SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

April 2012

Assessment Task 2
Year 12

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question if full marks are to be awarded.
- Answer in simplest exact form unless otherwise instructed.

Total Marks – 60

- Attempt sections A – C.
- Start each **NEW** section in a separate answer booklet.
- Hand in your answers in 3 separate bundles:
  - Section A
  - Section B
  - Section C

Examiner: *J. Chen*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x > 0$

START A NEW ANSWER BOOKLET

SECTION A [20 marks] **Marks**

For these 10 questions there is one correct answer per question. Write down in your answer booklet the question number and letter of your answer.

1. [1]

$$\int_1^2 \frac{dx}{2x+5}$$

equals

(a) $\ln\left(\frac{9}{7}\right)$

(b) $\frac{1}{2}\ln(63)$

(c) $\frac{1}{2}\ln\left(\frac{9}{7}\right)$

(d) $\ln(63)$

2. [1]

$$\lim_{x\to 0}\frac{\sin 2x}{x}$$

equals

(a) 2

(b) 1

(c) 0

(d) $\frac{1}{2}$

3. If $\log_m 64 + \log_m 4 = x\log_m 2$, then the value of x is: [1]

(a) 4

(b) 8

(c) 6

(d) 2

4. $\frac{d}{dx}\log_e(e^{3x}+2)$ equals [1]

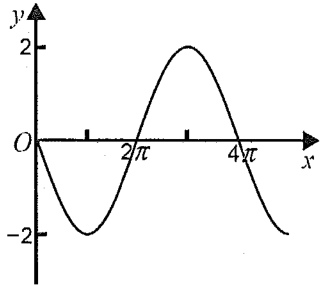
(a) $3e^{3x}$

(b) $e^{3x}+2$

(c) $\frac{1}{e^{3x}+2}$

(d) $\frac{3e^{3x}}{e^{3x}+2}$

5. The diagram below shows a part of the graph of a trigonometric function. [1]



A possible equation for the function is

(a) $y = 2\sin 2x$

(b) $y = -2\cos 2x$

(c) $y = -2\sin\frac{x}{2}$

(d) $y = 2\cos\frac{x}{2}$

6. [1]

$$\int \cos 6x \,.\, dx$$

equals

(a) $\frac{\sin 6x}{6} + C$

(b) $-\frac{\sin 6x}{6} + C$

(c) $6\sin 6x + C$

(d) $-6\sin 6x + C$

7. [1]

$$\int 8xe^{x^2}.dx$$

equals

(a) $4xe^{x^2} + C$

(b) $8e^{x^2} + C$

(c) $2xe^{x^2} + C$

(d) None of the above

8. What is the exact value of sin 75°? [1]

(a) $\frac{\sqrt{2}+\sqrt{6}}{4}$

(b) $\frac{\sqrt{2}-\sqrt{6}}{4}$

(c) $\frac{\sqrt{6}+\sqrt{2}}{4}$

(d) $\frac{\sqrt{6}-\sqrt{2}}{4}$

9. [1]

$$\int_{-\pi}^{\pi} 2\sin x\,.dx$$

equals

(a) 0

(b) 2

(c) $2\int_0^{\pi} 2\sin x\,.dx$

(d) $\left|\int_{-\pi}^{0} 2\sin x\,.dx\right| + \int_0^{\pi} 2\sin x\,.dx$

10. If $f(x) = \cos 2x$, then $f'\left(-\frac{\pi}{6}\right)$ is: [1]

(a) $\frac{\sqrt{3}}{2}$

(b) $\sqrt{3}$

(c) $-\frac{\sqrt{3}}{2}$

(d) None of the above

End of Multiple Choice Section

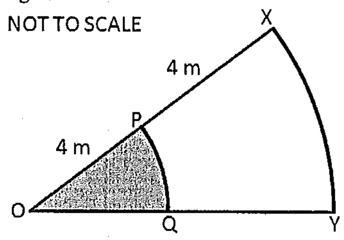
11. Differentiate $\cot x$. [2]

12. Solve the equation, [3]

$$3\ln(x+1) = \ln(x^3 + 19)$$

13. Find the equation of the tangent to the curve $y = \sin x$ at $x = \pi$. [2]

14. PQ and XY are arcs of concentric circles with centre O. OP = PX = 4 m. [3]
The shaded sector OPQ has area $\frac{2\pi}{3}$ square metres. Find ∠POQ in degrees.



