SYDNEY GRAMMAR SCHOOL

2011 Trial Examination

FORM VI
MATHEMATICS EXTENSION 1

Wednesday 10th August 2011

General Instructions

- Reading time — 5 minutes
- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Structure of the paper

- Total marks — 84
- All seven questions may be attempted.
- All seven questions are of equal value.

Collection

- Write your candidate number clearly on each booklet.
- Hand in the seven questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper inside your answer booklet for Question One.

Checklist

- SGS booklets — 7 per boy
- Candidature — 126 boys

Examiner
LYL

QUESTION ONE (12 marks) Use a separate writing booklet. Marks

(a) Simplify $\dfrac{(n+1)!}{n!}$. [1]

(b) Find $\displaystyle\int \frac{1}{9+x^2}\,dx$. [1]

(c) When the polynomial $P(x) = x^3 + 3x^2 + ax - 10$ is divided by $x - 2$, the remainder is 24. Find a. [2]

(d) Differentiate $y = \sin^{-1}(x^3)$. [2]

(e) Suppose that α, β and γ are the roots of the equation $x^3 - 3x^2 - 4x + 12 = 0$.

(i) Write down the value of $\alpha\beta + \alpha\gamma + \beta\gamma$. [1]

(ii) Hence find the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$. [1]

(f) (i) Without the use of calculus, sketch the polynomial $y = x(x+1)(x-4)$ showing all the intercepts with the axes. [2]

(ii) Hence, or otherwise, solve the inequation $\dfrac{x(x+1)}{x-4} \geq 0$. [2]

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) Find the exact value of $\sin^{-1}(\sin \frac{2\pi}{3})$. [1]

(b) Find $\displaystyle\lim_{x\to\infty} \frac{3-x}{2x+3}$. [1]

(c) The point A is $(2, -4)$ and the point B is $(5, 2)$. The point P divides the interval AB externally in the ratio $4:1$. Find the coordinates of P. [2]

(d) Find the gradient of the tangent to the curve $y = \tan^{-1}(\sin x)$ at $x = \pi$. [2]

(e) A ball is projected vertically upwards from the ground. After t seconds, the height of the ball is given by $h = 45t - 5t^2$ metres.

(i) At what time does the ball returns to the ground? [1]

(ii) When is the ball instantaneously at rest? [1]

(iii) What is the greatest height attained by the ball? [1]

(f) (i) Sketch the graph of the function $y = |x^2 - 4|$. [2]

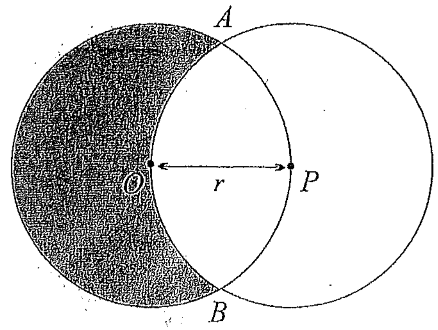
(ii) At what points is $f(x) = |x^2 - 4|$ not differentiable? [1]

Exam continues overleaf ...

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a) State the domain and range of $f(x) = 2\cos^{-1}\frac{x}{4}$. [2]

(b)



In the diagram above, two circles of equal radius r units are drawn such that their centres O and P are r units apart. The two circles intersect at A and B.

(i) Show that the quadrilateral $AOBP$ is a rhombus. [1]

(ii) Show that $\angle AOB = 120°$. [1]

(iii) Find the area of the shaded region in terms of r. [2]

(c) The function $f(x) = x\log x + x - 1{\cdot}1$ has a zero near $x = 1$. Take $x = 1$ as a first approximation and use Newton's method <u>once</u> to obtain a closer approximation to this zero. [3]

(d) Find the term independent of x in the expansion of $\left(4x^3 - \frac{1}{x}\right)^{12}$. [3]

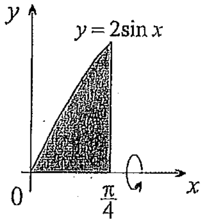
Exam continues next page ...

