SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2011

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 180 minutes.
- Write using black or blue pen.
  Pencil may be used for diagrams.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Start each **NEW** question in a separate answer booklet.

Total Marks - 120 Marks

- Attempt questions 1 – 8
- All questions are of equal value.

Examiners: *External Examiner*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks – 120
Attempt Questions 1 – 8
All questions are of equal value

Answer each section in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 Marks) Use a SEPARATE writing booklet.

(a) Using the standard table of integrals, find $\int \frac{dx}{\sqrt{4x^2+1}}$. **2**

(b) Find k, where k is a rational number and satisfies **3**

$$\int_4^6 \frac{x-2}{2x^2-8x+3}\,dx = k\ln 3.$$

(c) Use the substitution $u = 2x-1$ to find $\int \frac{x^2}{2x-1}\,dx$. **2**

(d) (i) Express $\frac{2x^2}{(x+5)(x-3)}$ in the form $A+\frac{B}{x+5}+\frac{C}{x-3}$, where A, B and C are real constants. **3**

(ii) Hence, find $\int \frac{2x^2}{(x+5)(x-3)}\,dx$. **2**

(e) Using integration by parts, find $\int x\sin(2x-1)\,dx$. **3**

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) A complex number is defined by $z = x + 2i$, where x is real.

(i) Express the following in the form $a + ib$, where a and b are real numbers.

(1) z^2. **1**

(2) $z^2 + 2\bar{z}$. **2**

(ii) Show that there is exactly one value of x for which $z^2 + 2\bar{z}$ is real. **1**

(b) The complex number $2 + 3i$ is a root of the quadratic equation

$$x^2 + bx + c = 0$$

where b and c are real numbers.

(i) Write down the other root of this equation. **1**

(ii) Find the values of b and c. **2**

(c) (i) A circle, C, in the Argand diagram has equation $|z + 5 - i| = \sqrt{2}$.
Write down its radius and the complex number representing its centre. **2**

(ii) A ray, R, in the Argand diagram has equation $\arg(z + 2i) = \dfrac{3\pi}{4}$.
Show that $z_1 = -4 + 2i$ lies on R. **1**

(iii) (1) Show that $z_1 = -4 + 2i$ also lies on C. **1**

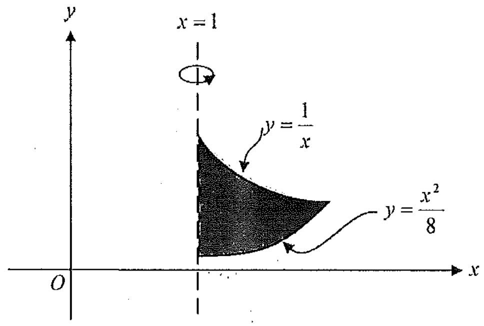
(2) Hence show that R is a tangent to C. **2**

(3) Sketch R and C on one Argand diagram. **1**

(iv) The complex number z_2 lies on C and is such that $\arg(z_2 + 2i)$ has as great a value as possible.
Indicate the position of z_2 on your sketch. **1**

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) The shaded region shown in the diagram below is bounded by $y = \dfrac{1}{x}$, $y = \dfrac{x^2}{8}$ and $x = 1$. This region is rotated about the line $x = 1$.



(i) Show that the volume of the resulting solid of revolution using the method of cylindrical shells $V = 2\pi \displaystyle\int_1^2 (x-1)\left(\frac{1}{x} - \frac{x^2}{8}\right) dx$. **3**

(ii) Find the volume of the solid of revolution. **2**

(b) The roots of the cubic equation $z^3 - 2z^2 + pz + 10 = 0$ are α, β and γ.
It is given that $\alpha^3 + \beta^3 + \gamma^3 = -4$.

(i) Write down the value of $\alpha + \beta + \gamma$ **1**

(ii) (1) Explain why $\alpha^3 - 2\alpha^2 + p\alpha + 10 = 0$ **1**

(2) Hence show that $\alpha^2 + \beta^2 + \gamma^2 = p + 13$ **3**

(iii) Deduce that $p = -3$. **2**

(iv) Solve $z^3 - 2z^2 - 3z + 10 = 0$ **3**

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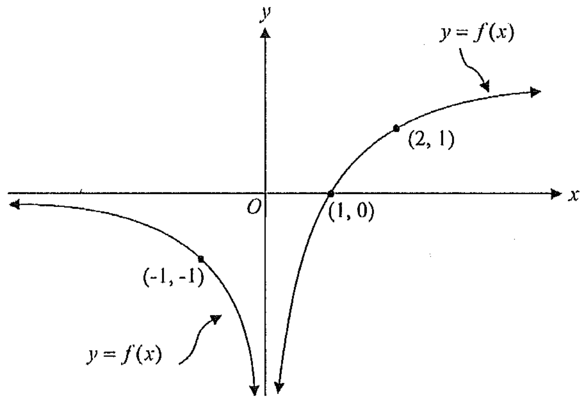
Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) The function f is a discontinuous function.
The diagram below shows the graph of $y = f(x)$.



Without using calculus, sketch the following graphs on the ANSWER sheet provided, clearly showing any asymptotes, intercepts and stationary points where necessary.

Make sure you attach this sheet to your writing booklet for Question 4.

(i) $y = \dfrac{1}{f(x-1)}$ **2**

(ii) $y = [f(x)]^2$ **1**

(iii) $y = xf(x)$ **2**

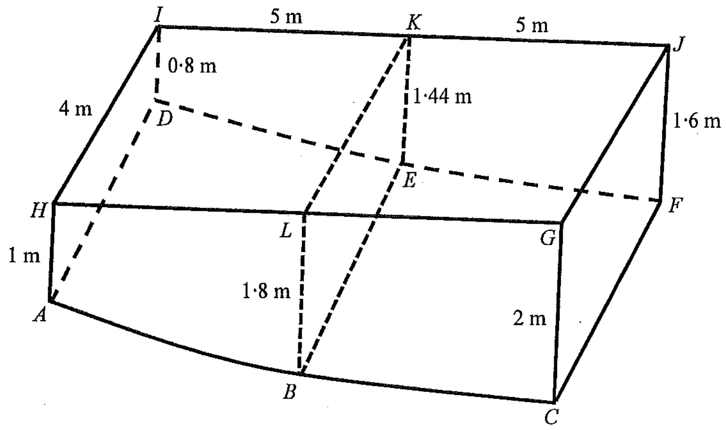
(iv) $y = \sin^{-1} f(x)$. **2**

(b) A curve is defined by the equation $x^2 + xy = e^y$. **3**
Find the gradient at the point $(-1, 0)$ on this curve.

Question 4 continues on page 7

Question 4 (continued)

(c) The surface of a backyard swimming pool is 10 metres long and 4 metres wide.



