the rocket is m_2 and the distance between the rocket and the centre of the planet is x. The gravitational force F, acting on the rocket is given by

$$F = \frac{Gm_1m_2}{x^2}$$

where G is the constant of gravitation on that planet. Assume that there are no other forces acting on the rocket.

(i) Write down an expression for the acceleration of the rocket in terms of x, taking the positive direction as away from the surface of the planet. **1**

(ii) Find an expression for the velocity v of the rocket in terms of x. **3**

(b) ABC is a triangle. The internal bisectors of $\angle CBA$ and $\angle ACB$ meet at D. DP, DQ and DR are the perpendiculars from D to BC, CA and AB respectively. Show that $DR = DQ$ and deduce that the internal bisectors of the three angles of a triangle are concurrent. **4**

(c) If α and β are the roots of $x^2 - 2x + 4 = 0$, prove that $\alpha^n - \beta^n = i2^{n+1}\sin\left[\frac{n\pi}{3}\right]$.

Hence find the value of $\alpha^9 - \beta^9$ **4**

(d) The quadratic equation $x^2 - (2\cos\theta)x + 1 = 0$ has roots α and β.

(i) Suppose $f(x) = \sqrt{1+x}$. Show that $f'(x) < \frac{1}{6}$ for $x > 8$. **1**

(ii) Using part (i), or otherwise, show that $\sqrt{1+x} \le 3 + \frac{x-8}{6}$ when $x \ge 8$. **2**

STANDARD INTEGRALS

$\int x^n dx$	$= \frac{1}{n+1}x^{n+1}, n \ne -1;\ x \ne 0$, if $n < 0$
$\int \frac{1}{x} dx$	$= \ln x,\ x > 0$
$\int e^{ax} dx$	$= \frac{1}{a}e^{ax},\ a \ne 0$
$\int \cos ax\,dx$	$= \frac{1}{a}\sin ax,\ a \ne 0$
$\int \sin ax\,dx$	$= -\frac{1}{a}\cos ax,\ a \ne 0$
$\int \sec^2 ax\,dx,$	$= \frac{1}{a}\tan ax,\ a \ne 0$
$\int \sec ax \tan ax\,dx$	$= \frac{1}{a}\sec ax,\ a \ne 0$
$\int \frac{1}{a^2+x^2}dx$	$= \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \ne 0$
$\int \frac{1}{\sqrt{a^2-x^2}}dx$	$= \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$
$\int \frac{1}{\sqrt{x^2-a^2}}dx$	$= \ln(x + \sqrt{x^2-a^2}),\ x > a > 0$
$\int \frac{1}{\sqrt{x^2+a^2}}dx$	$= \ln(x + \sqrt{x^2+a^2})$

NOTE : $\ln x = \log_e x, \quad x > 0$

2012 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

- **Solutions including marking scale**
- **Mapping grid**

We have endeavored to ensure that the solutions are free of errors and follow the spirit of the syllabus in the methods used to solve the problems. However, individual teachers may opt for alternate solutions and/or may choose a different marking system.

Marking Guidelines: Mathematics Extension II – Solutions

Section I

1A 2C 3A 4A 5B 6D 7D 8C 9C 10B

(1) $\left(cis\frac{\pi}{3}\right)^9 = cis\left(\frac{\pi}{3}\times 9\right) = cis3\pi = cis\pi = \cos\pi + i\sin\pi = -1$

(2)

$$\int \frac{2x}{\sqrt{1-x^4}}dx \quad \text{Let } u = x^2 \quad \therefore \frac{du}{dx} = 2x \ \ or \ \ dx = \frac{du}{2x}$$

$$= \therefore \int \frac{2x}{\sqrt{1-x^4}}dx = \int \frac{2x}{\sqrt{1-u^2}}\cdot\frac{du}{2x}$$

$$= \int \frac{1}{\sqrt{1-u^2}}du$$

$$= \sin^{-1} u + C$$

$$= \sin^{-1} x^2 + C$$

(3)