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) Given that α is an acute angle and $\cos\alpha = \frac{3}{4}$, find the exact value of $\tan\frac{\alpha}{2}$. $\boxed{2}$

(b) Using the substitution $u = 4x + 1$, evaluate $\displaystyle\int_0^1 \frac{4x}{(4x+1)^2}\,dx$. $\boxed{3}$

(c)



The diagram above shows the region bounded by the curve $y = 2\sin x$, the x-axis and the line $x = \frac{\pi}{4}$. Find the exact volume of the solid generated when the shaded region is rotated about the x-axis. $\boxed{3}$

(d) A particle is moving in a straight line according to the equation

$$x = \sqrt{3}\cos 3t - \sin 3t\,,$$

where x metres is its displacement from the origin after t seconds.

(i) Show that the particle is moving in simple harmonic motion. $\boxed{2}$

(ii) Find the time at which the particle first passes through the origin. $\boxed{2}$

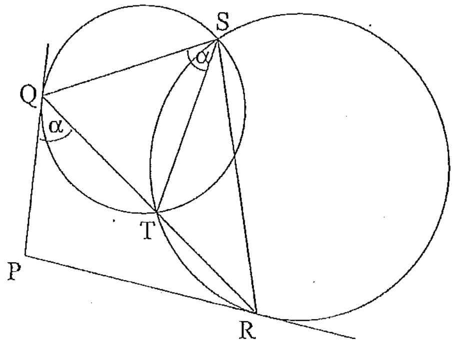
Exam continues overleaf ...

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a) Prove by mathematical induction that for all positive integer values of n, $\boxed{4}$

$$\frac{1}{3}\times\frac{1}{1}+\frac{1}{5}\times\frac{1}{3}+\frac{1}{7}\times\frac{1}{5}+\cdots+\frac{1}{(2n+1)}\times\frac{1}{(2n-1)}=\frac{n}{2n+1}\,.$$

(b)

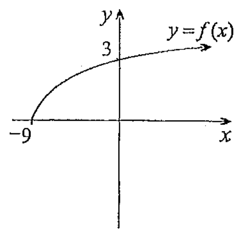


In the diagram above PQ and PR are tangents to the circles SQT and STR respectively, and the points Q, T and R are collinear.

(i) Given that $\angle QST = \alpha$, state a reason why $\angle PQT = \alpha$. $\boxed{1}$

(ii) Prove that $PQSR$ is a cyclic quadrilateral. $\boxed{2}$

(c)



The diagram above shows a sketch of $y = f(x)$ where $f(x) = \sqrt{x+9}$.

(i) Copy the diagram. On the same set of axes, sketch the graph of the inverse function $y = f^{-1}(x)$, clearly marking the x and y-intercepts. $\boxed{1}$

(ii) What is the domain of $f^{-1}(x)$? $\boxed{1}$

(iii) Find an expression for $f^{-1}(x)$. $\boxed{1}$

(iv) Given that the graphs of $y = f(x)$ and $y = f^{-1}(x)$ meet at the point P, find the x-coordinate of P. $\boxed{2}$

Exam continues next page ...

QUESTION SIX (12 marks) Use a separate writing booklet. Marks

(a) When an object falls from rest at $t = 0$ through a resisting liquid, the rate of change of its velocity at time t is given by $\dfrac{dv}{dt} = -k(v - 600)$, where k is a positive constant.

(i) Show that $v = 600 + Pe^{-kt}$ is a solution to the differential equation for some constant P. 1

(ii) If the velocity of the object at $t = 3\,\text{s}$ is $25\,\text{ms}^{-1}$, find P and k. 2

(iii) Find the velocity of the object at $t = 10\,\text{s}$. Give your answer correct to one decimal place. 1

(iv) What is the limiting value of v as $t \to \infty$? 1

(b) Let $(2x + y)^{12} = \sum_{k=0}^{12} T_k$ where $T_k = {}^{12}C_k \times (2x)^{12-k} \times y^k$.

