12 Extension 1 **Trigonometry, Inverse functions & Inverse Trigonometric Functions**

Term 21, 2011 | Week 6 *Time Allowed*: 50 mins *Marks*: 36

Show all working to gain maximum marks

Marks will be deducted for poorly presented or illegible work

Question 1 (12 marks) **Start a new booklet** **Marked by GHW**

(a) Find in terms of π the value of the expression
$\cos^{-1}(-\frac{1}{2})-\sin^{-1}(-\frac{1}{2})$ [2]

(b) Find $\int \frac{dx}{\sqrt{4-x^2}}$ [2]

(c) Find $\int \frac{dx}{9+25x^2}$ [2]

(d) Find the gradient function of $y=\cos^{-1}\cos x$ [2]

(e) For $f(x)=x^2+2x$

(i) State the domain of $f(x)=x^2+2x$ which restricts it to a monotonic increasing curve [2]

(ii) Find the function, $f^{-1}(x)$ and sketch it [2]

START A NEW BOOKLET

Question 2 (13 marks) **Marked by RDS**

(a) Show $$\frac{1-\sin(\frac{\pi}{2}-2\theta)}{\sin 2\theta}=\tan\theta$$ [3]

(b) Find the exact value of $\cos(\frac{\pi}{12})$ [3]

(c) Find $\int \cos^2 x+\frac{1}{\cos^2 x}dx$ [3]

(d) Find the general solution of: $\sqrt{3}\sin x-\cos x=\sqrt{3}$ [4]

START A NEW BOOKLET

Question 3 (11 marks) **Marked by HRK**

(a) A function is given by the rule $f(x)=\frac{x+1}{x+2}$. Find the rule for the inverse function $f^{-1}(x)$. **[2]**

(b) (i) Sketch the graph of $y=4\sin^{-1}(\frac{x}{2})$. [2]

(ii) Find the exact equation of the tangent to the curve $y=4\sin^{-1}(\frac{x}{2})$ at the point where $x=1$. [3]

(c) Find the first derivative of $\left(\cos^{-1}7x\right)^5$ [2]

(d) Find $\frac{d}{dx}[e^{\tan^{-1}\underbrace{(\ln[\cos 2x])}_{u}}]$ in its simplest form. [2]

Ext 1 Term 2, Test 1
2011 Exam - Solutions

Question 1

a) $\frac{2\pi}{3} - \left(-\frac{\pi}{6}\right)$

$= \frac{4\pi}{6} + \frac{\pi}{6}$

$= \frac{5\pi}{6}$ (2)

b) $\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\left[\frac{x}{2}\right] + C$ (2)

c) $\int \frac{dx}{9+25x^2} = \int \frac{dx}{3^2+(5x)^2} = \frac{1}{3}\tan^{-1}\left(\frac{5x}{3}\right) \div 5 + C$

$= \frac{1}{15}\tan^{-1}\left(\frac{5x}{3}\right) + C$ (2)

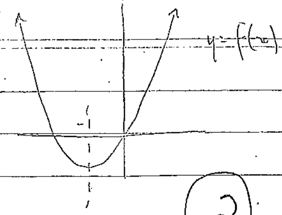
d) $y = \cos^{-1}\cos x$ $\qquad u = \cos x$

$y = \cos^{-1} u$ $\qquad \frac{du}{dx} = -\sin x$

$\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}}$

$\therefore \frac{dy}{dx} = \frac{\sin x}{\sqrt{1-\cos^2 x}} = \frac{\sin x}{\sin x} = 1$ (2)

e) $f(x) = x^2+2x$



i) increasing $\therefore y' \geq 0$

$x \geq -1$ (2)

ii) $y = x^2+2x$

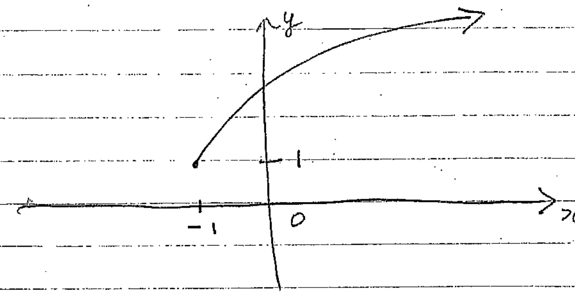
$y+1 = x^2+2x+1$
$= (x+1)^2$

$x+1 = \sqrt{y+1}$

$x = -1 \pm \sqrt{y+1}$ $\quad \therefore y = -1 \pm \sqrt{x+1}$

As this is restricted, $f^{-1}(x) = -1 + \sqrt{x+1}$ (2)

(1)



Question 2

a) $$\frac{1-\sin\left(\frac{\pi}{2}-2\theta\right)}{\sin 2\theta} = \tan\theta$$

$$\text{LHS} = \frac{1-\cos 2\theta}{\sin 2\theta} \quad (1)$$

$$= \frac{\cancel{1}-(\cancel{1}-\cancel{2}\sin^2\theta)}{\cancel{2}\sin\theta\cos\theta} \quad (1)$$

$$= \frac{\sin^{\cancel{2}}\theta}{\cancel{\sin\theta}\cos\theta} \quad (1)$$

$= \tan\theta$ As required.

b) $$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{4}-\frac{\pi}{3}\right) \quad (1)$$

$$= \cos\frac{\pi}{4}\sin\frac{\pi}{3} + \sin\frac{\pi}{4}\cos\frac{\pi}{3} \quad (1)$$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4} \quad (1)$$

$$\int \cos^2 x + \frac{1}{\cos^2 x}\, dx$$

$$= \frac{1}{2}\int \cos 2x + 1\, dx \;(1) + \int \sec^2 x\, dx \;(1)$$

$$= \frac{1}{2}\left[\frac{\sin 2x}{2} + x\right] + \tan x + C \quad (1)$$

$$\sqrt{3}\sin x - \cos x = \sqrt{3}$$

let $R\sin(x+\alpha) = \sqrt{3}\sin x - \cos x$

$\text{LHS} = R\sin x\cos\alpha \; \cancel{+} \; R\sin\alpha\cos x$

$R\cos\alpha = \sqrt{3}$ +ve cos and

$R\sin\alpha = +1$ -ve tan ∴ 4th quadrant.