Let $(a+ib)^2 = 8+6i$

or $a^2 + 2abi - b^2 = 8+6i$

$\therefore a^2 - b^2 = 8$ and $2ab = 6$ or $ab = 3$

By inspection we get $a = \pm 3$, $b = \pm 1$

or $\sqrt{8+6i} = \pm(3+i)$

(4) In $\frac{x^2}{25} + \frac{y^2}{16} = 1$, $a = 5$ and $b = 4$

We know that $b^2 = a^2(1-e^2)$

$\therefore 16 = 25(1-e^2)$ or $e^2 = 1 - \frac{16}{25}$ i.e. $e = \frac{3}{5}$

(5) Cartesian equation of the hyperbola is $x^2 - y^2 = 4$. Then $\frac{x^2}{4} - \frac{y^2}{4} = 1$. Hence $a = 2$ and $b = 2$. Therefore the hyperbola has parametric equations $x = 2\sec\theta$ and $y = 2\tan\theta$, $-\pi < \theta \le \pi, \theta \ne \pm\frac{\pi}{2}$.

(6) Resolving the forces the answer is $N + T\cos\theta - mg = 0$

(7) The four consonants can be arranged in 4!ways, then there are 5 spaces where each of the U's can be inserted (in front, in back and three in between). The U's can be inserted in ${}^5C_3$ ways. Thus total = $4!\times{}^5C_3$=240 ways

(8)

$$\left(\sqrt{x}\right)^3 + 2\sqrt{x} - 1 = 0$$

$$x\sqrt{x} + 2\sqrt{x} = 1$$

$$\left(\sqrt{x}(x+2)\right)^2 = 1$$

$$x\left(x^2 + 4x + 4\right) = 1$$

$$x^3 + 4x^2 + 4x - 1 = 0$$

(9) $\frac{{}^8C_4}{2} = 35$

(10)

$\int \frac{1}{x^2+2x+2}dx$ {Show first that $x^2+2x+2=(x+1)^2+1$}.

Make the substitution $x+1=u, dx=du$. Then we get

$\int \frac{1}{x^2+2x+2}dx = \int \frac{1}{(x+1)^2+1}dx = \int \frac{du}{u^2+1} = \tan^{-1} u + c = \tan^{-1}(x+1)+c$.

Section II

Question 11

Criteria
(a) One for both du and limits, one for integral and one for simplification. (b) One for substitution of $\cos\theta$ in terms of t, one for $\int_0^1 \frac{1}{\frac{2(1+t^2)}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2\,dt}{1+t^2}$ and one for simplification.(c) (i) One for process and one for simplification. (ii) One for splitting the denominator and one for simplification. (d) One for shape of curve, one for location of curve and one for critical point. (e) One for change of limits and one for simplification.

Answers:

(a) Let u = x^2 when x = $\frac{1}{\sqrt{2}}$

u = $\frac{1}{2}$ when x = 0 u = 0

du = 2 xdx

$\frac{1}{2}\int_0^{\frac{1}{2}} \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\left[\sin^{-1} u\right]_0^{1/2} = \frac{1}{2}\left[\frac{\pi}{6}-0\right] = \frac{\pi}{12}$

(b) $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$

Let $t=\tan\frac{x}{2}$

$\therefore \frac{dt}{dx}=\frac{1}{2}\sec^2\frac{x}{2}=\frac{1}{2}\left[1+\tan^2\frac{x}{2}\right]=\frac{1}{2}\left[1+t^2\right]$ or $dx=\frac{2\,dt}{1+t^2}$

and $\cos\theta=\frac{1-t^2}{1+t^2}$

$\therefore \int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x} = \int_0^1 \frac{1}{2+\frac{1-t^2}{1+t^2}}\cdot\frac{2\,dt}{1+t^2}$

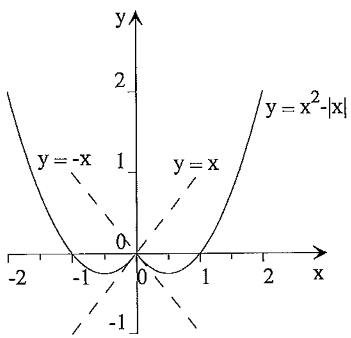
since $t=\tan\frac{\pi}{4}=1$ and $t=\tan\frac{0}{2}=0$

$= \int_0^1 \frac{1}{\frac{2(1+t^2)}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2\,dt}{1+t^2} = \int_0^1 \frac{1+t^2}{3+t^2}\cdot\frac{2\,dt}{1+t^2}$