(i) Show that $\dfrac{T_{k+1}}{T_k} = \dfrac{y(12-k)}{2x(k+1)}$. 1

(ii) Suppose that $x = 4$ and $y = 5$ in the expansion of $(2x + y)^{12}$. Show that there are two consecutive terms that are equal, and greater in value than any of the other terms. 2

(c) (i) Find the general solutions of the equation 3

$$2\cos 3x \sin 4x + 2\cos 3x - \sin 4x - 1 = 0\,.$$

(ii) Hence write down all the solutions in the domain $0 \le x \le \pi$. 1

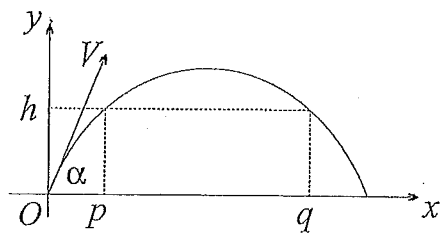
Exam continues overleaf ...

QUESTION SEVEN (12 marks) Use a separate writing booklet. Marks

(a) Using the identity $(1+x)^{2n} = (1+x)^n(1+x)^n$, show that 2

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 .$$

(b)



A particle is projected from a point O at an angle of elevation α with level ground at an initial velocity $V\,\text{ms}^{-1}$, as in the diagram above.

The particle just clears two vertical poles of height h metres at horizontal distances of p and q metres from O. Take acceleration due to gravity as $10\,\text{ms}^{-2}$ and ignore air resistance. You may assume the equations of motion:

$$x = Vt\cos\alpha$$
$$y = Vt\sin\alpha - 5t^2$$

(i) Find an expression for V^2 in terms of α, p and h. 2

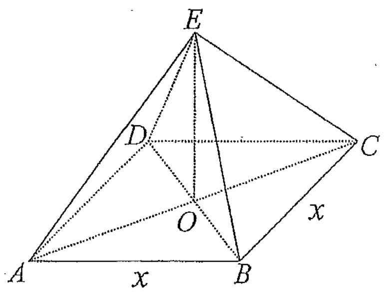
(ii) Hence show that $\tan\alpha = \dfrac{h(p+q)}{pq}$. 2

Question Seven continues on the next page

Exam continues next page ...

QUESTION SEVEN (Continued)

(c)



A square pyramid has its apex vertically above the centre of the base. The square base has side length x and the volume of the pyramid is V. The area of each triangular face is $\frac{S}{4}$ for some constant S.

(i) Show that $S^2 = x^4 + \dfrac{36V^2}{x^2}$. [2]

(ii) Prove that if V is constant and x is variable, then S has its minimum value when [2]

$$x^3 = (3\sqrt{2})V.$$

(iii) When S is at its minimum, show that each triangular face is equilateral. [2]

END OF EXAMINATION

2011 TRIAL. EXT 1

Q1

a) $\frac{(n+1)!}{n!} = n+1$ ✓

b) $\int \frac{1}{9+x^2}\,dx = \frac{1}{3}\tan^{-1}\frac{x}{3} + c$ ✓

c) $P(2)=24$

$2^3 + 3\times 2^2 + 2a - 10 = 24$

$8 + 12 + 2a - 10 = 24$ ✓

$2a + 10 = 24$

$2a = 14$ ✓

$a = 7$

d) $y = \sin^{-1} u \qquad u = x^3$

$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}} \qquad \frac{du}{dx} = 3x^2$ ✓

$\frac{dy}{dx} = \frac{3x^2}{\sqrt{1-x^6}}$ ✓

e) i) $\alpha\beta + \alpha\gamma + \beta\gamma = -4$ ✓

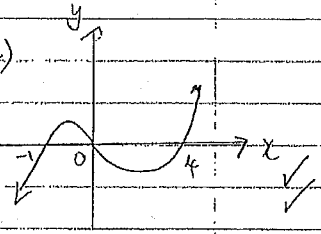
ii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$