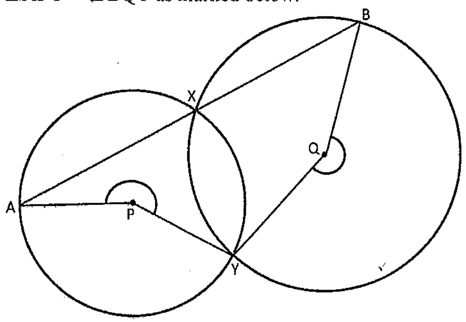
End of Section A

START A NEW ANSWER BOOKLET

SECTION B [20 marks] **Marks**

1. At any point on the curve $y = f(x)$ the gradient function is given by $\frac{dy}{dx} = \frac{x+1}{x+2}$. If $y = -1$ when $x = -1$, find the value of y when $x = 1$, correct your answer to the nearest 3 significant figures. **[4]**

2. P and Q are centres of the circles, AXB is a straight line. Prove that $\angle$APY = $\angle$BQY as marked below. **[3]**



3. Evaluate **[2]**

$$\int_0^{\frac{\pi}{6}} \sec^2 x \tan^8 x \, .dx$$

4. Consider the function $f(x) = \frac{\log_e x}{x^2}$. **[6]**
   (i) Find the x intercept of the curve.

   (ii) Find the coordinates of the turning point and the point of inflexion.

   (iii) Hence, sketch the curve $y = f(x)$ and label the critical points and any asymptotes.

5. Consider the function $f(x) = x - \sin x$. **[2]**
   $P(X, 1)$ is a point on the curve $y = f(x)$. Starting with an initial approximation of $X = 2$, use one application of Newton's Method to find an improved approximation to the value of X, giving the answer correct to 3 decimal places.

6. Prove by Mathematical Induction that $3^{3n} + 2^{n+2}$ is divisible by 5 for all integers $n \geq 1$. **[3]**

End of Section B

START A NEW ANSWER BOOKLET

SECTION C [20 marks] **Marks**

1. **[2]**
   (i) Show that there is a solution to the equation $x - 2 = \sin x$ between $x = 2.5$ and $x = 2.6$.

   (ii) By halving the interval, find the solution correct to 2 decimal places.

2. **[5]**
   (i) Use the Principle of Mathematical Induction to prove that
   $$\sin(x + n\pi) = (-1)^n \sin x$$
   for all positive integers n.

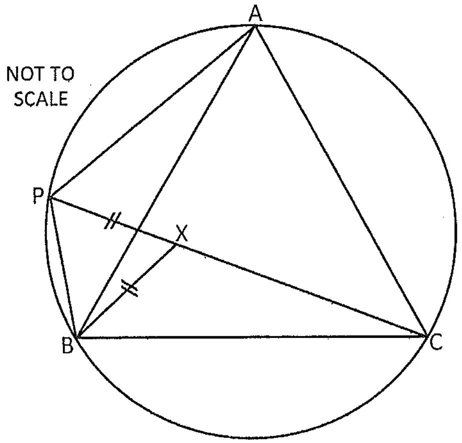
   (ii) If
   $$S = \sum_{k=1}^{n} \sin(x + k\pi)$$
   for $0 < x < \frac{\pi}{2}$ and for all positive integers n.
   Prove that $-1 < S \leq 0$.

3. Consider the function $f(x) = e^x\left(1 - \frac{x}{4}\right)^4$. **[8]**
   (i) Find the coordinates of the stationary points and determine their nature.

   (ii) Sketch the curve $y = f(x)$ and label the turning points and any asymptotes.

   (iii) Hence, prove that $\left(\frac{5}{4}\right)^4 \leq e \leq \left(\frac{4}{3}\right)^4$.

4. In the diagram, A, B, C and P are points on the circumference of the circle and ΔABC is an equilateral triangle. X is a point on the straight line PC such that PX = BX. Prove that PC = PA + PB. **[5]**



Copy or trace the diagram into your answer booklet.

End of Section C
End of Exam

SECTION A

1. $\int_1^2 \frac{dx}{2x+5} = \frac{1}{2}\left[\ln(2x+5)\right]_1^2$

$= \frac{1}{2}\{\ln 9 - \ln 7\}$

$= \frac{1}{2}\ln\frac{9}{7}$ (C)

2. $\lim_{x\to 0}\frac{\sin 2x}{x} = 2\lim_{2x\to 0}\frac{\sin 2x}{2x}$

$= 2\times 1$

$= 2$ (A)

3. $\log_m 64 + \log_m 4 = x\log_m 2$

LHS $= \log_m 256$

$= \log_m 2^8$

$= 8\log_m 2$

$x = 8$ (B)