$= \int_0^1 \frac{2}{3+t^2}dt = 2\int_0^1 \frac{1}{3+t^2}dt = 2\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$

$= \frac{\pi}{3\sqrt{3}}$

(d)



$y=x^2-|x|$

$y=\begin{cases} x^2-x, x\ge 0 \\ x^2+x, x<0 \end{cases}$ $\frac{dy}{dx}=\begin{cases} 2x-1, x\ge 0 \\ 2x+1, x<0 \end{cases}$

$\frac{dy}{dx}\to -1$ as $x\to 0^+$ $\frac{dy}{dx}\to 1$ as $x\to 0^-$

$\Rightarrow \frac{dy}{dx}$ is not defined at $x=0$, and (0, 0) is a critical point.

(c) (i)

$\frac{4}{x^2-1}=\frac{A}{x-1}+\frac{B}{x+1}$

$\Rightarrow \frac{4}{x^2-1}=\frac{A(x+1)+B(x-1)}{x^2-1}$

$\Rightarrow 4=A(x+1)+B(x-1)$

When $x=1, 4=2A$

$\Rightarrow A=2$

When $x=-1, 4=-2B$

$\Rightarrow B=-2$

$\Rightarrow \frac{4}{x^2-1}=\frac{2}{x-1}-\frac{2}{x+1}$

(ii)

$\int \frac{4}{x^2-1}dx$

$=2\int\frac{dx}{x-1}-2\int\frac{dx}{x+1}$

$=2\log_e(x-1)-2\log_e(x+1)$

$=2\log_e\frac{x-1}{x+1}$ +c

(e)

$\int_0^a f(a-x)\,dx$ Let $u=a-x$ $\therefore \frac{du}{dx}=-1$

If $x=a$ then $u=a-a=0$, $x=0$ then $u=a-0=a$

$\int_0^a f(u).-du=-\int_a^0 f(u)\,du$

$=\int_0^a f(u)\,du$

Question 12

Criteria
(a) One for $e^x\sin x-\int(\sin x)e^x dx$, one for $\int_0^{\frac{\pi}{2}} e^x\cos x dx=\left[e^x\sin x\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}(\sin x)e^x dx$, one for $2\int_0^{\frac{\pi}{2}} e^x\cos x dx=\left[e^x\sin x\right]_0^{\frac{\pi}{2}}+\left[e^x\cos x\right]_0^{\frac{\pi}{2}}$ and one for simplification. (b) (i) One for modulus and one for argument (ii) One for $=2^5 cis\frac{10\pi}{3}=32\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$ and one for simplification. (c) One for $x^2+y^2=1$ and one for simplification (d) One for location of lines and one for shaded area. (e) One for location ,one for shape of curve and one for x intercepts.

(a) $\int_0^{\frac{\pi}{2}} e^x \cos x\,dx$

Let $u = e^x$ and $dv = \cos x\,dx$

$\Rightarrow du = e^x dx$ and $v = \sin x$

$\int u\,dv = uv - \int v\,du$

$= e^x \sin x - \int (\sin x) e^x\,dx$

$\int_0^{\frac{\pi}{2}} e^x \cos x\,dx = \left[e^x \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (\sin x) e^x\,dx$

Let $u = e^x$ and $dv = \sin x\,dx$

$\Rightarrow du = e^x dx$ and $v = -\cos x$

$\Rightarrow \int_0^{\frac{\pi}{2}} e^x \cos x\,dx$

$= \left[e^x \sin x\right]_0^{\frac{\pi}{2}} + \left[e^x \cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (\cos x) e^x\,dx$

$\Rightarrow 2\int_0^{\frac{\pi}{2}} e^x \cos x\,dx$

$= \left[e^x \sin x\right]_0^{\frac{\pi}{2}} + \left[e^x \cos x\right]_0^{\frac{\pi}{2}}$

$= \left[e^{\frac{\pi}{2}} \times 1 - 0\right] + \left[e^{\frac{\pi}{2}} \times 0 - 1 \times 1\right]$

$= e^{\frac{\pi}{2}} - 1$

$\Rightarrow \int_0^{\frac{\pi}{2}} e^x \cos x\,dx = \frac{1}{2}(e^{\frac{\pi}{2}} - 1)$