$= \frac{-4}{-12}$ ✓

$= \frac{1}{3}$

f) i)



ii) $\frac{x(x+1)}{(x-4)} \geq 0 \qquad x \neq 4$

$\left[\times (x-4)^2\right]$

$(x-4)x(x+1) \geq 0$

$-1 \leq x \leq 0$ ✓ or $x > 4$ ✓

Note * $x \neq 4$

Q2

a) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\pi}{3}$ ✓

b) $\lim_{x\to\infty} \frac{3-x}{2x+3}$

$= \lim_{x\to\infty} \frac{\frac{3}{x} - \frac{x}{x}}{\frac{2x}{x} + \frac{3}{x}}$

$= -\frac{1}{2}$ ✓

c) $A(\underset{x_1}{2}, \underset{y_1}{-4}) \qquad B(\underset{x_2}{5}, \underset{y_2}{2}) \qquad \underset{m}{4} : \underset{n}{-1}$

$P = \left(\frac{mx_2 - nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$

$= \left(\frac{4\times5 - 1\times2}{4-1}, \frac{4\times2 - 1\times -4}{3}\right)$

$= \left(\frac{20-2}{3}, \frac{8+4}{3}\right)$

$= (6, 4)$ ✓ ✓

d) $y = \tan^{-1}(\sin x)$

$\frac{dy}{dx} = \frac{\cos x}{1 + \sin^2 x}$ ✓

At $x = \pi$

$\frac{dy}{dx} = \frac{\cos\pi}{1+\sin^2\pi}$ ✓

$= -1$

e) i) $45t - 5t^2 = 0$

$5t(9-t) = 0$ ✓

Ball returns at $t = 9s$.

ii) $h = 45t - 5t^2$

$\frac{dh}{dt} = 45 - 10t$

$45 - 10t = 0$

$-10t = -45$

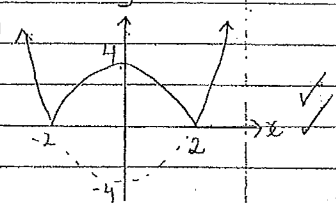
$t = 4.5$ ✓

iii) $t = 4.5$

$h = 45\times4.5 - 5\times(4.5)^2$

$= 101.25\,m$ ✓

f) i)



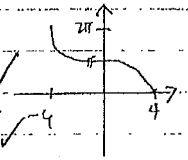
ii) Not differentiable at $(-2, 0)$ $(2, 0)$ ✓

Q3

a) $f(x) = 2\cos^{-1}\frac{x}{4}$

Stretched horizontally $\times 4$

vertically $\times 2$



Domain $-4 \le x \le 4$ ✓

Range $0 \le y \le 2\pi$ ✓

b) i) $OA = PA = OB = PB = r$ ✓

(radii of congruent circles)

$OAPB$ is a rhombus.

ii) $OP = OA = AP$ (radii)

$\triangle OAP$ is equilateral

$\angle AOP = \frac{\pi}{3}$

$OP = OB = BP$ (radii) ✓

$\triangle OBP$ is equilateral

$\angle POB = \frac{\pi}{3}$

$\angle AOB = \angle AOP + \angle POB$ (adjacent angles)

$= \frac{2\pi}{3}$

iii) Area of segment $= \frac{1}{2}r^2\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$

Shaded region $= \pi r^2 - 2 \times \frac{1}{2}r^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$ ✓

$= \pi r^2 - r^2\left(\frac{4\pi - 3\sqrt{3}}{6}\right)$

$= \frac{2\pi r^2 + 3\sqrt{3}r^2}{6}$ ✓

c) $f(x) = x\log x + x - 1.1$

$f'(x) = (1 + \log x) + 1$

$= 2 + \log x$ ✓

$f(1) = \log 1 + 1 - 1.1$

$= -0.1$ ✓

$f'(1) = 2$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 1 - \frac{(-0.1)}{2}$