Cross-sections parallel to the end face *ADIH* are all trapezia with one edge being $0\cdot8$ times the edge corresponding to it on the opposite face.
The four side faces are all vertical.
The depth at *A* is 1 metre.
At *B*, which is halfway along the deep edge of the pool the depth is $1\cdot8$ metres and at *C* the depth is 2 metres.
Take the positive *x*-axis and positive *y*-axis as edges *GH* and *GC* respectively, i.e. *A* corresponds to the point $(10,1)$, *B* to the point $(5,1\cdot8)$ and *C* to the point $(0,2)$.

(i) Show that the equation of the parabolic edge *ABC* is $y=2+0\cdot02x-0\cdot012x^2$. **3**

(ii) By summing volumes of slices parallel to the face *ADIH* find the capacity of the pool in litres if it is to be filled to within 10 cm of its top. **2**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) (i) If $I_n=\int_0^1 x^n e^{x^2}\,dx$, show that $2I_n+(n-1)I_{n-2}=e$, for $n\geq 2$. **2**

(ii) Evaluate I_5. **3**

(b) A particle, *P*, of mass *m kg* is dropped from point *A* and falls towards point *B*, which is directly underneath *A*.
At the instant when *P* is dropped, a second particle, *Q*, also of mass *m kg*, is projected upwards from *B* towards *A* with an initial velocity equal to twice the terminal velocity of *P*.
Each particle experiences a resistance of magnitude *mkv* as it moves, where *v* m/s is the velocity and *k* is a constant.

(i) Show that the terminal velocity of *P* is $\frac{g}{k}$, where *g* is acceleration due to gravity. **1**

(ii) For particle *Q*, show that $t=\frac{1}{k}\ln\left(\frac{3g}{g+kv}\right)$, where *v* m/s is the velocity after *t* seconds. **3**

(iii) Suppose the particles collide at the instant when *P* has reached 30% of its terminal velocity.
Find the velocity of *Q* when they collide.
Leave your answer in terms of *g* and *k*. **3**

(b) (i) Mathsland only has an alphabet consisting of the letters X, Y and Z.
A word is to be spelt using *p* Xs, *q* Ys and *r* Zs, where *p*, *q* and *r* are integers and $q<p+r+1$.
Show that there are ${}^{p+r}C_p \times {}^{p+r+1}C_q$ ways that a word can be spelt using all the letters, so that no two Ys are adjacent. **2**

(iii) How many permutations are there of the letters of the name **1**

BEBE LE BELLE

taken all at a time, subject to the restriction that no two Bs are adjacent?

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) Let $w=\cos\theta+i\sin\theta$

(i) Show that $\left(z^4-w\right)\left(z^4-\frac{1}{w}\right)=z^8-2z^4\cos\theta+1$. **1**

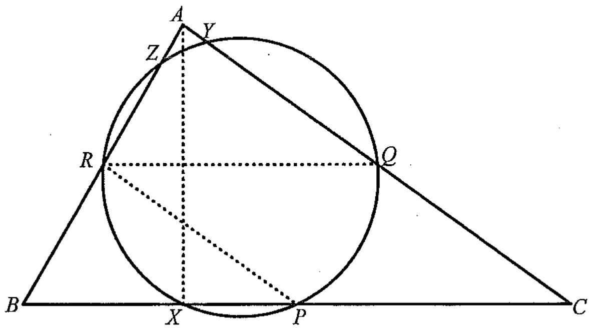
(ii) Hence solve the equation $z^8-z^4+1=0$, leaving your answers in modulus –argument form. **3**

(iii) Indicate these roots on an Argand diagram. **1**

(b) (i) If $y=\ln(\cos\theta+i\sin\theta)$, show that $\frac{dy}{d\theta}=i$. **2**

(ii) Hence deduce that $\cos\theta+i\sin\theta=e^{i\theta}$. **1**

(c) In the diagram below, *P*, *Q* and *R* are the midpoints of the sides *BC*, *CA* and *AB* respectively of an acute angled triangle *ABC*.
The circle drawn through the points *P*, *Q* and *R* meets the sides *BC*, *CA* and *AB* again at *X*, *Y* and *Z* respectively.



(i) Explain why *RPCQ* is a parallelogram. **1**

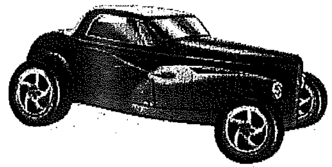
(ii) Show that ΔXQC is isosceles. **2**

(iii) Show that $AX\perp BC$. **2**

(iv) The lines *AX*, *BY* and *CZ* meet the circle again at *K*, *J* and *L* respectively. Show that *PK*, *QJ* and *RL* are concurrent. **2**

Question 7 (15 marks) Use a SEPARATE writing booklet.

A dragster racing car accelerates uniformly over a straight line course and completes a 'standing' (that is, starting from rest) 400 metres in eight seconds.



(a) (i) Starting with $\frac{d^2x}{dt^2}=k$, where *x* is in metres, *t* in seconds and *k* a constant, find the acceleration (in m/s^2) of the dragster over the 400 metres. **3**

(ii) Show that the dragster reaches a speed of 100 m/s at the end of the 400 metre course. **2**

At the 400 metre mark, the dragster stops accelerating.
At this instant, the dragster's brakes are applied and, in addition, a small parachute opens at the rear to slow the car down.

The retarding force applied by the brakes (including friction) is 5000 N.
The retarding force due to the parachute is $0\cdot5v^2$ N where *v* is the velocity of the car *x* metres *beyond* the 400 metre mark.
The mass of the dragster (car and driver) is 400 kg.

(b) (i) Show that the equation of motion for the dragster during this stage is given by **3**

$$\frac{dx}{dv}=-\frac{800v}{10^4+v^2}.$$

(ii) Hence show, to the nearest metre, that the distance the dragster takes to stop from the instant the brakes are applied is 277 m. **3**

(c) Show that it takes 2π seconds to bring the dragster to rest from the 400 metre mark. **4**

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Use de Moivre's theorem, or otherwise, to show that **2**

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$

[You may use the expansion $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$]

(ii) Given that $\alpha = \tan^{-1}\left(\frac{1}{4}\right)$, show that $4\alpha = \tan^{-1}\left(\frac{240}{161}\right)$. **1**

(iii) Write $161 + 240i$ in the form $r(\cos\theta + i\sin\theta)$, expressing θ in terms of α. **1**

(iv) Hence find, in the form $a + ib$ where a and b are integers, the four fourth roots of $161 + 240i$. **2**

(b) Let $f(x) = \dfrac{1}{1+x^2}$.