(b) (i) $-1 + \sqrt{3}i = 2cis\frac{2\pi}{3}$

(ii) $\left(-1+\sqrt{3}i\right)^5 = \left(2cis\frac{2\pi}{3}\right)^5$

$= 2^5 cis\frac{10\pi}{3} = 32\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$= 32\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= 16\left(-1 - \sqrt{3}i\right)$

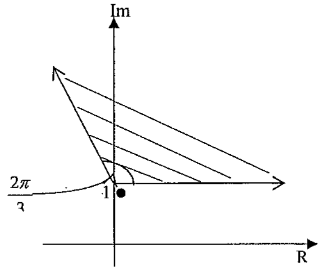
(c) Let $z = x + iy$.

$|z| = 1$, so $x^2 + y^2 = 1$

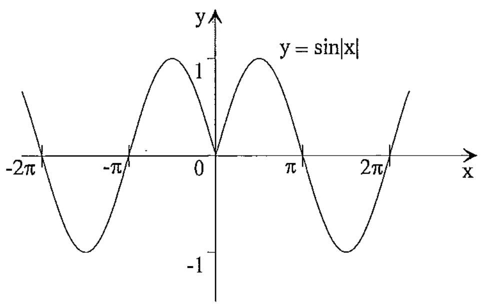
$\text{LHS} = z^{-1} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{x-iy}{1} = \bar{z}$

=RHS

(d)



(e)



Question 13

Criteria

(a) (i) One for derivative, one for $y - y_1 = \frac{-b^2 x_1}{a^2 y_1}(x - x_1)$ and one for simplification (ii) One for complements of $\sin x$ and $\cos x$ and one for simplification (b) (i) One for $cis\frac{\pi}{3}$, $cis\frac{2\pi}{3}$, $cis\frac{4\pi}{3}$ and $cis\frac{5\pi}{3}$. and one for simplification. (ii) One for $(z-1)$ and $(z+1)$.and one for simplification. (c) (i) One for correct answer.(ii) One for correct answer. (iii) One for location and one for shape of curve. (iv) One for location and one for shape of curve.

(a) (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Now $\frac{2x}{a^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{-b^2 x}{a^2 y}$

$\therefore$ Gradient of tangent at $(x_1, y_1) = \frac{-b^2 x_1}{a^2 y_1}$

$\therefore Eqn \quad y - y_1 = \frac{-b^2 x_1}{a^2 y_1}(x - x_1)$

$a^2 y_1 y - a^2 y_1^{\,2} = -b^2 x_1 x + b^2 x_1^2$

$b^2 x_1 x + a^2 y_1 y = b^2 x_1^2 + a^2 y_1^{\,2}$ Divide by $a^2 b^2$

$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$

$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$

(ii) $M(\frac{a}{2}(\cos\theta + \cos\phi), \frac{b}{2}(\sin\theta + \sin\phi))$

$\phi = \frac{\pi}{2} - \theta \therefore \cos\phi = \sin\theta \;\&\; \sin\phi = \cos\theta$

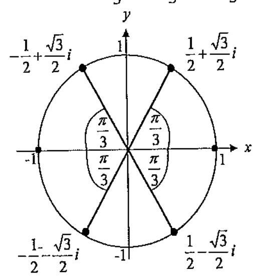
$\therefore M(\frac{a}{2}(\cos\theta + \sin\theta), \frac{b}{2}(\sin\theta + \cos\theta))$

(b) (i) $z^6 = 1$

We are looking for the sixth roots of unity. We know that one root is 1 and another is −1. The 6 roots of unity are evenly spaced around the circumference of a circle of radius 1 unit.

So, the other four must be

$cis\frac{\pi}{3}$, $cis\frac{2\pi}{3}$, $cis\frac{4\pi}{3}$ and $cis\frac{5\pi}{3}$.