$= 1.05$ ✓

d) $\left(4x^3 - \frac{1}{x}\right)^{12}$

General term

${}^{12}C_r (4x^3)^{12-r}(-x^{-1})^r$ ✓

$= {}^{12}C_r\, 4^{12-r} x^{36-3r} (-1)^r x^{-r}$

For term independent of x

$x^{36-3r}x^{-r} = x^0$

$36 - 4r = 0$

$4r = 36$

$r = 9$ ✓

$\therefore$ Term independent of x is

$-{}^{12}C_9\, 4^3$

$= -14080$ ✓

Q4

a) $\cos\alpha = \frac{1-t^2}{1+t^2}$ $\quad \cos\alpha = \frac{3}{4}$

$\frac{1-t^2}{1+t^2} = \frac{3}{4}$

$4 - 4t^2 = 3 + 3t^2$ ✓

$7t^2 = 1$

$t = \pm\frac{1}{\sqrt{7}}$

Since α is acute

$t = \frac{1}{\sqrt{7}}$ ✓

$\tan\frac{\alpha}{2} = \frac{1}{\sqrt{7}}$

b) $\int_0^1 \frac{4x}{(4x+1)^2}\,dx$ $\quad u = 4x+1$, $\frac{du}{dx} = 4$, $du = 4dx$, $dx = \frac{du}{4}$, $x = \frac{u-1}{4}$, $x=0$ $u=1$, $x=1$ $u=5$

$= \int_1^5 \frac{4\left(\frac{u-1}{4}\right)}{u^2}\,\frac{du}{4}$

$= \int_1^5 \frac{u-1}{4u^2}\,du$

$= \int_1^5 \left(\frac{1}{4u} - \frac{1}{4u^2}\right)du$

$= \frac{1}{4}\left[\log u + \frac{1}{u}\right]_1^5$

$= \frac{1}{4}\left[\left(\log 5 + \frac{1}{5}\right) - (\log 1 + 1)\right]$

$= \frac{1}{4}\left[\log 5 - \frac{4}{5}\right]$

c) $y = 2\sin x$ $\quad y^2 = 4\sin^2 x$

$V = \int_0^{\frac{\pi}{4}} \pi y^2\,dx$

$= 4\pi\int_0^{\frac{\pi}{4}} \sin^2 x\,dx$

$= 4\pi\int_0^{\frac{\pi}{4}} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx$ ✓

$= 4\pi\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{4}}$ ✓

$= 4\pi\left[\left(\frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2}\right) - (0-0)\right]$

$= 4\pi\left(\frac{\pi}{8} - \frac{1}{4}\right)$

$= \frac{4\pi^2}{8} - \frac{4\pi}{4}$ ✓

$= \left(\frac{\pi^2}{2} - \pi\right)u^3$

d) i) $x = \sqrt{3}\cos 3t - \sin 3t$

$\dot{x} = -3\sqrt{3}\sin 3t - 3\cos 3t$ ✓

$\ddot{x} = -9\sqrt{3}\cos 3t + 9\sin 3t$

$= -9(\sqrt{3}\cos 3t - \sin 3t)$

$= -9x$ ✓

$= -n^2x$

ii) $\sqrt{3}\cos 3t - \sin 3t = 0$

$\sqrt{3}\cos 3t = \sin 3t$ ✓

$\tan 3t = \sqrt{3}$

$3t = \frac{\pi}{3}, \frac{4\pi}{3}, \ldots$

$t = \frac{\pi}{9}, \frac{4\pi}{9}, \ldots$

$\therefore$ Particle first passes the origin at $t = \frac{\pi}{9}$ ✓

Q5

a) A. When $n=1$

$RHS = \frac{1}{2\times1+1}$

$= \frac{1}{3}$

$LHS = \frac{1}{(2\times1+1)} \times \frac{1}{(2\times1-1)}$

$= \frac{1}{3} \times \frac{1}{1}$

$= \frac{1}{3}$

$= RHS$ ✓

$\therefore$ Statement true for $n=1$

B. Assume statement true for $n=k$ where k is a positive integer

ie.

** $\frac{1}{3}\times\frac{1}{1} + \frac{1}{5}\times\frac{1}{3} + \dots + \frac{1}{(2k+1)}\times\frac{1}{(2k-1)} = \frac{k}{2k+1}$ ✓

Must prove statement for $n=k+1$

ie.