(i) (1) Prove that $f(x)$ is a decreasing function for all $x > 0$. **1**

(2) Hence or otherwise prove that if $0 < x < 1$ then $\dfrac{1}{2} < \dfrac{1}{1+x^2} < 1$. **1**

(ii) Find the sixth-degree polynomial $P(x)$ and the constant A such that **3**

$$x^4(1-x)^4 \equiv (1+x^2)P(x) + A.$$

(iii) Hence show that $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi$. **2**

(iv) Use (i) to deduce that $\dfrac{22}{7} - \dfrac{1}{630} < \pi < \dfrac{22}{7} - \dfrac{1}{1260}$. **2**

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0$$

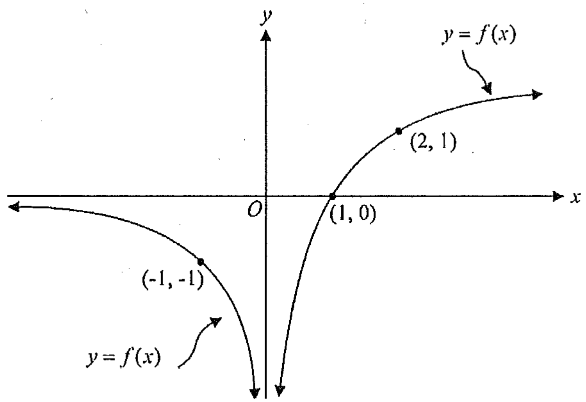
$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

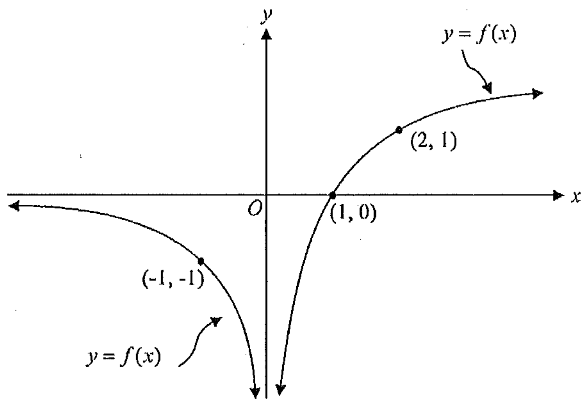
SBHS 2011 Trial HSC Extension 2 Mathematics Exam Answer Sheet for Question 4 (a)

STUDENT NUMBER ________________________________

(i)



(ii)

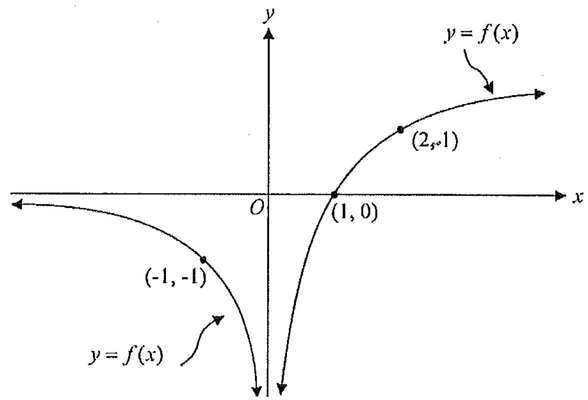


Parts (iii) and (iv) are on the back

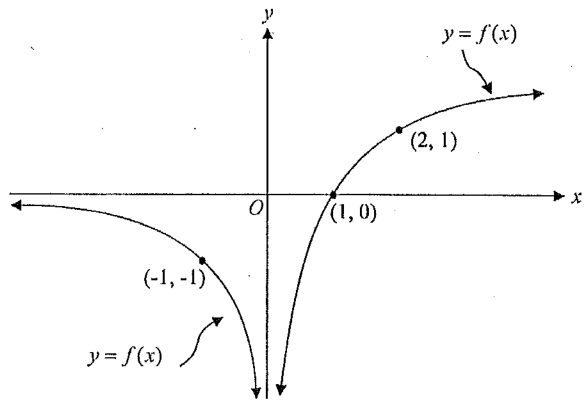
SBHS 2011 Trial HSC Extension 2 Mathematics Exam Answer Sheet for Question 4 (a)

STUDENT NUMBER ________________________________

(iii)



(iv)



Now place these sheets <u>INSIDE</u> your booklet for Question 4

SBHS THSC Ext 2 2011

Question 1

a) $\int \frac{dx}{\sqrt{4x^2+1}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2+\frac{1}{4}}}$

$= \frac{1}{2}\ln\left(x+\sqrt{x^2+\frac{1}{4}}\right)+C$

or $\frac{1}{2}\ln\left(2x+\sqrt{4x^2+1}\right)+k$

b) $\int_4^6 \frac{x-2}{2x^2-8x+3}\,dx = \frac{1}{4}\int_4^6 \frac{4x-8}{2x^2-8x+3}\,dx$

$= \left[\frac{1}{4}\ln\left(2x^2-8x+3\right)\right]_4^6$

$= \frac{1}{4}\ln\left(2(6)^2-8(6)+3\right) - \frac{1}{4}\ln\left(2(4)^2-8(4)+3\right)$

$= \frac{1}{4}\ln 27 - \frac{1}{4}\ln 3$

$= \frac{1}{4}\ln\left(\frac{27}{3}\right)$

$= \frac{1}{4}\ln 9$

$= \frac{1}{4}\ln 3^2$

$= \frac{1}{2}\ln 3$

$\therefore k = \frac{1}{2}$

c) $\int \frac{x^2}{2x-1}\,dx$

$u = 2x-1$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$

$2x = u+1$, $x = \frac{u+1}{2}$, $x^2 = \left(\frac{u+1}{2}\right)^2 = \frac{u^2+2u+1}{4}$

$= \int \frac{\left(\frac{u^2+2u+1}{4}\right)}{u}\cdot\frac{du}{2}$

$= \int\left(\frac{u}{8}+\frac{1}{4}+\frac{1}{8}\cdot\frac{1}{u}\right)du$

$= \frac{u^2}{16}+\frac{u}{4}+\frac{1}{8}\ln u + C$

$= \frac{(2x-1)^2}{16}+\frac{2x-1}{4}+\frac{1}{8}\ln(2x-1)+C$

d) i) $\frac{2x^2}{(x+5)(x-3)} \equiv A + \frac{B}{x+5} + \frac{C}{x-3}$

$2x^2 \equiv A(x+5)(x-3) + B(x-3) + C(x+5)$

let $x=3$

$2(3)^2 = C(3+5)$

$18 = 8C$

$C = \frac{9}{4}$

let $x=-5$

$2(-5)^2 = B(-5-3)$

$50 = -8B$

$B = -\frac{25}{4}$

equate coefficients of x^2:

$2 = A$

$\therefore \frac{2x^2}{(x+5)(x-3)} \equiv 2 + \frac{\left(-\frac{25}{4}\right)}{x+5} + \frac{\left(\frac{9}{4}\right)}{x-3}$

ii) $\int \frac{2x^2}{(x+5)(x-3)}\,dx = \int\left(2 - \frac{25}{4}\cdot\frac{1}{x+5} + \frac{9}{4}\cdot\frac{1}{x-3}\right)dx$