So, th

$\pm 1, +\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

(ii) $z^6 - 1 = 0$

$\left(z^3 - 1\right)\left(z^3 + 1\right) = 0$

$(z-1)\left(z^2 + z + 1\right)(z+1)\left(z^2 - z + 1\right) = 0$

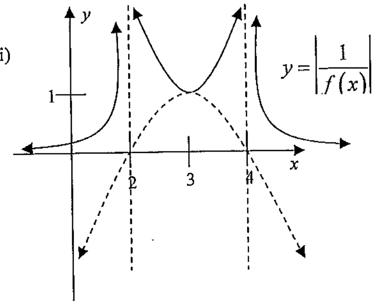
The two real roots of the equation are revealed by the factors $(z-1)$ and $(z+1)$. The four non-real roots are revealed by the factors $\left(z^2 + z + 1\right)$and $\left(z^2 - z + 1\right)$.

So $\left(\omega^2 + \omega + 1\right)\left(\omega^2 - \omega + 1\right) = 0$

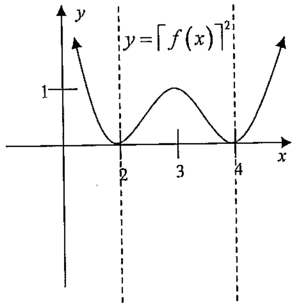
$\omega^4 + \omega^2 + 1 = 0$

So $\omega^4 + \omega^2 = -1$ as required.

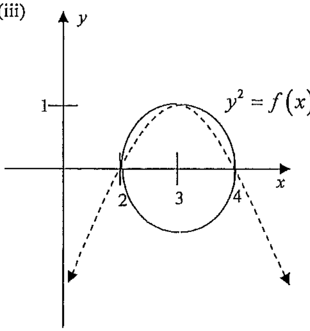
(i)



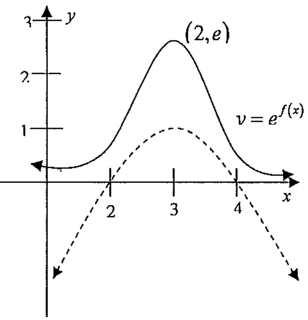
(ii)



(iii)



(iv)



Question 14

Criteria

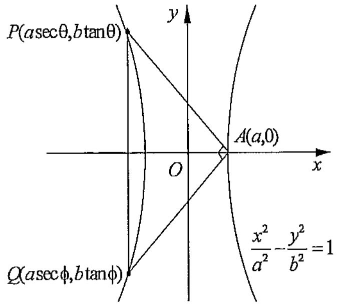
(a) (i) One for finding both a and b and one for simplification (b) One for $\cos\theta + \cos\phi - 1 - \cos\theta\cos\phi = \frac{b^2}{a^2}\sin\theta\sin\phi$, one for $\frac{b^2}{a^2}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\left(2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right)$ one for $-4\sin^2\frac{\theta}{2}\sin^2\frac{\theta}{2} = \frac{b^2}{a^2}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\left(2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right)$ and one for simplification.(c) One for correct answer (d) (i) One for correct answer. (ii) One for $\sin 2x(2\cos x + 1) = 0$ and one for simplification (e) One for finding a and b, one for splitting integral and one for simplification. (f) One for $\therefore 2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx}$ =0 and one for simplification.

Answers

(a) Eccentricity $e = \frac{3}{2}$ and the directrices $x = \pm 4$ of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. But the directrices have equations $x = \pm\frac{a}{e}$. Thus $a = 4 \cdot \frac{3}{2} = 6$. Now $b^2 = a^2(e^2 - 1) = 36 \cdot \left(\frac{9}{4} - 1\right) = 45$.

Cartesian equation of the hyperbola is $\frac{x^2}{36} - \frac{y^2}{45} = 1$.

(b)



It is a right-angled triangle. Therefore $AP^2 + AQ^2 = PQ^2$.

$a^2(\sec\theta - 1)^2 + b^2\tan^2\theta + a^2(\sec\phi - 1)^2 + b^2\tan^2\phi$

$= a^2(\sec\theta - \sec\phi)^2 + b^2(\tan\theta - \tan\phi)^2$

Then

$-2a^2\sec\theta + a^2 - 2a^2\sec\phi + a^2 = -2a^2\sec\theta\sec\phi - 2b^2\tan\theta\tan\phi$,

$\cos\theta + \cos\phi - 1 - \cos\theta\cos\phi = \frac{b^2}{a^2}\sin\theta\sin\phi$,

