ASCHAM - 2011 TASK 1

YEAR 12 EXTENSION 2 MATHEMATICS April 1 2011

HSC Assessment: Conics and Curve Sketching

Question 1 (12 Marks)

Consider $f(x)=\frac{x}{x^2-4}$ On separate half page diagrams, sketch the following graphs.

Indicate clearly any obvious critical points and asymptotes.

i) $y=f(x)$ [2]

ii) $y=\frac{1}{f(x)}$ [2]

iii) $y=f(|x|)$ [2]

iv) $y=f(-x)$ [2]

v) $y^2=f(x)$ [2]

vi) $y=[f(x)]^2$ [2]

Question 2 (14 Marks)

a) Consider the ellipse: $\frac{x^2}{5}+\frac{y^2}{4}=1$.

Find:

i) the eccentricity [1]

ii) the foci [1]

iii) the directrices [1]

iv) and hence sketch the conic showing these features on your diagram. [3]

Please Turn Over

b) i) $P(a\sec\theta, a\tan\theta)$ lies on the rectangular hyperbola $x^2-y^2=a^2$. A is the point $(a,0)$. M is the midpoint of AP. Find the locus of M. [4]

ii) $P\left(ct,\frac{c}{t}\right)$ is any point on the rectangular hyperbola $xy=c^2$. U and V are the points where the tangent at P intersects the asymptotes. O is the centre of the hyperbola. Prove that U, V and O are concyclic points of the circle centre P. [4]

Question 3 (6 Marks)

i) Show that if $y=mx+k$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $P(a\sec\theta, b\tan\theta)$, then $m^2a^2-b^2=k^2$. [3]

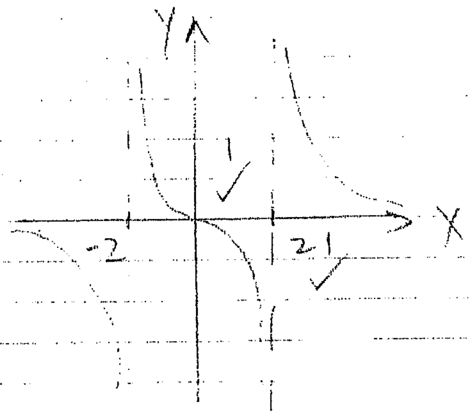
ii) Hence find the equations of the tangents from the point (1,3) to the hyperbola $\frac{x^2}{4}-\frac{y^2}{15}=1$ and the coordinates of their points of contact. [3]

End of Assessment

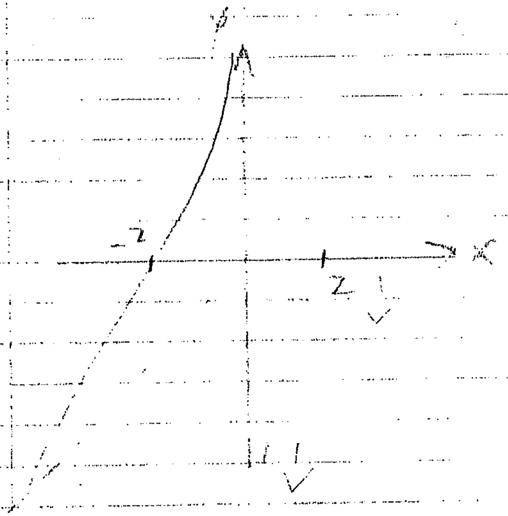
Yr. 12 Ext 2 HSC Task 1/4/2011

Q1 $y=f(x)=\dfrac{x}{x^2-4}$

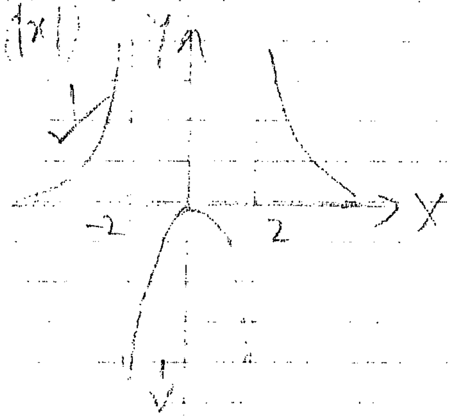
i)