$= 2x - \frac{25}{4}\ln(x+5) + \frac{9}{4}\ln(x-3) + C$

e) $\int x\sin(2x-1)\,dx$

$u = x$, $v' = \sin(2x-1)$

$u' = 1$, $v = -\frac{1}{2}\cos(2x-1)$

$= -\frac{x}{2}\cos(2x-1) + \frac{1}{2}\int\cos(2x-1)\,dx$

$= -\frac{x}{2}\cos(2x-1) + \frac{1}{4}\sin(2x-1) + C$

Question 2

(a) $z = x + 2i$

(i) (1) $z^2 = (x+2i)^2$

$= x^2 + 4ix - 4$

$= (x^2-4) + 4x.i$

$a = x^2-4 \quad b = 4x$ [1]

(2) $z^2 + 2\bar{z} = (x^2-4) + 4ix + 2x - 4i$

$= (x^2+2x-4) + (4x-4)i$ [2]

(ii) $z^2 + 2\bar{z}$ is real when $\text{Im}(z^2+2\bar{z}) = 0$

ie $4x-4=0$ [1]

$x = +1$

(b) $2+3i$ is a root of $x^2+bx+c=0$

b, c, real.

(i) Other root $2-3i$ (by conj. root theorem) [1]

(ii) $(x-(2+3i))(x-(2-3i)) = 0$

$x^2 - 4x + 13 = 0$ [2]

$b = -4, \; c = 13$

(c)(i) $|z+5-i| = \sqrt{2}$

$\therefore |z-(-5+i)| = \sqrt{2}$ [2]

Radius $= \sqrt{2}$ Centre $= -5+i$

(ii) R is $\arg(z+2i) = \frac{3\pi}{4}$

(iii) $\arg(z_1+2i)$

$= \arg(-4+4i)$

$= \tan^{-1}\frac{4}{-4}$

$= \tan^{-1} -1$

$= \frac{3\pi}{4}$ [1]

$\therefore z_1$ lies on R

(iii) (1) $|z+5-i|$

$= |-4+2i+5-i|$

$= |1+i|$

$= \sqrt{2}$ [1]

$\therefore z_1$ lies on C

(c)(iii)(2) Distance of z_1 from centre $z_3 = -5+i$

$= |-5+i-(-4+2i)|$

$= |-1-i|$

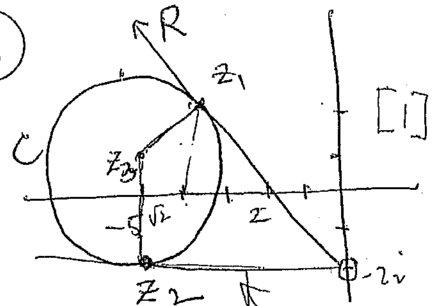
$= \sqrt{2} = r$

And $\arg(-1-i) = -\frac{3\pi}{4}$

$\therefore z_1 - z_3 \perp R$ and distance $\sqrt{2}$ from z_2.

$\therefore R$ is tangent. [2]

(3)

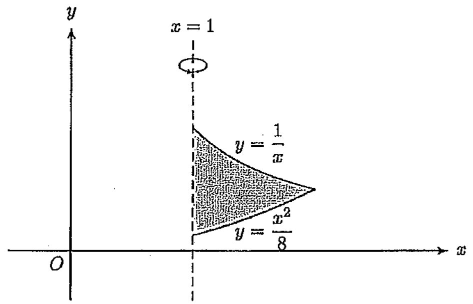


[1]

(iv) See above [1]

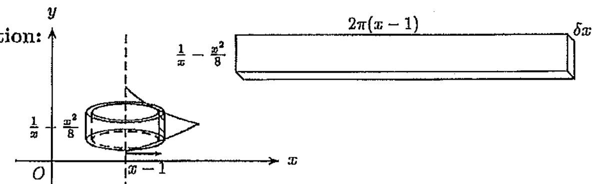
2011 Extension 2 Mathematics Task :
Solutions— Question 3

3. (a) The shaded region shown in the diagram below is bounded by $y = \frac{1}{x}$, $y = \frac{x^2}{8}$ and $x = 1$. This region is rotated about the line $x = 1$.



(i) Show that the volume of the resulting solid of revolution using the method of cylindrical shells is $V = 3\pi \int_1^2 (x-1)\left(\frac{1}{x} - \frac{x^2}{8}\right) dx$. [3]

Solution:



$\delta V = 2\pi(x-1)(\frac{1}{x} - \frac{x^2}{8})\delta x,$

$\therefore V = \lim_{\delta x \to 0} \sum_{x=1}^{2} 2\pi(x-1)\left(\frac{1}{x} - \frac{x^2}{8}\right)\delta x,$

$= 2\pi \int_1^2 (x-1)\left(\frac{1}{x} - \frac{x^2}{8}\right) dx.$

At the intersection:

$\frac{1}{x} = \frac{x^2}{8},$

$x^3 = 8,$

$x = 2.$

(ii) Find the volume of the solid of revolution. [2]

Solution: $V = 2\pi \int_1^2 \left(1 - \frac{x^3}{8} - \frac{1}{x} + \frac{x^2}{8}\right) dx,$

$= 2\pi \left[x - \frac{x^4}{32} - \ln x + \frac{x^3}{24}\right]_1^2,$

$= 2\pi\{2 - \frac{1}{2} - \ln 2 + \frac{1}{3} - (1 - \frac{1}{32} - 0 + \frac{1}{24})\},$

$= \left(\frac{79}{48} - 2\ln 2\right)\pi.$

(b) The roots of the cubic equation $z^3 - 2z^2 + pz + 10 = 0$ are α, β and γ. It is given that $\alpha^3 + \beta^3 + \gamma^3 = -4$.

