SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

SEPTEMBER 2008

Yearly Examination

YEAR 11

Mathematics (2 unit) Accelerated

General Instructions

- There is **NO** reading time.
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Start each **NEW** question in a separate answer booklet.

Total Marks – 68

- Attempt questions 1 – 5
- Questions are not of equal value.

Examiner: *A. Fuller*

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right)\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x > 0$

Total marks 68

Attempt questions 1 to 5

Answer each **Question** in a **Separate** writing booklet

(Use a SEPARATE writing booklet)

Question 1 (15 marks)

(a) Evaluate $\sin 2$ correct to 2 decimal places. $\boxed{1}$

(b) Convert $75°$ to radians (in terms of π). $\boxed{1}$

(c) Differentiate the following with respect to x: $\boxed{7}$

i. $x^3 + e^3$

ii. xe^{3x}

iii. $\sin^3 x$

iv. $\ln\left[\dfrac{(x+1)^3}{x-1}\right]$

(d) If $f(x) = \tan x$, find the exact value of: $\boxed{3}$

i. $f\left(\frac{\pi}{6}\right)$

ii. $f'\left(\frac{\pi}{6}\right)$

(e) i. Show that $\left(\dfrac{\sqrt{x}+1}{\sqrt{x}}\right)^2 = 1 + \frac{2}{\sqrt{x}} + \frac{1}{x}$. $\boxed{3}$

ii. If $f'(x) = \left(\dfrac{\sqrt{x}+1}{\sqrt{x}}\right)^2$ and $f(1) = 5$, find $f(x)$.

(Use a SEPARATE writing booklet)

Question 2 (15 marks)

(a) Consider the series: $195 + 191 + 187 + 183 + \ldots$. $\boxed{6}$

i. Write an expression for the nth term of the series.

ii. What is the first term less than zero?

iii. What is the sum of all the terms that are greater than zero?

(b) Consider the series: $2 - 6x + 18x^2 - 54x^3 + \ldots$. $\boxed{3}$

i. For what values of x will a limiting sum exist for the geometric series?

ii. For what value of x is the limiting sum 9?

(c) The area bounded by the curve $y = 4 - x^2$ and the x-axis is rotated about the y-axis. $\boxed{3}$

Find the volume of the solid formed in terms of π.

(d) A tangent to the curve $y = e^{-x}$ meets the x and y axes at equal (positive) distances $\boxed{3}$

from the origin. Find the equation of this tangent in general form.

(Use a SEPARATE writing booklet)

Question 3 (13 marks)

(a) The population of a small town is increasing at a decreasing rate. Given that P is the population of the town at a given time t, what does this statement imply about $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$? [2]

(b) Find the fifth term of a series whose sum to n terms is given by $S_n = 11n - n^2$. [2]

(c) If $\int_0^5 f(x)dx = 3$, find $\int_0^5 3[f(x) + 2]dx$. [2]

(d) Solve for x: $\log_{25}(6x - 9) = \log_5 x$. [2]

(e) A particle, moving in a straight line, at time t seconds has velocity $v = 3t^2 - 18t + 24$ m/s. Initially the particle is at the origin. [5]

i. When is the particle at rest?

ii. State the displacement function.

iii. How far has the particle travelled in the first 4 seconds?

iv. Does the particle return to the origin?

(Use a SEPARATE writing booklet)

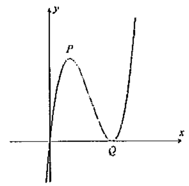
Question 4 (13 marks)

(a) Find a primitive of $\dfrac{1 + e^{-2x}}{1 + e^{2x}}$ [2]

(b) i. Sketch the curve $y = \log_e(x + 2)$, showing any asymptotes and intercepts with the x and y axes. [5]

ii. Find the exact area enclosed by the curve and the x and y axes.

(c) A graph of the function, $y = x(x - a)^2$, for constant a, has a local maximum at P and a local minimum at Q. [6]



i. Determine the coordinates of P and Q in terms of a.

ii. Determine the area bound by the curve and the x-axis, between the origin and the point Q, in terms of a, and hence find the value of a if the area is $\frac{4}{3}$ square units.