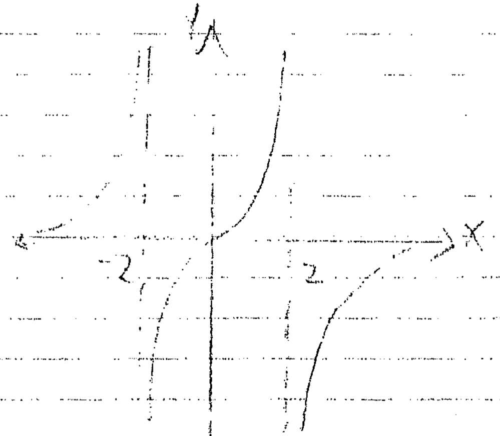
ii) $\dfrac{1}{f(x)}=\dfrac{x^2-4}{x}=x-\dfrac{4}{x}=y$



(iii) $y=f(|x|)$



(iv) $y=f(-x)$



Q2 (a) $\frac{x^2}{(\sqrt{5})^2}+\frac{y^2}{2^2}=1$ $\qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(i) $\frac{b^2}{a^2}=1-e^2$

$e^2=1-\frac{b^2}{a^2}=1-\frac{4}{5}$

$=\frac{1}{5}$

$e=\frac{1}{\sqrt{5}}$ 1 ✓ is the eccentricity.

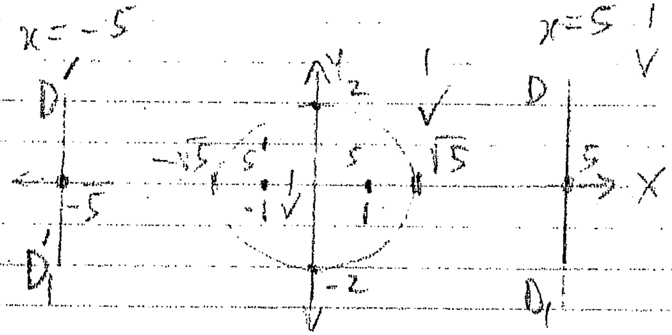
(ii) $S(\pm ae, 0)$ $\qquad ae=\sqrt{5}\times\frac{1}{\sqrt{5}}=1$

$\therefore S(\pm 1, 0)$ are the foci. 1 ✓

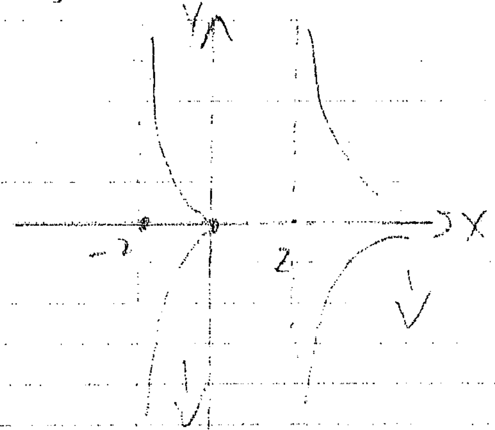
(iii) $x=\pm\frac{a}{e}=\pm\frac{\sqrt{5}}{\left(\frac{1}{\sqrt{5}}\right)}=\pm 5$

$x=\pm 5$ are the directrices 1 ✓

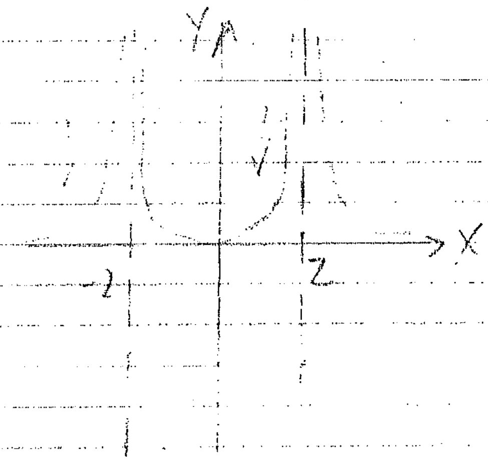
(iv)