$\left(1 - 2\sin^2\frac{\theta}{2}\right) + \left(1 - 2\sin^2\frac{\phi}{2}\right) - 1 - \left(1 - 2\sin^2\frac{\theta}{2}\right)\left(1 - 2\sin^2\frac{\phi}{2}\right)$,

$= \frac{b^2}{a^2}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\left(2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right)$

(c)

$P(x) = (x+1)(x-3)^3$

$= (x+1)(x^3 - 9x^2 + 27x - 27) = x^4 - 8x^3 + 18x^2 - 27$

(d) (i)

$\sin x + \sin 3x$

$= \sin(2x - x) + \sin(2x + x)$

$= \sin 2x\cos x - \cos 2x\sin x +$

$\sin 2x\cos x + \cos 2x\sin x$

$= 2\sin 2x\cos x$

(d) (ii)

$\sin x + \sin 2x + \sin 3x = 0$

$2\sin 2x\cos x + \sin 2x = 0$

$\sin 2x(2\cos x + 1) = 0$

$\therefore \sin 2x = 0, \quad \cos x = -\frac{1}{2}$

$2x = 0, \pi, 2\pi, 3\pi, 4\pi$

$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{2\pi}{3}, \frac{4\pi}{3}$

(e)

$\frac{x^2 + 4x + 5}{(x+1)(x+3)} \equiv 1 + \frac{2}{(x+1)(x+3)}$.

Let $\frac{2}{(x+1)(x+3)} \equiv \frac{a}{(x+1)} + \frac{b}{(x+3)}$

$2 \equiv a(x+3) + b(x+1)$.

Let $x = -1 : 2 = 2a \Rightarrow a = 1$. Let $x = -3 : 2 = -2b \Rightarrow b = -1$.

$\int_0^3 \frac{x^2 + 4x + 5}{(x+1)(x+3)}dx = \int_0^3 1dx + \int_0^1 \frac{2}{(x+1)(x+3)}dx$

$= [x]_0^3 + \int_0^3 \frac{1}{x+1}dx - \int_0^3 \frac{1}{x+3}dx$

$$-4\sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2}=\frac{b^2}{a^2}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\left(2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right).$$

Hence $\tan\frac{\theta}{2}\tan\frac{\phi}{2}=-\frac{b^2}{a^2}$.

$$=3+\left[\ln|x+1|\right]_0^3-\left[\ln|x+3|\right]_0^3$$

$$=3+\ln 4-\ln 1-(\ln 6-\ln 3)=3+\ln 2$$

(f)

$$x^2+xy+y^2=1\quad\therefore 2x+y+x\frac{dy}{dx}+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}(x+2y)=-(2x+y)\quad\therefore\frac{dy}{dx}=\frac{-(2x+y)}{x+2y}$$

Question 15

Criteria

(a)(i) One for $\int(\sec^2 x-1)\tan^{n-2}x\,dx$, one for $\int\sec^2 x\tan^{n-2}x\,dx-\int\tan^{n-2}x\,dx$, and one for simplification. (ii) One for $\frac{1}{2}[1-0]+[\ln(\cos x)]_0^{\frac{\pi}{4}}$ and one for simplification. (b) One for finding velocity and one for simplification. (c) One for $=\lim_{\delta x\to 0}\sum_{x=0}^{\frac{\pi}{2}}\sin^2 x\delta x=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^2 x\,dx$, one for $\frac{1}{2}\left[x-\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$ and one for simplification. (d) (i) One mark for line (A), one for line (B) and one for simplification.(ii) One mark for line (A) and one mark for simplification.

Answers

(a) (i) $I_n=\int\tan^n x\,dx$

$$=\int\tan^2 x\,.\tan^{n-2}x\,dx$$

$$=\int(\sec^2 x-1)\tan^{n-2}x\,dx$$

$$=\int\sec^2 x\tan^{n-2}x\,dx-\int\tan^{n-2}x\,dx$$

$$=\frac{1}{n-1}\tan^{n-1}x-I_{n-2}$$

(ii)

$$\int_0^{\frac{\pi}{4}}\tan^3 x\,dx=\left[\frac{1}{2}\tan^2 x\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\tan x\,dx$$

$$=\frac{1}{2}[1-0]+[\ln(\cos x)]_0^{\frac{\pi}{4}}$$

$$=\frac{1}{2}+\left[\ln\left(\frac{1}{\sqrt{2}}\right)-\ln 1\right]$$

$$=\frac{1}{2}+\ln\frac{1}{\sqrt{2}}$$

(d)