$\tan\alpha = \frac{+1}{\sqrt{3}}$ ∴ $\alpha = +\frac{\pi}{6}$

$R = 2$

$$2\sin\left(x-\frac{\pi}{6}\right) = \sqrt{3}$$

$$\sin\left(x-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$x-\frac{\pi}{6} = n\pi + (-1)^n\cdot\frac{\pi}{3}$$

$$= n\pi + (-1)^n\cdot\frac{\pi}{3} + \frac{\pi}{6}$$

Question 3

$\checkmark = 1$ mark.

a) $f(x) = \dfrac{x+1}{x+2}$

let $y = \dfrac{x+1}{x+2}$ ✓

For a number of students basic algebra skills need polishing!!! ☹

$yx + 2y = x + 1$

$yx - x = 1 - 2y$

$x(y-1) = 1 - 2y$

$x = \dfrac{1-2y}{y-1}$

$\therefore f^{-1}(x) = \dfrac{1-2x}{x-1}$ ✓

b) $y = 4\sin^{-1}\left(\dfrac{x}{2}\right)$

If $y = \sin^{-1} X$

Domain $-1 \le X \le 1$

Range $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$

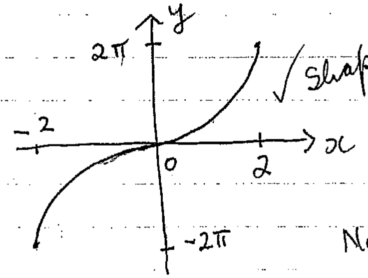
$\therefore$ Domain $-1 \le \dfrac{x}{2} \le 1$

$-2 \le x \le 2$

Range $-\dfrac{\pi}{2} \le \dfrac{y}{4} \le \dfrac{\pi}{2}$

$-2\pi \le y \le 2\pi$

i)



✓ shape ✓ values

NOTE axes should be labelled

ii) $y = 4\sin^{-1}\left(\dfrac{x}{2}\right)$

$\dfrac{d}{dx}\sin^{-1}\left(\dfrac{x}{a}\right) = \dfrac{1}{\sqrt{a^2 - x^2}}$

$\dfrac{dy}{dx} = 4\left[\dfrac{1}{\sqrt{1-\frac{x^2}{4}}}\right] \times \dfrac{1}{2}$ OR $4 \times \dfrac{1}{\sqrt{4 - x^2}}$

$= \dfrac{2}{\sqrt{1-\frac{x^2}{4}}}$

when $x = 1$ $M_T = 4 \times \dfrac{1}{\sqrt{3}}$

$= \dfrac{4}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{4\sqrt{3}}{3}$

When $x = 1$

$\dfrac{dy}{dx} = \dfrac{2}{\sqrt{1-\frac{1}{4}}} = \dfrac{2}{\sqrt{\frac{3}{4}}} = 2 \times \dfrac{2}{\sqrt{3}}$

$= \dfrac{4\sqrt{3}}{3}$

When $x=1$ $y = 4\sin^{-1}\left(\dfrac{1}{2}\right)$

$= 4 \cdot \dfrac{\pi}{6}$

$= \dfrac{2\pi}{3}$

$\therefore \dfrac{y - \frac{2\pi}{3}}{x - 1} = \dfrac{4\sqrt{3}}{3}$

$3y - 2\pi = 4\sqrt{3}x - 4\sqrt{3}$

$\therefore 4\sqrt{3}x - 3y - 4\sqrt{3} + 2\pi = 0$

c)

$$y = \left(\cos^{-1} 7x\right)^5 \qquad u = \cos^{-1} 7x$$

$$\frac{du}{dx} = \frac{-7}{\sqrt{1-x^2}}$$

$$y = u^5$$

$$\frac{dy}{du} = 5u^4$$

$$\therefore \frac{dy}{dx} = \frac{-7}{\sqrt{1-x^2}} \cdot 5\left(\cos^{-1} 7x\right)^4$$

$$= \frac{-35\left(\cos^{-1} 7x\right)^4}{\sqrt{1-x^2}}$$

d) $$\frac{d}{dx}\left[e^{\tan^{-1}(\ln(\cos 2x))}\right]$$

$$\frac{d}{dx}\ln(\cos 2x) = \frac{-2\sin 2x}{\cos 2x} = -2\tan x$$

$$\frac{d}{dx}\tan^{-1}\left[\ln(\cos 2x)\right] = \frac{1}{1+\left[\ln(\cos 2x)\right]^2} \times \frac{d}{dx}\left(\ln(\cos 2x)\right)$$

$$= \frac{-2\tan x}{1+\left[\ln(\cos 2x)\right]^2}$$

$$\therefore \frac{d}{dx} e^{\tan^{-1}(\ln(\cos 2x))} = \frac{-2\tan x}{1+\left[\ln(\cos 2x)\right]^2} \cdot e^{\tan^{-1}\left[\ln(\cos 2x)\right]}$$