(Use a SEPARATE writing booklet)

Question 5 (12 marks)

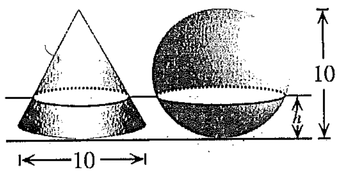
(a) The parabola $y = ax^2 + bx + c$ passes through the points $(-1, 2)$, $(1, 3)$ and $(3, 9)$. Evaluate $\int_{-1}^{3}(ax^2 + bx + c)dx$. 3

(b) AB is a diameter of a semicircle and the chord AP makes an angle of x radians with AB. If AP divides the semicircle into two equal areas, prove that $2x + \sin 2x = \frac{\pi}{2}$. 3

(c) In the diagram, the sphere has a diameter of 10cm. Also, the right circular cone has a height of 10cm, and its base has a diameter of 10cm. The sphere and cone sit on a horizontal surface. If a horizontal plane cuts both the sphere and the cone, the cross-sections will both be circles, as shown. Let the radii of the cross-sections of the cone and sphere at height h be r_1 and r_2 respectively. 6

i. Show that $r_1^2 = \left(5 - \frac{h}{2}\right)^2$ and $r_2^2 = 10h - h^2$.

ii. Hence, or otherwise, find the height of the horizontal plane that gives the greatest sum of the cross-sectional areas.



End of paper

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a) $\sin 2 = 0.91$

b) $75° \Rightarrow \frac{5\pi}{12}$.

c) i) $\frac{d}{dx}(x^3 + e^3)$
$= 3x^2$.

ii) $\frac{d}{dx}(x e^{3x})$
$= e^{3x} + x.3e^{3x}$.
$= e^{3x}(1 + 3x)$

iii) $\frac{d}{dx}(\sin^3 x)$
$= 3\sin^2 x \cos x$.

iv) $\frac{d}{dx}\left(\ln \frac{(x+1)^3}{x-1}\right)$
$= \frac{d}{dx}(3\ln(x+1) - \ln(x-1))$
$= \frac{3}{x+1} - \frac{1}{x-1}$.
$= \frac{2(x-2)}{x^2-1}$.

d) i $f(\frac{\pi}{6}) = \tan(\frac{\pi}{6})$
$= \frac{1}{\sqrt{3}}$.
$f'(x) = \sec^2 x$.
$= \frac{1}{\cos^2(\frac{\pi}{6})}$
$= \left(\frac{2}{\sqrt{3}}\right)^2$.
$= \frac{4}{3}$.

e) $\left(\frac{\sqrt{x}+1}{\sqrt{x}}\right)^2 = \frac{(\sqrt{x}+1)^2}{x}$.
$= \frac{x + 2\sqrt{x} + 1}{x}$.
$= 1 + \frac{2}{\sqrt{x}} + \frac{1}{x}$.

ii). $\int \frac{\sqrt{x}+1}{\sqrt{x}} dx$
$= \int\left(1 + \frac{2}{\sqrt{x}} + \frac{1}{x}\right)dx$.
$= x + 4\sqrt{x} + \ln x + C$.

$f(1) = 1 + 4\sqrt{1} + \ln 1 + C = 5$.
$C = 0$.

$f(x) = x + 4\sqrt{x} + \ln x$.

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QUESTION 2

(a) $195 + 191 + 187 + \dots$

AS: $a = 195 \quad d = -4$

(i) $U_n = a + (n-1)d$

$= 195 - 4(n-1)$

$= 195 - 4n + 4$

$U_n = 199 - 4n$ [2]

(ii) $199 - 4n < 0$

$199 < 4n$

$n > \frac{199}{4}$

> 49.75

$\therefore$ 1st term < 0 is when

$n = 50$, $U_{50} = 199 - 200$

$= -1$ [2]