(i) LHS $=\tan[(\alpha+\beta)+\gamma]$

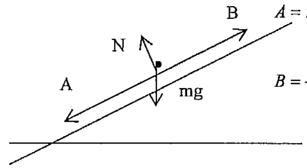
$$=\frac{\tan(\alpha+\beta)+\tan\gamma}{1-\tan(\alpha+\beta)\tan\gamma}$$

$$=\frac{\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}+\tan\gamma}{1-\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\tan\gamma}\quad\text{.....A}$$

$$=\frac{\dfrac{\tan\alpha+\tan\beta+\tan\gamma(1-\tan\alpha\tan\beta)}{1-\tan\alpha\tan\beta}}{\dfrac{(1-\tan\alpha\tan\beta)-\tan\gamma(\tan\alpha+\tan\beta)}{1-\tan\alpha\tan\beta}}\quad\text{.....B}$$

$$=\frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\beta-\tan\beta\tan\gamma-\tan\gamma\tan\alpha}$$

(b)



$A=mg\sin\theta$

$B=\frac{mV^2}{r}\cos\theta$

For no sideways movement

$A-B=0$

$$\therefore\frac{mV^2}{r}\cos\theta=mg\sin\theta$$

$$\therefore\tan\theta=\frac{V^2}{gr}$$

$$V=\frac{100\times1000}{3600}=\frac{250}{9}ms^{-1}\qquad\therefore\tan\theta=\frac{(\frac{250}{9})^2}{10\times100}$$

$\theta=37°39'$

Angle of banking is $37°39'$

(c)

Area of cross section= $\sin^2 x$

Volume of cross section $\sin^2 x\delta x$

Therefore Vol of solid

$$=\lim_{\delta x\to 0}\sum_{x=0}^{\frac{\pi}{2}}\sin^2 x\delta x=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^2 x\,dx$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}(1-\cos 2x)dx=\frac{1}{2}\left[x-\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\left(\frac{\pi}{2}-\frac{\sin\pi}{2}\right)-\frac{1}{2}\left(0-\frac{\sin 0}{2}\right)=\frac{\pi}{4}\text{ cubic units}$$

(ii) Let $\alpha=\tan^{-1}\frac{1}{2}\Rightarrow\tan\alpha=\frac{1}{2}$

$\beta=\tan^{-1}\frac{1}{4}\Rightarrow\tan\beta=\frac{1}{4}$ and

$\gamma=\tan^{-1}\frac{1}{13}\Rightarrow\tan\gamma=\frac{1}{13}$

$$\tan\left(\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{13}\right)\text{......A}$$

$$=\frac{\frac{1}{2}+\frac{1}{4}+\frac{1}{13}-\frac{1}{2}\times\frac{1}{4}\times\frac{1}{13}}{1-\frac{1}{2}\times\frac{1}{4}-\frac{1}{4}\times\frac{1}{13}-\frac{1}{13}\times\frac{1}{2}}=1$$

$$\therefore\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{13}=\tan^{-1}1=\frac{\pi}{4}$$

Question 16

Criteria

(a) (i) One for correct answer (b) One for line (A), one for line (B) and one for simplification (c) One for finding α^n, one for finding β^n, one for subtraction and one for simplification. (d) (i) One for correct answer (ii) One for $f'(x) < g'(x)$ when x>8 and one for conclusion.

Answers

(a) (i)

$$m_2\ddot{x} = -\frac{Gm_1m_2}{x^2}$$

so, $$\ddot{x} = -\frac{Gm_1}{x^2}$$

(ii) $$\ddot{x} = -\frac{Gm_1}{x^2}$$

So $$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{Gm_1}{x^2}$$

$$\frac{1}{2}v^2 = -Gm_1\int x^{-2}dx$$

$$v^2 = 2Gm_1x^{-1} + c \quad (A)$$

When $x = R, v = V$

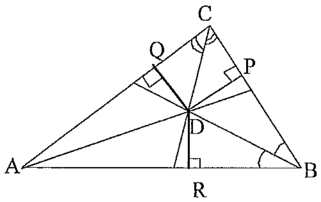
So $$V^2 = \frac{2Gm_1}{R} + c$$

$$c = V^2 - \frac{2Gm_1}{R} \quad (B)$$

So $$v^2 = \frac{2Gm_1}{x} + V^2 - \frac{2Gm_1}{R}$$

$$v = \left(V^2 + \frac{2Gm_1}{x} - \frac{2Gm_1}{R}\right)^{\frac{1}{2}}$$

(b)