(i) Write down the value of $\alpha + \beta + \gamma$. [1]

Solution: $\alpha + \beta + \gamma = 2.$

(ii) (α) Explain why $\alpha^3 - 2\alpha^2 + p\alpha + 10 = 0$. [1]

Solution: As α is a root of $P(z) = z^3 - 2z^2 + pz + 10 = 0$, then $P(\alpha) = \alpha^3 - 2\alpha^2 + p\alpha + 10 = 0.$

(β) Hence show that $\alpha^2 + \beta^2 + \gamma^2 = p + 13$. [3]

Solution:

$$\begin{aligned}
\alpha^3 - 2\alpha^2 + p\alpha + 10 &= 0,\\
\beta^3 - 2\beta^2 + p\beta + 10 &= 0,\\
\gamma^3 - 2\gamma^2 + p\gamma + 10 &= 0,\\
\alpha^3 + \beta^3 + \gamma^3 - 2(\alpha^2 + \beta^2 + \gamma^2) + p(\alpha + \beta + \gamma) + 30 &= 0,\\
-4 - 2(\alpha^2 + \beta^2 + \gamma^2) + 2p + 30 &= 0,\\
\alpha^2 + \beta^2 + \gamma^2 &= p + 13.
\end{aligned}$$

(iii) Deduce that $p = -3$. [2]

Solution:

$$\begin{aligned}
(\alpha + \beta + \gamma)^2 &= \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha),\\
&(\text{now } \alpha\beta + \beta\gamma + \gamma\alpha = p)\\
\alpha^2 + \beta^2 + \gamma^2 &= 2^2 - 2p,\\
p + 13 &= 4 - 2p \text{ (from part (ii))},\\
3p &= -9,\\
p &= -3.
\end{aligned}$$

(iv) Solve $z^3 - 2z^2 + pz + 10 = 0$. [3]

Solution: $P(z) = z^3 - 2z^2 + pz + 10$, possible zeroes $\pm 1, \pm 2, \pm 5$.

$P(2) = 8 - 8 - 6 + 10 \neq 0,$

$P(-2) = -8 - 8 + 6 + 10 = 0.$

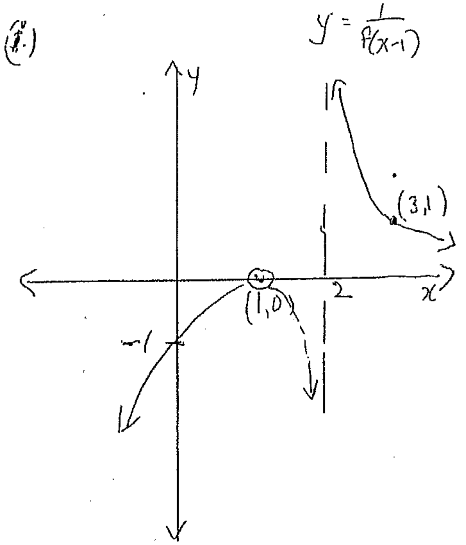
	1	−2	−3	10
−2		−2	8	−10
	1	−4	5	0

$$\begin{aligned}
z^2 - 4z + 4 + 1 &= 0,\\
(z-2)^2 &= -1,\\
z - 2 &= \pm i,\\
z &= 2 \pm i.
\end{aligned}$$

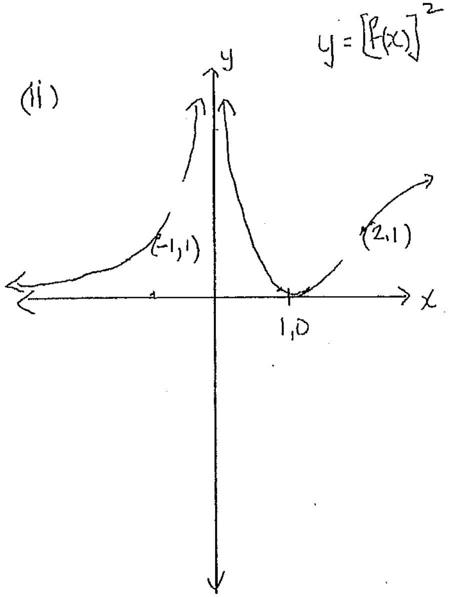
The roots are -2, $2 + i$, $2 - i$.

QUESTION 4 (a) X2

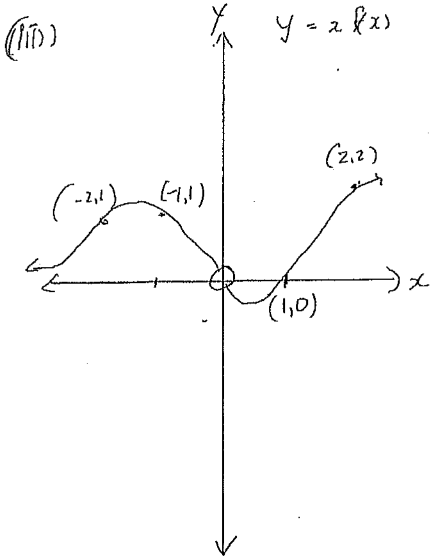
(i)



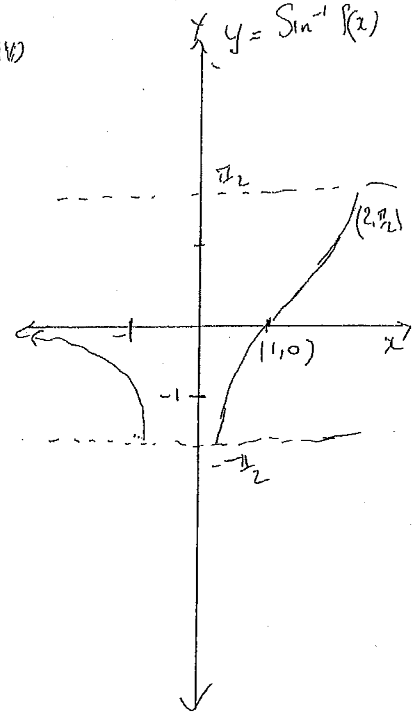
(ii)



(iii)



(iv)



QUESTION 4 X2

(b) $x^2 + xy = e^y$

$2x + y + x\frac{dy}{dx} = e^y \frac{dy}{dx}$

$\frac{dy}{dx} = (e^y - x) = 2x + y$

$\frac{dy}{dx} = \frac{2x+y}{e^y - x}$

at $(1,0)$ $\frac{dy}{dx} = \frac{-2+0}{1+1} = -1$

(c) (i) $y = 2 + 0.02x - 0.012x^2$

$(10,1)$ $1 = 2 + 0.2 - 1.2$

$1 = 1$ (satisfies eqn)

$(5,1.8)$ $1.8 = 2 + 0.1 - 0.030$

$1.8 = 1.8$ satisfies eqn

$(0,2)$ $2 = 2$ satisfies eqn.

$\therefore$ y is the equation of the parabolic edge

(ii) each slice of pool is a trapezium

Slice $= \frac{4}{2}(y + 0.8y)$

$= 3.6y$

(ii) cont.

