



SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

April 2012

Assessment Task 2
Year 12

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question if full marks are to be awarded.
- Answer in simplest exact form unless otherwise instructed.

Total Marks – 60

- Attempt sections A – C.
- Start each **NEW** section in a separate answer booklet.
- Hand in your answers in 3 separate bundles:
 - Section A
 - Section B
 - Section C

Examiner: *J. Chen*

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax,$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left(x + \sqrt{x^2 - a^2} \right), x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

NOTE: $\ln x = \log_e x, x > 0$

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

START A NEW ANSWER BOOKLET

SECTION A [20 marks]

For these 10 questions there is one correct answer per question. Write down in your answer booklet the question number and letter of your answer.

1.

$$\int_1^2 \frac{dx}{2x+5}$$

equals

- (a) $\ln\left(\frac{9}{7}\right)$
- (b) $\frac{1}{2}\ln(63)$
- (c) $\frac{1}{2}\ln\left(\frac{9}{7}\right)$
- (d) $\ln(63)$

Marks

[1]

2.

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{x}$$

equals

- (a) 2
- (b) 1
- (c) 0
- (d) $\frac{1}{2}$

[1]

3. If $\log_m 64 + \log_m 4 = x \log_m 2$, then the value of x is:

- (a) 4
- (b) 8
- (c) 6
- (d) 2

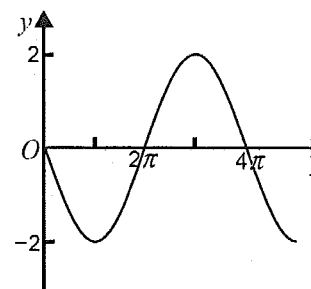
[1]

4. $\frac{d}{dx} \log_e(e^{3x} + 2)$ equals

- (a) $3e^{3x}$
- (b) $e^{3x} + 2$
- (c) $\frac{1}{e^{3x} + 2}$
- (d) $\frac{3e^{3x}}{e^{3x} + 2}$

[1]

5. The diagram below shows a part of the graph of a trigonometric function.



A possible equation for the function is

- (a) $y = 2 \sin 2x$
- (b) $y = -2 \cos 2x$
- (c) $y = -2 \sin \frac{x}{2}$
- (d) $y = 2 \cos \frac{x}{2}$

[1]

6.

$$\int \cos 6x \cdot dx$$

equals

- (a) $\frac{\sin 6x}{6} + C$
- (b) $-\frac{\sin 6x}{6} + C$
- (c) $6 \sin 6x + C$
- (d) $-6 \sin 6x + C$

[1]

7. $\int 8xe^{x^2} \cdot dx$

[1]

equals

(a) $4xe^{x^2} + C$

(b) $8e^{x^2} + C$

(c) $2xe^{x^2} + C$

(d) None of the above

8. What is the exact value of $\sin 75^\circ$?

[1]

(a) $\frac{\sqrt{2}+\sqrt{6}}{4}$

(b) $\frac{\sqrt{2}-\sqrt{6}}{4}$

(c) $\frac{\sqrt{6}+\sqrt{2}}{4}$

(d) $\frac{\sqrt{6}-\sqrt{2}}{4}$

9. $\int_{-\pi}^{\pi} 2 \sin x \cdot dx$

[1]

equals

(a) 0

(b) 2

(c) $2 \int_0^{\pi} 2 \sin x \cdot dx$

(d) $\left| \int_{-\pi}^0 2 \sin x \cdot dx \right| + \int_0^{\pi} 2 \sin x \cdot dx$

10. If $f(x) = \cos 2x$, then $f'(-\frac{\pi}{6})$ is:

[1]

(a) $\frac{\sqrt{3}}{2}$

(b) $\sqrt{3}$

(c) $-\frac{\sqrt{3}}{2}$

(d) None of the above

End of Multiple Choice Section

11. Differentiate $\cot x$.

[2]

12. Solve the equation,

$$3 \ln(x + 1) = \ln(x^3 + 19)$$

[3]

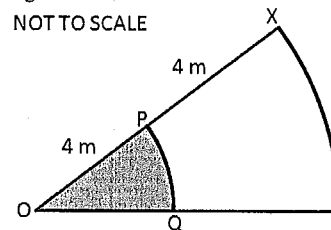
13. Find the equation of the tangent to the curve $y = \sin x$ at $x = \pi$.