(iii) $S_n = \frac{n}{2}(2a + (n-1)d)$

$S_{49} = \frac{49}{2}(390 - 4(48))$

$= \cancel{5047}\ 4851$

OR $S_n = \frac{n}{2}(a + l)$

$= \frac{49}{2}(195 + 3)$

$= 4851$ [2]

(b) $2 - 6x + 18x^2 - 54x^3 + \dots$

GS: $a = 2 \quad r = -3x$

(i) S_∞ exists for $|r| < 1$

ie $|-3x| < 1$

$|x| < \frac{1}{3}$ [1]

(ii) $S_\infty = \frac{a}{1-r}$

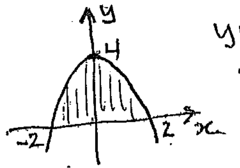
$9 = \frac{2}{1+3x}$

$9 + 27x = 2$

$27x = -7$

$x = -\frac{7}{27}$ [2]

(iii)



$y = 4 - x^2$

$x^2 = 4 - y$

$V = \pi \int_0^4 x^2\, dy$

$= \pi \int_0^4 (4 - y)\, dy$

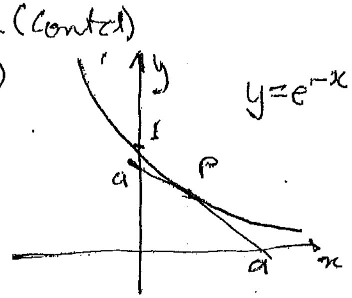
$= \pi \left[4y - \frac{y^2}{2}\right]_0^4$

$= \pi[(16 - 8) - 0]$

$= 8\pi$ [3]

Q2 (Contd) (4)

(d)



Let the tangent meet the curve at P, and the x-axis in $(a, 0)$ and the y-axis in $(0, a)$.

Clearly $m = -1$

$\frac{dy}{dx} = -e^{-x}$

$\therefore -1 = -e^{-x}$

$1 = e^{-x}$

$x = 0$

$\therefore$ Tgt passes thro' $(0, 1)$

$y - 1 = -1(x - 0)$

$= -x$

$x + y - 1 = 0$ [3]

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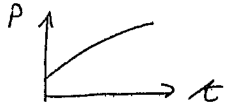
QUESTION 3

(a) $\frac{dP}{dt} > 0$, $\frac{d^2P}{dt^2} < 0$ ✓ ✓



(b) $T_n = S_n - S_{n-1}$

$\therefore T_5 = S_5 - S_4$

$= 11\times 5 - 5^2 - (11\times 4 - 4^2)$

$= 55 - 25 - (44 - 16)$

$= 30 - 28$

$\boxed{= 2.}$ ✓✓

(c) $\int_0^5 3\left(f(x)+2\right)dx = 3\int_0^5 f(x)\,dx + 3\int_0^5 2\,dx$

$= 3\times 3 + 3\left[2x\right]_0^5$

$= 9 + 3\left[10 - 0\right]$

$= \boxed{39.}$ ✓✓

(e) (i) $3t^2 - 18t + 24 = 0$

$3(t^2 - 6t + 8) = 0$

$3(t-4)(t-2) = 0$

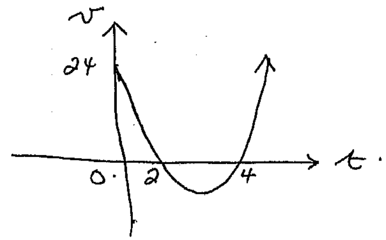
$\boxed{t = 2, 4}$ ✓ $\therefore$ at rest after 2 and 4 secs.

(ii) $x = t^3 - 9t^2 + 24t + c.$

when $t=0, v=0 \therefore c=0$ ✓

$\therefore \boxed{x = t^3 - 9t^2 + 24t}$

(iii)



distance travelled in first 4 secs.