$V = \int_0^{10} 3.6y \, dx$

$= 3.6\int_0^{10} (2 + 0.02x - 0.012x^2) dx$

$= 3.6\left[2x + 0.01x^2 - 0.004x^3\right]_0^{10}$

$= 3.6(20 + 1 - 4) - 0$

$= 61.2 \text{ m}^3 - 10 \times 4 \times 0.1$

$= 57.2 \text{ m}^3$

$= 57200 \text{ L}$

Q5 (a) (i) $I_n = \int_0^1 x^n e^{x^2}\,dx$

$= \frac{1}{2}\int_0^1 x^{n-1}\cdot(2x\,e^{x^2})\,dx$

$= \frac{1}{2}\left[x^{n-1}e^{x^2}\right]_0^1 - \frac{1}{2}\int_0^1 (n-1)\,x^{n-2}e^{x^2}\,dx$

$= \frac{1}{2}\left[1\times e - 0\times 1\right] - \frac{1}{2}(n-1)\int_0^1 x^{n-2}e^{x^2}\,dx$

$\therefore I_n = \frac{1}{2}e - \frac{1}{2}(n-1)\,I_{n-2}$

$\therefore 2I_n + (n-1)I_{n-2} = e.$

(ii) $I_5 = \frac{e}{2} - \frac{4}{2}I_3$

& $I_3 = \frac{e}{2} - I_1$ where $I_1 = \int_0^1 x\,e^{x^2}\,dx$

$= \frac{1}{2}\left[e^{x^2}\right]_0^1$

$= \frac{1}{2}(e-1)$

$\therefore I_5 = \frac{e}{2} - 2\left(\frac{e}{2} - \left(\frac{1}{2}(e-1)\right)\right)$

$= \frac{e}{2} - e + (e-1)$

$= \frac{e}{2} - 1$

(b) Down from B

↓ $m\ddot{x}$ ↓ mg ↑ mkv.

$m\ddot{x} = mg - mkv$ when $t=0,\ v=0$

$\therefore \ddot{x} = g - kv.$

(i) For terminal velocity let $\ddot{x} = 0$

$g - kv = 0$

$v = \frac{g}{k}.$

(ii) For particle Q. (up from A)

$m\ddot{x}$ ↑ ↓ mg ↓ mkv.

when $t=0$, $v = \frac{2g}{k}$.

$m\ddot{x} = -mg - mkv.$

$\ddot{x} = -(g+kv)$

$\frac{dv}{dt} = -(g+kv)$

$\frac{dt}{dv} = \frac{-1}{g+kv}$

$t = -\int \frac{dv}{g+kv}$

$t = -\frac{1}{k}\ln(g+kv) + c.$

when $t=0$ $v = \frac{2g}{k}$.

$0 = -\frac{1}{k}\ln\left(g + \frac{k\times 2g}{k}\right) + c$

$0 = -\frac{1}{k}\ln 3g + c$

$c = \frac{1}{k}\ln 3g$

$\therefore t = -\frac{1}{k}\ln(g+kv) + \frac{1}{k}\ln 3g$

$t = \frac{1}{k}\ln\left(\frac{3g}{g+kv}\right)$

(iii) For P. $\ddot{x} = g - kv$

$\frac{dv}{dt} = g - kv.$

now to find t when $v = 30\%$ of $\frac{g}{k}$

$= \frac{3g}{10k}$.

$$t = \int_0^{\frac{3g}{10k}} \frac{dv}{g-kv}.$$

$$= -\frac{1}{k}\Big[\ln(g-kv)\Big]_0^{\frac{3g}{10k}}.$$

$$= -\frac{1}{k}\left[\ln\left(g - \frac{k \times 3g}{10k}\right) - \ln g\right]$$

$$= -\frac{1}{k}\left(\ln \frac{7g}{10} - \ln g\right)$$

$$= -\frac{1}{k} \ln \frac{7}{10}.$$

Set $-\frac{1}{k}\ln\frac{7}{10} = \frac{1}{k}\ln \frac{3g}{g+kv}$

$$\frac{10}{7} = \frac{3g}{g+kv}.$$

$$10(g+kv) = 21g.$$

$$10kv = 11g$$

$$\boxed{v = \frac{11g}{10k}.}$$

(b) (I) First we need to place the x's & Z's in line. This can be done in

$$\frac{(p+r)!}{p!\,r!} = \frac{(p+r)!}{((p+r)-r)!\,r!} \text{ ways}$$

$$= {}^{p+r}C_r \text{ OR } {}^{p+r}C_p$$

ways.

we then place the q lots of Y. into the gaps

$\uparrow$ x $\uparrow$ x Z x Z x x Z - - - . . x $\uparrow$.

etc.

as there are $(p+r-1)$ gaps + 2 end spots.

giving $p+r-1+2 = p+r+1$ positions to place the q Y's.

ie. $\binom{p+r+1}{q}$

OR ${}^{p+r+1}C_q$.

$\therefore$ Total ways is

$$\binom{p+r}{p} \times \binom{p+r+1}{q}$$

$$\text{OR. } \boxed{{}^{p+r}C_p \times {}^{p+r+1}C_q}$$

(II.) 3 B's 5 E's & 3 L's.

Lay out E's & L's $= \binom{8}{3}$

Insert B's in $\binom{9}{3}$ ways

$\therefore \binom{8}{3} \times \binom{9}{3} = 4704$ ways.

Question 6

a) i) $LHS = (z^4 - \omega)(z^4 - \frac{1}{\omega})$

$= z^8 - \frac{1}{\omega}.z^4 - \omega.z^4 + 1$

$= z^8 - (\omega + \frac{1}{\omega})z^4 + 1$

since $|\omega| = 1$

$\frac{1}{\omega} = \bar{\omega}$

$\omega + \frac{1}{\omega} = 2Re(\omega)$

$= 2\cos\theta$

$\therefore LHS = z^8 - 2\cos\theta z^4 + 1$

$= RHS$

ii) $z^8 - z^4 + 1 = 0$

$(z^4 - \omega)(z^4 - \frac{1}{\omega}) = 0$ where $2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

$\theta = \pm\frac{\pi}{3}$

$z^4 - \omega = 0$

$z^4 = \omega$

let $z = r\,cis\beta$

Note: $cis\beta = \cos\beta + i\sin\beta$

$(r\,cis\beta)^4 = cis\frac{\pi}{3}$

$r^4 cis 4\beta = cis\frac{\pi}{3}$

equate

$r^4 = 1$ $\quad 4\beta = \frac{\pi}{3} + 2k\pi$ $\quad$ where $k = 0, 1, 2, 3$

$r = 1$ $\quad \beta = \frac{\pi}{12} + \frac{k\pi}{2}$

$z_1 = cis\frac{\pi}{12}$

$z_2 = cis\frac{7\pi}{12}$

$z_3 = cis\frac{13\pi}{12} = cis(\frac{-11\pi}{12})$

$z_4 = cis\frac{19\pi}{12} = cis(-\frac{5\pi}{12})$

since $\frac{1}{\omega} = \bar{\omega}$

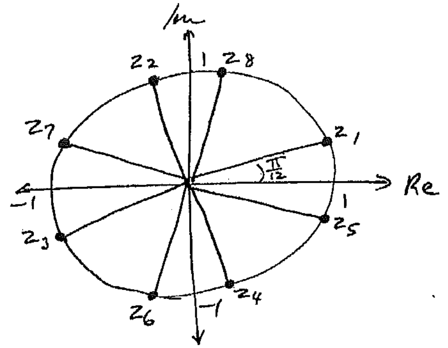
$z_5 = cis(-\frac{\pi}{12})$

$z_6 = cis(-\frac{7\pi}{12})$

$z_7 = cis(\frac{11\pi}{12})$

$z_8 = cis(\frac{5\pi}{12})$

iii)



b) i) $y = \ln(\cos\theta + i\sin\theta)$

$\frac{dy}{d\theta} = \frac{-\sin\theta + i\cos\theta}{\cos\theta + i\sin\theta}$