(v) $y^2=f(x)$



(vi) $y=[f(x)]^2=\frac{x^2}{(x^2-4)^2}$



(b) (i) $P(a\sec\theta, a\tan\theta)$ $A(a,0)$

$M\left(\frac{a\sec\theta + a}{2}, \frac{a\tan\theta + 0}{2}\right)$

$x = \frac{a(\sec\theta + 1)}{2}$ (1), $y = \frac{a\tan\theta}{2}$ (2) ✓

From (1) $\frac{2x}{a} = \sec\theta + 1$

$\frac{2x}{a} - 1 = \sec\theta$ (3)

From (2) $\frac{2y}{a} = \tan\theta$ (4) ✓

Now $1 + \tan^2\theta = \sec^2\theta$

$1 + \left(\frac{2y}{a}\right)^2 = \left(\frac{2x}{a} - 1\right)^2$ ✓

$1 + \frac{4y^2}{a^2} = \frac{4x^2}{a^2} - \frac{4x}{a} + 1$

$a^2 + 4y^2 = 4x^2 - 4ax + a^2$

$4y^2 = 4x^2 - 4ax$

$y^2 = x^2 - ax$

$= (x^2 - ax + \frac{a^2}{4}) - \frac{a^2}{4}$

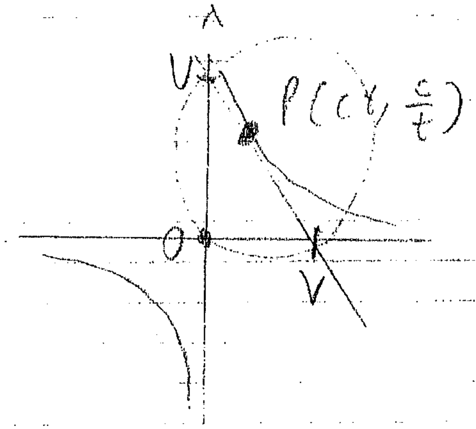
$y^2 + \frac{a^2}{4} = (x - \frac{a}{2})^2$

$\frac{a^2}{4} = (x - \frac{a}{2})^2 - y^2$

$1 = \frac{4}{a^2}(x - \frac{a}{2})^2 - \frac{4y^2}{a^2}$ ✓

which is a rectangular hyperbola

(ii)



tangent at P: $x = ct$ $y = ct^{-1}$

$\frac{dx}{dt} = c$ $\frac{dy}{dt} = -ct^{-2}$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{c}{t^2} \times \frac{1}{c} = -t^{-2}$ ✓

$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

$t^2y - ct = -x + ct$

$x + t^2y = 2ct$

so at U, $x = 0$, $y = \frac{2ct}{t^2} = \frac{2c}{t}$

$\therefore U(0, \frac{2c}{t})$ ✓

at V, $y = 0$, $x = 2ct$

$\therefore V(2ct, 0)$ ✓

and $O(0,0)$

Q3

(i) $\frac{d}{dx}\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = \frac{d}{dx}(1)$

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{2x}{a^2} = \frac{2y}{b^2}\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{b^2 x}{a^2 y} = \frac{b^2 x_1}{b^2 y_1} \text{ at } (x_1, y_1)$$

tangent: $m = \frac{b^2 x_1}{a^2 y_1}$ (x_1, y_1)

$$y - y_1 = m(x - x_1)$$

$$y - y_1 = \frac{b^2 x_1}{a^2 y_1}(x - x_1)$$

$$a^2 y y_1 - a^2 y_1^2 = b^2 x_1 x - b^2 x_1^2$$

$$\frac{b^2 x_1^2}{b^2 a^2} - \frac{a^2 y_1^2}{b^2 a^2} = \frac{b^2 x_1 x}{b^2 a^2} - \frac{a^2 y_1 y}{b^2 a^2}$$