[2]

14. PQ and XY are arcs of concentric circles with centre O. $OP = PX = 4$ m.

[3]

The shaded sector OPQ has area $\frac{2\pi}{3}$ square metres. Find $\angle POQ$ in degrees.



End of Section A

START A NEW ANSWER BOOKLET

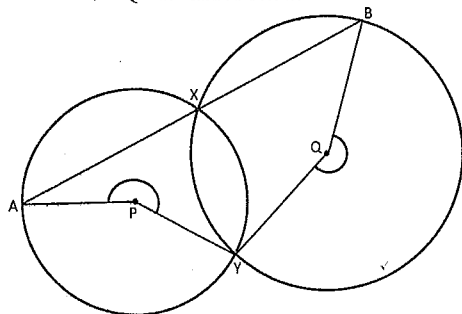
SECTION B [20 marks]

1. At any point on the curve $y = f(x)$ the gradient function is given by $\frac{dy}{dx} = \frac{x+1}{x+2}$. If $y = -1$ when $x = -1$, find the value of y when $x = 1$, correct your answer to the nearest 3 significant figures.

Marks
[4]

2. P and Q are centres of the circles, AXB is a straight line. Prove that $\angle APY = \angle BQY$ as marked below.

[3]



3. Evaluate

$$\int_0^{\frac{\pi}{6}} \sec^2 x \tan^3 x \cdot dx$$

[2]

4. Consider the function $f(x) = \frac{\log_e x}{x^2}$.

[6]

(i) Find the x intercept of the curve.

(ii) Find the coordinates of the turning point and the point of inflexion.

(iii) Hence, sketch the curve $y = f(x)$ and label the critical points and any asymptotes.

5. Consider the function $f(x) = x - \sin x$.

[2]

$P(X, 1)$ is a point on the curve $y = f(x)$. Starting with an initial approximation of $X = 2$, use one application of Newton's Method to find an improved approximation to the value of X , giving the answer correct to 3 decimal places.

6. Prove by Mathematical Induction that $3^{3n} + 2^{n+2}$ is divisible by 5 for all integers $n \geq 1$.

[3]

End of Section B

START A NEW ANSWER BOOKLET

SECTION C [20 marks]

Marks
[2]

1. (i) Show that there is a solution to the equation $x - 2 = \sin x$ between $x = 2.5$ and $x = 2.6$.

(ii) By halving the interval, find the solution correct to 2 decimal places.

2. (i) Use the Principle of Mathematical Induction to prove that $\sin(x + n\pi) = (-1)^n \sin x$ for all positive integers n . [5]

(ii) If

$$S = \sum_{k=1}^n \sin(x + k\pi)$$

for $0 < x < \frac{\pi}{2}$ and for all positive integers n .
Prove that $-1 < S \leq 0$.

3. Consider the function $f(x) = e^x \left(1 - \frac{x}{4}\right)^4$. [8]

(i) Find the coordinates of the stationary points and determine their nature.

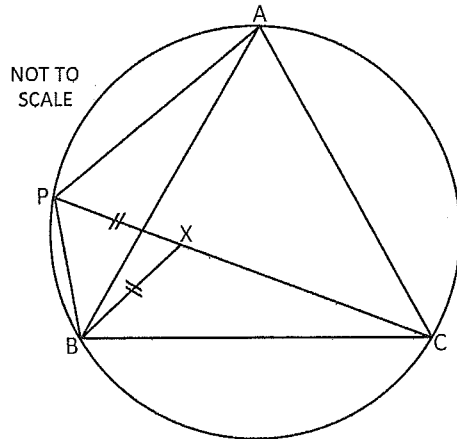
(ii) Sketch the curve $y = f(x)$ and label the turning points and any asymptotes.