$= \frac{i(\cos\theta + i\sin\theta)}{\cos\theta + i\sin\theta}$

$= i$

ii) $\int i\,d\theta = i\theta + c$

$\ln(\cos\theta + i\sin\theta) = i\theta + c$

when $\theta = 0$

$\ln(\cos 0 + i\sin 0) = i(0) + c$

$\ln(1) = c$

$c = 0$

$\therefore \ln(\cos\theta + i\sin\theta) = i\theta$

$\cos\theta + i\sin\theta = e^{i\theta}$

c) i) The line joining the midpoints of two sides of a triangle is parallel to the third side.

$\therefore PR \parallel QC$ & $RQ \parallel PC$

$\therefore RPCQ$ is a parallelogram.

ii) let $Q\hat{C}X = x$

$Q\hat{R}P = x$ (opposite angles in parallelogram)

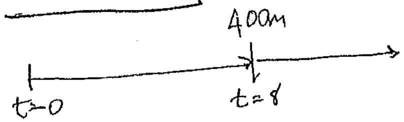
$Q\hat{X}C = x$ (angles in same segment)

$\therefore \triangle XQC$ is isosceles $(Q\hat{C}X = Q\hat{X}C)$

iii) $QX = QC$ ($\triangle XQC$ is isosceles)
$QC = AQ$ (Q is midpoint of AC)
$\therefore Q$ is the centre of a circle through points A, X & C.
$A\hat{X}C = 90^\circ$ (angle in semicircle)
$\therefore AX \perp BC$

iv) since $K\hat{X}P = 90^\circ$
KP is a diameter
similarly, JQ & RL are diameters
since PK, QJ & RL are all diameters of the same circle they are concurrent with the point of concurrency being the centre of the circle.

Question 7



(a)(i) $\frac{d^2x}{dt^2} = k$

$\therefore \frac{dx}{dt} = kt + C$ (Integ. w.r.t. t)

When $t=0$, $v=0$

$\therefore 0 = 0 + C$; $\therefore C = 0$

Thus $\frac{dx}{dt} = kt$

Integrate w.r.t. t:

$x = \frac{1}{2}kt^2 + D$

When $t=8$, $x=400$

$\therefore 400 = \frac{1}{2}k(64) + D$

$400 = 32k + D$

Also when $t=0$, $x=0$. $\therefore D=0$

So $400 = 32k$

$k = \frac{25}{2}$

But this is acceleration.

$\therefore \frac{d^2x}{dt^2} = \frac{25}{2}$ m/s^2 [3]

(ii) $v = \frac{dx}{dt} = kt$

When $t=8$

$v = \frac{25}{2} \times 8$

$= 100$ m/s [2]

(b)(i) $m = 400$kg
Braking Force $F_B = -5000$ N
Parachute Force $F_P = -\frac{1}{2}v^2$.
$F_B + F_P = -(5000 + \frac{1}{2}v^2)$.

$\therefore m\left(v\frac{dv}{dx}\right) = -(5000 + \frac{1}{2}v^2)$

$\therefore 400v\frac{dv}{dx} = -(5000 + \frac{1}{2}v^2)$

$\therefore 800v\frac{dv}{dx} = -(10^4 + v^2) \leftarrow (*)$

$\therefore \frac{dx}{dv} = -\frac{800v}{10^4 + v^2}$ [3]
as required

(ii) Integrate w.r.t. v:

$x = -\int \frac{800v}{10^4 + v^2} + C$

$x = -400\ln(10^4 + v^2) + C$

When $x=0$, $v=100$

$\therefore 0 = -400\ln(10^4 + 10^4) + C$

$\therefore C = 400\ln(2 \times 10^4)$

So $x = 400\ln\left(\frac{2 \times 10^4}{10^4 + v^2}\right)$

When $v=0$

$x = 400\ln 2$

$\doteqdot 227$ m. [3]

(c) From (*) above

$800\left(\frac{dv}{dt}\right) = -(10^4 - v^2)$

$\frac{dt}{dv} = -\frac{800}{10^4 + v^2}$

$\therefore t = -800\int \frac{dv}{10^4 + v^2} + C$

$t = -8\tan^{-1}\left(\frac{v}{100}\right) + C$

Q7 (Contd)

When $t=0$, $v=100$

$\therefore 0 = -8\tan^{-1}(1)+C$

$0 = -8\left(\frac{\pi}{4}\right)+C$

$C = 2\pi$

$\therefore t = 2\pi - 8\tan^{-1}\left(\frac{v}{100}\right)$

When $v=0$, $t = 2\pi - 0$

$\therefore t = 2\pi$ is time to come to rest. [4]

2011 Extension 2 Mathematics Task :
Solutions— Question 8

8. (a) (i) Using de Moivre's theorem or otherwise, show that [2]

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}.$$

[You may use the expansion $(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$.]

Solution:

$$\begin{aligned}
\text{cis}\,4\theta &= (\cos\theta + i\sin\theta)^4,\\
&= \cos^4\theta + 4\cos^3\theta i\sin\theta - 6\cos^2\theta\sin^2\theta - 4\cos\theta i\sin^3\theta + \sin^4\theta,\\
\cos 4\theta &= 4\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta, \text{ (equating real parts)}\\
\sin 4\theta &= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta, \text{ (equating imaginary parts)}\\
\therefore \tan 4\theta &= \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{4\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta} \div \frac{\cos^4\theta}{\cos^4\theta},\\
&= \frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}.
\end{aligned}$$

Alternative Solution:

$$\begin{aligned}
\tan 4\theta &= \frac{2\tan 2\theta}{1-(\tan 2\theta)^2},\\
&= \frac{2\times\frac{2\tan\theta}{1-\tan^2\theta}}{1-\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)^2},\\
&= \frac{4\tan\theta(1-\tan^2\theta)}{1-2\tan^2\theta+\tan^4\theta-4\tan^2\theta},\\
&= \frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}.
\end{aligned}$$

(ii) Given that $\alpha = \tan^{-1}\left(\frac{1}{4}\right)$, show that $4\alpha = \tan^{-1}\left(\frac{240}{161}\right)$. [1]