$BD = BD$(common)
$\angle DBR = \angle DBP$(given)
$\angle DRB = \angle DPB = 90^0$(given)
$\therefore \Delta DRB \equiv \Delta DPB$(AAS)
$\therefore DR = DP$(corresponding sides of congruent Δ's)

$CD = CD$(common)
$\angle DCQ = \angle DCP$(given)
$\angle CQD = \angle CPD = 90^0$(given)
$\therefore \Delta DQC \equiv \Delta DPC$(AAS)
$\therefore DQ = DP$(corresponding sides of congruent Δ's)

$\therefore DR = DQ$
$AD = AD$(common)
$\angle DQA = \angle DRA = 90^0$(given)
$\therefore \Delta DAQ \equiv \Delta DAR$(RHS)
$\therefore DQ = DR$(corresponding sides of congruent Δ's)
$\angle DAQ = \angle DAR$(corresponding angles of congruent Δ's)
$\therefore$ AD is the bisector of $\angle QAR$
$\therefore$ the internal bisectors of the three triangles are congruent

(c)

Given $x^2 - 2x + 4 = 0$,

then $x = \frac{--2\pm\sqrt{(-2)^2-4(1)(4)}}{2\times1} = \frac{2\pm\sqrt{4-16}}{2} = \frac{2^1 \pm 2^1\sqrt{-3}}{2^1}$

$= \frac{1\pm 3i}{1}$

The roots are $1+\sqrt{3}i$ and $1-\sqrt{3}i$

Let $\alpha = 1+\sqrt{3}i$ and $\beta = 1-\sqrt{3}i$

$$|\alpha| = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$$

Pricipal value of $\alpha = \tan^{-1}\left|\frac{\sqrt{3}}{1}\right| = \frac{\pi}{3}$

$\theta = \frac{\pi}{3}$ as $1+\sqrt{3}i$ is in the first quadrant

$\therefore \alpha = 1+\sqrt{3}i = 2(\text{Cos}\frac{\pi}{3} + \text{Sin}\frac{\pi}{3})$

Now $\alpha^n = \left\{2(\text{Cos}\frac{\pi}{3} + \text{i Sin}\frac{\pi}{3})\right\}^n$

$= 2^n\left[\text{Cos}\frac{\pi}{3} + \text{i Sin}\frac{\pi}{3}\right]^n$

$= 2^n\left[\text{Cos}\frac{n\pi}{3} + \text{i Sin}\frac{n\pi}{3}\right]$

Similarly $\beta^n = 2^n\left[\text{Cos}\frac{\pi}{3} - i\text{Sin}\frac{\pi}{3}\right]^n$

$= 2^n\left[\text{Cos}\frac{n\pi}{3} - i\text{Sin}\frac{n\pi}{3}\right]$

$\therefore \alpha^n - \beta^n = 2^n\left[2i\text{Sin}\frac{n\pi}{3}\right] = 2^{n+1}i\text{Sin}\frac{n\pi}{3}$

Putting $n = 9$, we get

$\alpha^n - \beta^n = 2^{10}i\ \text{Sin}\frac{9\pi}{3}$

$= 2^{10}i.0 = 0$

(d) (i)

$$f'(x) = \frac{1}{2\sqrt{1+x}}$$

$$< \frac{1}{2\sqrt{1+8}} \quad (\text{since } x > 8)$$

$$= \frac{1}{6}$$

So $f'(x) < \frac{1}{6}$ when x>8

(ii)

Let $g(x) = 3 + \frac{x-8}{6}$

Now, $g'(x) = \frac{1}{6}$ for all x

Also, $f(8) = 3 = g(8)$

So we have $f(8)=g(8)$

and $f'(x) < g'(x)$ when x>8

It follows that $f(x) \le g(x)$ for all $x \ge 8$