(iii) Hence, prove that $\left(\frac{5}{4}\right)^4 \leq e \leq \left(\frac{4}{3}\right)^4$.

SECTION A

4. In the diagram, A, B, C and P are points on the circumference of the circle and $\triangle ABC$ is an equilateral triangle. X is a point on the straight line PC such that $PX = BX$. Prove that $PC = PA + PB$.

[5]



Copy or trace the diagram into your answer booklet.

End of Section C
End of Exam

$$1. \int_1^2 \frac{dx}{2x+5} = \frac{1}{2} [\ln(2x+5)]_1^2$$

$$= \frac{1}{2} \{ \ln 9 - \ln 7 \}$$

$$= \frac{1}{2} \ln \frac{9}{7} \quad \text{(C)}$$

$$2. \lim_{x \rightarrow 0} \frac{\sin 2x}{x} = 2 \lim_{2x \rightarrow 0} \frac{\sin 2x}{2x}$$

$$= 2 \times 1$$

$$= 2 \quad \text{(A)}$$

$$3. \log_m 64 + \log_m 4 = x \log_m 2$$

$$\text{LHS} = \log_m 256$$

$$= \log_m 2^8$$

$$= 8 \log_m 2$$

$$x = 8 \quad \text{(B)}$$

$$4. \frac{d}{dx} \log_e (e^{3x} + 2)$$

$$= \frac{1}{e^{3x} + 2} \cdot e^{3x} \cdot 3$$

$$= \frac{3e^{3x}}{e^{3x} + 2} \quad \text{(D)}$$

$$5. \quad \text{C} \quad y = -2 \sin \frac{x}{2}$$

$$6. \int \cos 6x \, dx$$

$$= \frac{1}{6} \sin 6x + c \quad \text{(A)}$$

$$7. \int 8x e^{x^2} \, dx$$

Let $u = x^2$
 $du = 2x \, dx$

$$= 4 \int e^u \, du$$

$$= 4e^u + c$$

$$= 4e^{x^2} + c \quad \text{(D) None of the above}$$

$$8. \sin 75^\circ = \sin(45^\circ + 30^\circ)$$

$$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

$$= \frac{\sqrt{6} + \sqrt{2}}{4} \quad \text{(A) or (C)}$$

$$9. \int_{-\pi}^{\pi} 2 \sin x \, dx$$

$$= 2 \left[-\cos x \right]_{-\pi}^{\pi}$$

$$= 2 \left[-\cos \pi - (-\cos(-\pi)) \right]$$

$$= 2 \left[1 + -1 \right]$$

$$= 0 \quad \text{(A)}$$

$$10. f(x) = \cos 2x$$

$$f'(x) = -\sin 2x \cdot 2$$

$$f'(-\frac{\pi}{6}) = -\sin(-\frac{\pi}{3}) \cdot 2$$

$$= -2 \times -\sin \frac{\pi}{3}$$

$$= 2 \times \frac{\sqrt{3}}{2}$$

$$= \sqrt{3} \quad \text{(B)}$$

$$11. \frac{d}{dx} (\cot x) = \frac{d}{dx} \left(\frac{\cos x}{\sin x} \right)$$

$$= \frac{\sin x \cdot -\sin x - \cos x \cdot \cos x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x}$$