$D = \int_0^2 (3t^2 - 18t + 24)\,dt + \left|\int_2^4 (3t^2 - 18t + 24)\,dt\right|$

$= \left[t^3 - 9t^2 + 24t\right]_0^2 + \left|\left[t^3 - 9t^2 + 24t\right]_2^4\right|$

$= (8 - 36 + 48) + \left|64 - 144 + 96 - (8 - 36 + 48)\right|$

$= 20 + \left|16 - 20\right|$

$= 20 + 4$

$\boxed{D = 24\text{ m}}$ ✓✓

(iv) now $x = t(t^2 - 9t + 24)$

If $x = 0$ $t = 0$ or $t^2 - 9t + 24 = 0$

$\Delta = 81 - 96$ ✓

$= -15$

$\therefore$ NO ROOTS.

$\boxed{\text{Hence NO!!}}$ [* must show a reason]

Question (4).

(a) $\int \frac{1+e^{-2x}}{1+e^{2x}}\,dx$

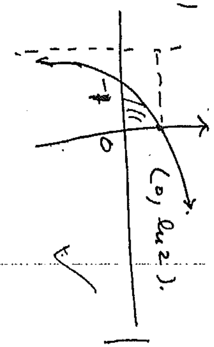
$= \int \frac{\frac{e^{2x}+1}{e^{2x}}}{1+e^{2x}}\,dx$

$= -\frac{1}{2}\int e^{-2x}\,dx$

$= -\frac{1}{2}e^{-2x} + c$

2

(b)



$y = \ln(x+2)$

$x = e^y - 2$

$\int_0^{\ln 2} (e^y - 2)\,dy$

$[e^y - 2y]_0^{\ln 2}$

$= e^{\ln 2} - 2\ln 2 - 1$

$= |2 - 1 - 2\ln 2|$

$A = 2\ln 2 - 1$

5

(c). $y = x(x-a)^2$

$\frac{dy}{dx} = (x-a)^2 + x\cdot 2(x-a)$

$= (x-a)[3x - a]$

$\therefore \frac{dy}{dx} = 0 \quad 3x^2 - 4ax + a^2$

$\Rightarrow x = a, \ x = \frac{a}{3}$

When $x = a$, $y = 0$

$P(a, 0)$.

$P\left(\frac{a}{3}, \frac{4a^3}{27}\right)$.

$\therefore A = \int_0^a x(x^2 - 2ax + a^2)\,dx$

$A = \int_0^a (x^3 - 2ax^2 + a^2x)\,dx$

$= \left[\frac{x^4}{4} - \frac{2ax^3}{3} + \frac{a^2x^2}{2}\right]_0^a$

$= \left(\frac{a^4}{4} - \frac{2a\cdot a^3}{3} + \frac{a^2 \times a^2}{2}\right)$

$= a^4\left[\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right]$

$= \frac{a^4}{12}$

$\therefore \frac{a^4}{12} = \frac{4}{3}$

$3a^4 = 48$

$a^4 = 16$

$\therefore a = 2$

6

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Solutions— Question 5

5. (a) The parabola $y = ax^2 + bx + c$ passes through the points $(-1, 2)$, $(1, 3)$ and $(3, 9)$. Evaluate $\int_{-1}^{3} (ax^2 + bx + c)\,dx$. $\boxed{3}$

Solution: Using Simpson's rule, which is exact for quadratics—

x	-1	1	2
y	2	3	9

$h = 2.$

$\therefore$ Integral $= \frac{2}{3}\Big(2 + 4 \times 3 + 9\Big),$

$= \frac{46}{3}$ or $15^1/_3$.

Solution: Alternative method—

$2 = a - b + c$ $\boxed{1}$

$3 = a + b + c$ $\boxed{2}$

$9 = 9a + 3b + c$ $\boxed{3}$

$\boxed{2} - \boxed{1}$: $1 = 2b,$

$b = {}^1/_2.$

$\boxed{3} - \boxed{2}$: $6 = 8a + 1,$

$a = {}^5/_8.$

Sub. in $\boxed{1}$: $2 = {}^5/_8 - {}^1/_2 + c,$

$c = \frac{15}{8}.$

$I = \frac{1}{8}\int_{-1}^{3} (5x^2 + 4x + 15)\,dx,$

$= \frac{1}{8}\left[\frac{5x^3}{3} + 2x^2 + 15x\right]_{-1}^{3},$

$= \frac{1}{8}\left\{45 + 18 + 45 - \left(\frac{-5}{3} + 2 - 15\right)\right\},$

$= \frac{46}{3}$ or $15^1/_3$.