Solution:

$$\begin{aligned}
\tan\alpha &= \tfrac{1}{4}.\\
\tan 4\alpha &= \frac{4\times\frac{1}{4} - 4\times\frac{1}{4^3}}{1-6\times\frac{1}{4^2}+\frac{1}{4^4}},\\
&= \frac{256-16}{256-96+1},\\
&= \frac{240}{161}.\\
\therefore 4\alpha &= \tan^{-1}\left(\tfrac{240}{161}\right).
\end{aligned}$$

(iii) Write $161 + 240i$ in the form $r(\cos\theta + i\sin\theta)$, expressing θ in terms of α. [1]

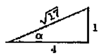
Solution:

$$\begin{aligned} r &= \sqrt{161^2 + 240^2}, \\ &= \sqrt{83521}, \\ &= 289. \\ 161 + 240i &= 289\left(\frac{161}{289} + \frac{240i}{289}\right), \\ \cos\theta &= \frac{161}{289}, \quad \sin\theta = \frac{240}{289}, \\ \tan\theta &= \frac{240}{161} \ \textit{i.e.}\ \tan 4\alpha. \\ \text{So } 161 + 240i &= 289(\cos 4\alpha + i\sin 4\alpha). \end{aligned}$$

(iv) Hence find in the form $a + ib$, where a and b are integers, the four fourth roots of $161 + 240i$. [2]

Solution: $161 + 240i = 289(\cos(4\alpha + 2n\pi) + i\sin(4\alpha + 2n\pi)),\ n \in \mathbb{J}$, (from part(iii))

$$\begin{aligned} (a+ib)^4 &= (\sqrt{17})^4\big(\cos(\alpha + n\tfrac{\pi}{2}) + i\sin(\alpha + n\tfrac{\pi}{2})\big)^4, \\ a + ib &= \sqrt{17}\big(\cos(\alpha + n\tfrac{\pi}{2}) + i\sin(\alpha + n\tfrac{\pi}{2})\big), \\ &= \sqrt{17}\left(\frac{4}{\sqrt{17}} + i.\frac{1}{\sqrt{17}}\right), \text{(the principal value)} \\ &= 4 + i. \end{aligned}$$



$\therefore$ The four roots are $\pm(4+i)$, $\pm(1-4i)$.

(b) Let $f(x) = \dfrac{1}{1+x^2}$.

(i) (α) Prove that $f(x)$ is a decreasing function for all $x > 0$. [1]

Solution: $f'(x) = \dfrac{-2x}{(1+x^2)^2}$,
< 0 for all $x > 0$.
$\therefore f(x)$ is a decreasing function when $x > 0$.

(β) Hence or otherwise prove that, if $0 < x < 1$, then $\dfrac{1}{2} < \dfrac{1}{1+x^2} < 1$. [1]

Solution: When $x = 0$, $f(x) = 1$,
$x = 1$, $f(x) = \frac{1}{2}$.
$\therefore$ If $0 < x < 1$, then $\frac{1}{2} < f(x) < 1$.

(ii) Find the sixth-degree polynomial $P(x)$ and the constant A such that [3]

$$x^4(1-x)^4 \equiv (1+x^2)P(x) + A.$$

Solution:

$$\begin{array}{r|l} & x^6 - 4x^5 + 5x^4 - 4x^2 + 4 \\ \hline x^2+1 & x^8 - 4x^7 + 6x^6 - 4x^5 + x^4 \\ & -x^8 \qquad\ \ - x^6 \\ \hline & -4x^7 + 5x^6 - 4x^5 \\ & 4x^7 \qquad\quad + 4x^5 \\ \hline & 5x^6 \qquad + x^4 \\ & -5x^6 \qquad - 5x^4 \\ \hline & -4x^4 \\ & 4x^4 + 4x^2 \\ \hline & 4x^2 \\ & -4x^2 - 4 \\ \hline & -4 \end{array}$$

$\therefore P(x) = x^6 - 4x^5 + 5x^4 - 4x^2 + 4$,
$A = -4$.

Alternative Solution:

$$\begin{aligned} x^4(1-x)^4 &= x^8 - 4x^7 + 6x^6 - 4x^5 + x^4, \\ &= (x^2+1)(ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g) + A, \\ &= ax^8 + bx^7 + (a+c)x^6 + (b+d)x^5 + (c+e)x^4 + (d+f)x^3 \\ &\quad + (e+g)x^2 + fx + (g+A). \end{aligned}$$

Equating coefficients,

$$\begin{aligned} a &= 1, & b &= -4, \\ a+c &= 6, & c &= 5, \\ b+d &= -4, & d &= 0, \\ c+e &= 1, & e &= -4, \\ d+f &= 0, & f &= 0, \\ e+g &= 0, & g &= 4, \\ g+A &= 0, & A &= -4. \end{aligned}$$

$\therefore P(x) = x^6 - 4x^5 + 5x^4 - 4x^2 + 4$,
$A = -4$.

(iii) Hence show that $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi$. [2]

Solution:

$$\begin{aligned}\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx &= \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}\right) dx,\\ &= \left[\frac{x^7}{7} - \frac{4x^6}{6} + \frac{5x^5}{5} - \frac{4x^3}{3} + 4x - 4\tan^{-1} x\right]_0^1,\\ &= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - 4.\frac{\pi}{4} - (0 - 4 \times 0),\\ &= \frac{22}{7} - \pi.\end{aligned}$$

(iv) Use (i) to deduce that $\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}$. [2]

Solution: Noticing that $x^4(1-x)^4 > 0$ for all x, we have from (i):

$$\begin{aligned}\tfrac{1}{2}x^4(1-x)^4 \quad &< \quad \frac{x^4(1-x)^4}{1+x^2} \quad < \quad x^4(1-x)^4,\\ \text{so } \tfrac{1}{2}\int_0^1 x^4(1-x)^4 dx \quad &< \quad \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx \quad < \quad \int_0^1 x^4(1-x)^4 dx.\end{aligned}$$

$$\begin{aligned}\text{Now } \int_0^1 x^4(1-x)^4 dx &= \int_0^1 (x^4 - 4x^5 + 6x^6 - 4x^7 + x^8)\,dx,\\ &= \left[\frac{x^5}{5} - \frac{4x^6}{6} + \frac{6x^7}{7} - \frac{4x^8}{8} + \frac{x^9}{9}\right]_0^1,\\ &= \frac{1}{630}.\end{aligned}$$

$$\begin{aligned}\therefore \frac{1}{1260} \quad &< \quad \frac{22}{7} - \pi \quad < \quad \frac{1}{630},\\ \frac{1}{1260} - \frac{22}{7} \quad &< \quad -\pi \quad < \quad \frac{1}{630} - \frac{22}{7},\\ \frac{22}{7} - \frac{1}{630} \quad &< \quad \pi \quad < \quad \frac{22}{7} - \frac{1}{1260}.\end{aligned}$$