$$= -\operatorname{cosec}^2 x. \quad \text{(2)}$$

$$12. 3 \ln(x+1) = \ln(x^3+19)$$

$$\therefore \ln((x+1)^3) = \ln(x^3+19)$$

$$\therefore x^3 + 3x^2 + 3x + 1 = x^3 + 19$$

$$\therefore 3x^2 + 3x - 18 = 0$$

$$x^2 + x - 6 = 0$$

$$(x+3)(x-2) = 0$$

$$x = -3 \text{ or } x = 2$$

As $x > 0$, $x \neq -3$

$$\therefore \text{Soln: } x = 2 \quad \text{(3)}$$

$$13. \quad y = \sin x$$

$$y' = \cos x$$

$$\text{When } x = \pi, \quad y = \sin \pi = 0$$

$$y' = \cos \pi = -1$$

$$\therefore \text{Eqn of tangent: } y - 0 = -1(x - \pi)$$

$$y = -x + \pi$$

$$y + x = \pi$$

2

$$14. \quad \text{Area} = \frac{1}{2} r^2 \theta$$

$$= \frac{1}{2} \times 16 \times \theta = \frac{2\pi}{3}$$

$$\therefore \theta = \frac{2\pi}{3}$$
~~$$\therefore \theta = \frac{2\pi}{24}$$~~

$$= \frac{\pi}{12}$$

$$= 15^\circ$$

$$\therefore \angle POQ = 15^\circ$$

3

2012 Extension 1 Mathematics Task 2:
Solutions— Section B

1. At any point on the curve $y = f(x)$ the gradient function is given by $\frac{dy}{dx} = \frac{x+1}{x+2}$. If $y = -1$ when $x = -1$, find the value of y when $x = 1$, correct your answer to the nearest 3 significant figures.

4

Solution: $\frac{dy}{dx} = 1 - \frac{1}{x+2}$

$$y = x - \ln(x+2) + c$$

$$-1 = -1 - \ln 1 + c,$$

$$c = 0.$$

$$y = x - \ln(x+2),$$

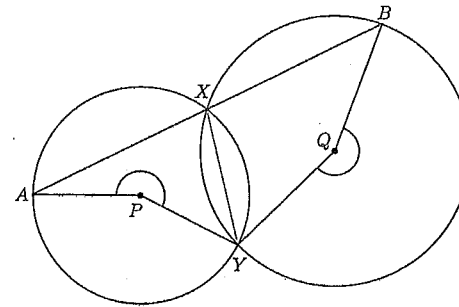
$$= 1 - \ln 3 \text{ when } x = 1,$$

$$\approx -0.0986 \text{ (3 sig. fig.)}$$

$$-2 \begin{array}{|c|c|} \hline 1 & 1 \\ \hline -2 & -2 \\ \hline 1 & -1 \\ \hline \end{array}$$

2. P and Q are centres of the circles, AXB is a straight line. Prove that $\angle APY = \angle BQY$ as marked below.

3



Solution:

$$\widehat{APY} = 2\widehat{AXY} \text{ (}\angle \text{ at centre } 2 \times \angle \text{ at circumf.)},$$

$$360^\circ - \widehat{BQY} = 2\widehat{BXY} \text{ (}\angle \text{ at centre } 2 \times \angle \text{ at circumf.)},$$

$$\widehat{AXY} + \widehat{BXY} = 180^\circ \text{ (}\widehat{AXB} \text{ is straight)},$$

$$\widehat{APY} + 360^\circ - \widehat{BQY} = 2 \times 180^\circ,$$

$$\widehat{APY} = \widehat{BQY},$$

$$\therefore \text{reflex } \widehat{APY} = \text{reflex } \widehat{BQY}.$$

3. Evaluate $\int_0^{\frac{\pi}{6}} \sec^2 x \tan^8 x \, dx$

2

Solution: $I = \int_0^{\frac{\pi}{6}} u^8 \, du,$ put $u = \tan x$
 $= \frac{u^9}{9} \Big|_0^{\frac{1}{\sqrt{3}}},$ $\frac{du}{dx} = \sec^2 x$
when $x = \frac{\pi}{6},$ $u = \frac{1}{\sqrt{3}}$
 $= \frac{1}{9} \times \frac{1}{81\sqrt{3}} - 0,$ $x = 0, \quad u = 0$
 $= \frac{\sqrt{3}}{2187}.$