4. $\frac{d}{dx}\log_e(e^{3x}+2)$

$= \frac{1}{e^{3x}+2}\cdot e^{3x}\cdot 3$

$= \frac{3e^{3x}}{e^{3x}+2}$ (D)

5. C $y = -2\sin\frac{x}{2}$

6. $\int \cos 6x \, dx$

$= \frac{1}{6} \sin 6x + c$ (A)

7. $\int 8x e^{x^2} dx$

Let $u = x^2$

$du = 2x\,dx$

$= 4 \int e^u du$

$= 4e^u + c$

$= 4e^{x^2} + c$ (D). None of the above

(8) $\sin 75^\circ = \sin(45+30)^\circ$

$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$ (A) or (C)

9. $\int_{-\pi}^{\pi} 2 \sin x \, dx$

$= 2 \left[-\cos x \right]_{-\pi}^{\pi}$

$= 2 \left[-\cos\pi - -\cos(-\pi) \right]$

$= 2. \left[+1 + -1 \right]$

$= 0$ (A)

10 $f(x) = \cos 2x$

$f'(x) = -\sin 2x \cdot 2$

$f'(-\frac{\pi}{6}) = -\sin(-\frac{\pi}{3}) \cdot 2$

$= -2 \times -\sin\frac{\pi}{3}$

$= 2 \times \frac{\sqrt{3}}{2}$

$= \sqrt{3}$ (B)

11. $\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$

$= \frac{\sin x \cdot -\sin x - \cos x \cdot \cos x}{\sin^2 x}$

$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$= \frac{-1}{\sin^2 x}$

$= -\operatorname{cosec}^2 x.$ (2)

12. $3 \ln(x+1) = \ln(x^3+19)$

$\therefore \ln\left((x+1)^3\right) = \ln(x^3+19)$

$\therefore x^3 + 3x^2 + 3x + 1 = x^3 + 19$

$\therefore 3x^2 + 3x - 18 = 0$

$x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

$x = -3$ or $x = 2$

As $x+1 > 0$, $x \neq -3$ (3)

$\therefore$ Soln : $x = 2$

13. $y = \sin x$

$y' = \cos x$

When $x = \pi$, $y = \sin \pi = 0$

$y' = \cos \pi = -1$

$\therefore$ Eqn of tangent is $y - 0 = -1(x - \pi)$

$y = -x + \pi$

$y + x = \pi$

(2)

14. Area $= \frac{1}{2} r^2 \theta$

$= \frac{1}{2} \times 16 \times \theta = \frac{2\pi}{3}$

$\therefore 8\theta = \frac{2\pi}{3}$

$\therefore \theta = \frac{2\pi}{24}$

$= \frac{\pi}{12}$

$= 15^\circ$

$\therefore \angle POQ = 15^\circ$

(3)

2012 Extension 1 Mathematics Task 2:
Solutions— Section B

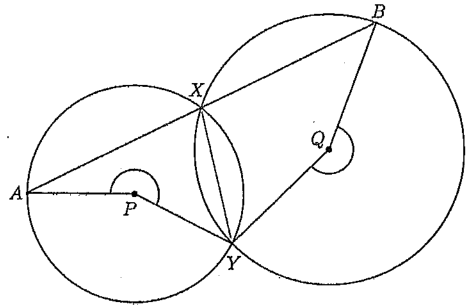
1. At any point on the curve $y = f(x)$ the gradient function is given by $\frac{dy}{dx} = \frac{x+1}{x+2}$. If $y = -1$ when $x = -1$, find the value of y when $x = 1$, correct your answer to the nearest 3 significant figures. [4]

Solution:

$$\begin{aligned} \frac{dy}{dx} &= 1 - \frac{1}{x+2}, \\ y &= x - \ln(x+2) + c. \\ -1 &= -1 - \ln 1 + c, \\ c &= 0. \\ y &= x - \ln(x+2), \\ &= 1 - \ln 3 \text{ when } x = 1, \\ &\approx -0.0986 \text{ (3 sig. fig.)} \end{aligned}$$

	1	1
−2		−2
	1	−1

2. P and Q are centres of the circles, AXB is a straight line. Prove that $\angle APY = \angle BQY$ as marked below. [3]



Solution:

$$\begin{aligned} A\hat{P}Y &= 2A\hat{X}Y \text{ } (\angle \text{ at centre } 2 \times \angle \text{ at circumf.}), \\ 360^\circ - B\hat{Q}Y &= 2B\hat{X}Y \text{ } (\angle \text{ at centre } 2 \times \angle \text{ at circumf.}), \\ A\hat{X}Y + B\hat{X}Y &= 180^\circ \text{ } (A\hat{X}B \text{ is straight}), \\ A\hat{P}Y + 360^\circ - B\hat{Q}Y &= 2 \times 180^\circ, \\ A\hat{P}Y &= B\hat{Q}Y, \\ \therefore \text{reflex } A\hat{P}Y &= \text{reflex } B\hat{Q}Y. \end{aligned}$$

3. Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \sec^2 x \tan^8 x.dx$ [2]

Solution: $\displaystyle I = \int_0^{\frac{1}{\sqrt{3}}} u^8.du,$ put $u = \tan x$; $du/dx = \sec^2 x$; when $x = \pi/6$, $u = 1/\sqrt{3}$; $x = 0$, $u = 0$

$$= \left[\frac{u^9}{9}\right]_0^{\frac{1}{\sqrt{3}}},$$
$$= \frac{1}{9} \times \frac{1}{81\sqrt{3}} - 0,$$
$$= \frac{\sqrt{3}}{2187}.$$

4. Consider the function $f(x) = \dfrac{\log_e x}{x^2}$. [6]

(a) Find the x intercept of the curve.

Solution: $\ln x = 0$ when $x = 1$, so the x-intercept is at $(1, 0)$.

(b) Find the coordinates of the turning point and the point of inflexion.

Solution:
$$f'(x) = \frac{\frac{x^2}{x} - 2x\ln x}{x^4},$$
$$= \frac{1 - 2\ln x}{x^3},$$
$$= 0 \text{ when } x = e^{1/2}.$$

$$f''(x) = \frac{x^3\left(\frac{-2}{x}\right) - 3x^2(1 - 2\ln x)}{x^6},$$
$$= \frac{-2 - 3 + 6\ln x}{x^4},$$
$$= \frac{6\ln x - 5}{x^4},$$
$$= 0 \text{ when } x = e^{5/6}.$$

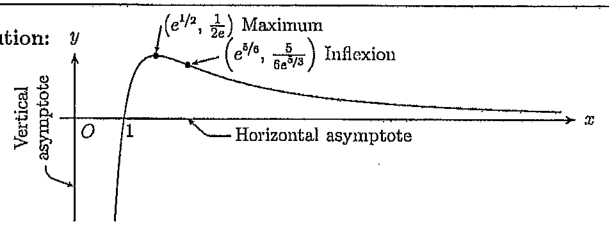
$$f''(e^{1/2}) = \frac{-2}{e^2} < 0.$$

$\therefore$ Maximum $\left(e^{1/2}, \dfrac{1}{2e}\right)$,

Inflexion $\left(e^{5/6}, \dfrac{5}{6e^{5/3}}\right)$,

(c) Hence sketch the curve $y = f(x)$, and label the critical points and any asymptotes.

Solution:



5. Consider the function $f(x) = x - \sin x$. [2]
$P(X, 1)$ is a point on the curve $y = f(x)$. Starting with an initial approximation of $X = 2$, use one application of Newton's Method to find an improved approximation to the value of X, giving the answer correct to 3 decimal places.

Solution: $f'(x) = 1 - \cos x.$
$$a_1 = 2 - \frac{2 - \sin 2 - 1}{1 - \cos 2},$$
$$\approx 1.936 \text{ (3 dec. pl.)}$$

6. Prove by Mathematical Induction that $3^{3n} + 2^{n+2}$ is divisible by 5 for all integers $n \geqslant 1$. [3]

Solution: $S_n = 3^{3n} + 2^{n+2}$.
Test $n = 1$, $S_1 = 3^3 + 2^3$,
$= 27 + 8$,
$= 35$.
$\therefore$ True for $n = 1$.
Assume true for $n = k$,
i.e. $S_k = 5p$ where $p \in \mathbb{Z}$.
Test $n = k + 1$,
i.e. $S_{k+1} = 5q$ where $q \in \mathbb{Z}$.
L.H.S. $= 3^{3(k+1)} + 2^{k+1+2}$,
$= 3^{3k+3)} + 2^{k+3}$,
$= 27.3^{3k} + 2.2^{k+2}$,
$= 27\left(3^{3k} + 2^{k+2}\right) - 25.2^{k+2}$,
$= 27S_k - 25.2^{k+2}$,
$= 27.5p - 25.2^{k+2}$ (using the assumption),
$= 5\left(27p - 5.2^{k+2}\right)$,
$= 5q$.

So, true for $n = k + 1$ if true for $n = k$; true for $n = 1$, and so true for $n = 2, 3, \ldots$, for all $n \geqslant 1$.