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = \frac{x_1 x}{a^2} - \frac{y_1 y}{b^2}$$

$$\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1 \text{ (can be quoted)}$$

at $P(a\sec\theta, b\tan\theta)$, tangent is

$$\frac{a\sec\theta}{a^2}x - \frac{b\tan\theta}{b^2}y = 1$$

$$PU^2 = (ct - 0)^2 + \left(\frac{c}{t} - \frac{2c}{t}\right)^2$$

$$= c^2t^2 + \frac{c^2}{t^2}$$

$$PV^2 = (ct - 2ct)^2 + \left(\frac{c}{t} - 0\right)^2$$

$$= c^2t^2 + \frac{c^2}{t^2}$$

$$PO^2 = (ct)^2 + \left(\frac{c}{t}\right)^2$$

$$= c^2t^2 + \frac{c^2}{t^2}$$

so $PO = PV = PU$ ✓

and P is the centre of the circle which passes through U, V and O.

or $\frac{\tan\theta}{b}y = \frac{\sec\theta}{a}x - 1$

$y = \frac{b\sec\theta}{a\tan\theta}x - \frac{b}{\tan\theta}$

but $y = mx + k$ ✓

$\therefore m = \frac{b\sec\theta}{a\tan\theta} = \frac{b\cos\theta}{a\sin\theta\cos\theta} = \frac{b\,\mathrm{cosec}\,\theta}{a}$ (1)

$k = -\frac{b}{\tan\theta} = -b\cot\theta$ ✓ (2)

from (1): $\mathrm{cosec}\,\theta = \frac{am}{b}$ (3)

(2): $\cot\theta = -\frac{k}{b}$ (4)

now $1 + \cot^2\theta = \mathrm{cosec}^2\theta$

$1 + \left(-\frac{k}{b}\right)^2 = \left(\frac{am}{b}\right)^2$ (from (3) and (4))

$1 + \frac{k^2}{b^2} = \frac{a^2m^2}{b^2}$ ✓

$b^2 + k^2 = a^2m^2$

or $k^2 = m^2a^2 - b^2$ as required

(ii) tangents pass through $(1,3)$

$\therefore (1,3)$ satisfies $y = mx + k$

$3 = m + k$

$\therefore k = 3 - m$ (1)

in $\frac{x^2}{4} - \frac{y^2}{15} = 1$ (A) $a^2 = 4$, $a = 2$; $b^2 = 15$, $b = \sqrt{15}$

$m^2a^2 - b^2 = k^2$

$4m^2 - 15 = k^2$ (2) ✓

Sub (1) into (2):

$4m^2 - 15 = (3-m)^2$

$4m^2 - 15 = 9 - 6m + m^2$

$3m^2 + 6m - 24 = 0$

$3(m^2 + 2m - 8) = 0$

$\Rightarrow (m+4)(m-2) = 0$

$m = 2, -4$

when $m = 2$, $k = 3 - 2 = 1 \Rightarrow y = 2x + 1$

$m = -4$, $k = 3 - -4 = 7 \Rightarrow y = -4x + 7$

So tangents are $y = 2x + 1$ (3) ✓

and $y = -4x + 7$ (4)

Sub (3) into (A): $\frac{x^2}{4} - \frac{(2x+1)^2}{15} = 1$

$15x^2 - 4(2x+1)^2 = 60$

$15x^2 - 4(4x^2 + 4x + 1) = 60$

$15x^2 - 16x^2 - 16x - 4 = 60$

$0 = x^2 + 16x + 64$

$0 = (x+8)^2$

$x=-8,\ y=2\times(-8)+1=-15$

$\therefore y=2x+1$ has point of contact $(-8,-15$

Sub ④ into Ⓐ: $\frac{x^2}{4}-\frac{(-4x+7)^2}{15}=1$

$15x^2-4(16x^2-56x+49)=60$

$15x^2-64x^2+224x-196=60$

$0=49x^2+224x+256$

$=(7x)^2+2\times7\times16x+16^2$

$=(7x+16)^2$

$\therefore x=-\frac{16}{7},\ y=2\times\left(-\frac{16}{7}\right)+1=-\frac{32}{7}+\frac{7}{7}$

$=-\frac{25}{7}$

$\therefore y=-4x+7$ has point of contact $\left(-\frac{16}{7},-\frac{2}{7}\right.$