4. Consider the function $f(x) = \frac{\log_e x}{x^2}.$

6

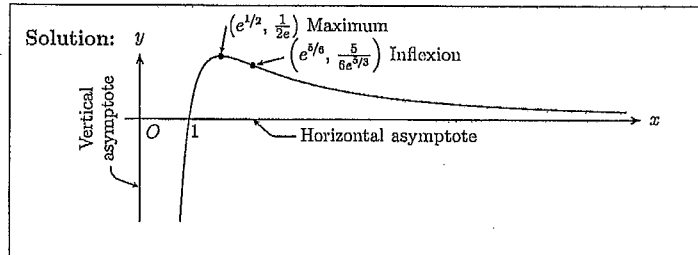
(a) Find the x intercept of the curve.

Solution: $\ln x = 0$ when $x = 1$, so the x -intercept is at $(1, 0).$

(b) Find the coordinates of the turning point and the point of inflexion.

Solution: $f'(x) = \frac{x^2 - 2x \ln x}{x^4},$ $f''(x) = \frac{x^3 \left(\frac{-2}{x}\right) - 3x^2(1 - 2 \ln x)}{x^8},$
 $= \frac{1 - 2 \ln x}{x^3},$ $= \frac{-2 - 3 + 6 \ln x}{x^4},$
 $= 0$ when $x = e^{1/2}.$ $= \frac{6 \ln x - 5}{x^4},$
 $= 0$ when $x = e^{5/6}.$
 \therefore Maximum $\left(e^{1/2}, \frac{1}{2e}\right),$ $f''(e^{1/2}) = \frac{-2}{e^2} < 0.$
Inflexion $\left(e^{5/6}, \frac{5}{6e^{5/3}}\right),$

(c) Hence sketch the curve $y = f(x)$, and label the critical points and any asymptotes.



5. Consider the function $f(x) = x - \sin x.$

2

$P(X, 1)$ is a point on the curve $y = f(x)$. Starting with an initial approximation of $X = 2$, use one application of Newton's Method to find an improved approximation to the value of X , giving the answer correct to 3 decimal places.

Solution: $f'(x) = 1 - \cos x.$
 $a_1 = 2 - \frac{2 - \sin 2 - 1}{1 - \cos 2},$
 ≈ 1.936 (3 dec. pl.)

6. Prove by Mathematical Induction that $3^{3n} + 2^{n+2}$ is divisible by 5 for all integers $n \geq 1.$

3

Solution: $S_n = 3^{3n} + 2^{n+2}.$
Test $n = 1, S_1 = 3^3 + 2^3,$
 $= 27 + 8,$
 $= 35.$
 \therefore True for $n = 1.$
Assume true for $n = k,$
i.e. $S_k = 5p$ where $p \in \mathbb{Z}.$
Test $n = k + 1,$
i.e. $S_{k+1} = 5q$ where $q \in \mathbb{Z}.$
L.H.S. $= 3^{3(k+1)} + 2^{k+1+2},$
 $= 3^{3k+3} + 2^{k+3},$
 $= 27 \cdot 3^{3k} + 2 \cdot 2^{k+2},$
 $= 27(3^{3k} + 2^{k+2}) - 25 \cdot 2^{k+2},$
 $= 27S_k - 25 \cdot 2^{k+2},$
 $= 27 \cdot 5p - 25 \cdot 2^{k+2}$ (using the assumption),
 $= 5(27p - 5 \cdot 2^{k+2}),$
 $= 5q.$

So, true for $n = k + 1$ if true for $n = k$; true for $n = 1,$
and so true for $n = 2, 3, \dots,$ for all $n \geq 1.$