(b) AB is a diameter of a semicircle and the chord AP makes an angle of x radians with AB. If AP divides the semicircle into two equal areas, prove that $2x + \sin 2x = \frac{\pi}{2}$. [3]

Solution: Area I $= \dfrac{r^2\theta}{2} - \dfrac{r^2 \sin\theta}{2}$,

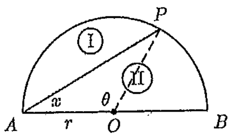
$= \dfrac{r^2}{2}((\pi - 2x) - \sin(\pi - 2x))$,

$= \dfrac{r^2}{2}(\pi - 2x - \sin 2x)$,

i.e. $\dfrac{1}{2} \times \dfrac{\pi r^2}{2} = \dfrac{r^2}{2}(\pi - 2x - \sin 2x)$,

$\dfrac{\pi}{2} = \pi - 2x - \sin 2x$,

$2x + \sin 2x = \dfrac{\pi}{2}$.



Solution: Alternative method—

Area I $= \dfrac{r^2}{2}(\pi - 2x) - \dfrac{r^2}{2}\sin(\pi - 2x)$,

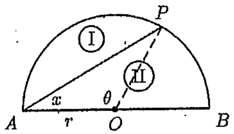
Area II $= \dfrac{r^2}{2}2x + \dfrac{r^2}{2}\sin(\pi - 2x)$,

Equating areas,

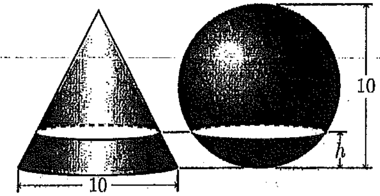
$2x + \sin(\pi - 2x) = \pi - 2x - \sin(\pi - 2x)$,

$4x + 2\sin 2x = \pi$,

$2x + \sin 2x = \dfrac{\pi}{2}$.

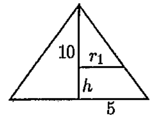


(c) In the diagram, the sphere has a diameter of 10cm. Also, the right circular cone has a height of 10cm, and its base has a diameter of 10cm. The sphere and cone sit on a horizontal surface. If a horizontal plane cuts both the sphere and the cone, the cross-sections will both be circles, as shown. Let the radii of the cross-sections of the cone and sphere at height h be r_1 and r_2 respectively. [6]



(i) Show that $r_1^2 = (5 - \frac{h}{2})^2$ and $r_2^2 = 10h - h^2$.

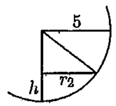
Solution:



$\dfrac{r_1}{10 - h} = \dfrac{5}{10} = \dfrac{1}{2}$,

$r_1 = 5 - h/2$.

$\therefore\ r_1{}^2 = (5 - h/2)^2$.



$r_2{}^2 + (5 - h)^2 = 25$,

$r_2{}^2 = 25 - 25 + 10h - h^2$,

$= 10h - h^2$.

(ii) Hence, or otherwise, find the height of the horizontal plane that gives the greatest sum of the cross-sectional areas.

Solution: Area sum, $A = \pi r_1{}^2 + \pi r_2{}^2$,

$= \pi\left((5 - h/2)^2 + 10h - h^2\right)$,

$= \pi\left(25 - 5h + \dfrac{h^2}{4} + 10h - h^2\right)$,

$= \pi\left(25 + 5h - \dfrac{3h^2}{4}\right)$,

$= \frac{\pi}{4}(100 + 20h - 3h^2)$.

$\dfrac{dA}{dh} = \frac{\pi}{4}(20 - 6h)$,

$= 0$ when $h = \frac{10}{3}$.

$\dfrac{d^2A}{dh^2} = \frac{\pi}{4} \times {}^-6$,

< 0.

$\therefore$ The height is $10/3$ cm for maximum area.