$\frac{1}{3}\times\frac{1}{1} + \frac{1}{5}\times\frac{1}{3} + \dots + \frac{1}{(2(k+1)+1)}\times\frac{1}{(2(k+1)-1)} = \frac{k+1}{2(k+1)+1}$

$= \frac{k+1}{2k+3}$

$LHS =$

$\underbrace{\frac{1}{3}\times\frac{1}{1} + \frac{1}{5}\times\frac{1}{3} + \dots + \frac{1}{(2k+1)}\times\frac{1}{(2k-1)}}_{**} + \frac{1}{(2(k+1)+1)}\times\frac{1}{(2(k+1)-1)}$ ✓

$= \frac{k}{2k+1} + \frac{1}{(2k+3)} \times \frac{1}{(2k+1)}$ By induction hypothesis

$= \frac{k(2k+3)+1}{(2k+3)(2k+1)}$ ✓

$= \frac{2k^2+3k+1}{(2k+3)(2k+1)} = \frac{\cancel{(2k+1)}(k+1)}{(2k+3)\cancel{(2k+1)}} = \frac{k+1}{2k+3} = RHS$

C. It follows from parts A and B by mathematical induction that the statement is true for all positive integers n.

b) i) $\angle PQT = \alpha$ (alternate segment theorem) ✓

ii) let $\angle PRQ = \theta$

$\angle TQS = \theta$ (alternate segment theorem)

In ΔPRQ

$\angle QPR = 180 - (\alpha + \theta)$ (angle sum of Δ) ✓

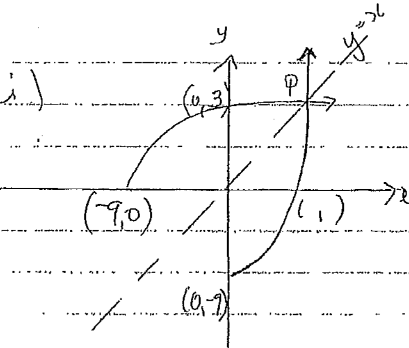
$\therefore \angle QSR = \alpha + \theta$ (adjacent $\angle$)

$\angle QPR + \angle QSR = 180$

$\therefore PQSR$ is cyclic quad as opposite angles in a cyclic quad are supplementary. ✓

Q5

c) i)



✓

ii) Domain of $f^{-1}(x) \geq 0$ ✓

iii) $y = \sqrt{x+9}$

interchange x & y

$x = \sqrt{y+9}$

$x^2 = y+9$ ✓

$y = x^2 - 9$ where $x \geq 0$

iv) $x = \sqrt{x+9}$

$x^2 = x+9$

$x^2 - x - 9 = 0$ ✓

$a = 1$ $\Delta = b^2 - 4ac$

$b = -1$ $= 1 - 4\times1\times-9$

$c = -9$ $= 37$

$x = \frac{1 \pm \sqrt{37}}{2}$ $x \geq 0$ ✓

$\therefore x = \frac{1+\sqrt{37}}{2}$ x-coordinate of P.

Q6

a) i) $v = 600 + Pe^{-kt}$

$\frac{dv}{dt} = \frac{d}{dt}(600 + Pe^{-kt})$

$= -kPe^{-kt}$ ✓

$= -k(v - 600)$

ii) $t=0 \quad v=0$

$t=3 \quad v=25$

$0 = 600 + Pe^0$

$P = -600$ ✓

$25 = 600 - 600e^{-3k}$

$-575 = -600e^{-3k}$

$e^{-3k} = \frac{575}{600}$

$e^{3k} = \frac{24}{23}$

$\ln e^{3k} = \ln\left(\frac{24}{23}\right)$

$3k = \ln\left(\frac{24}{23}\right)$ ✓

$k = \frac{1}{3}\ln\left(\frac{24}{23}\right)$

iii) $t = 10$

$v = 600 - 600e^{-10k}$

$\doteqdot 79.4\,ms^{-1}$ ✓

iv) $v = 600(1 - e^{-kt})$

$t \to \infty \quad e^{-kt} \to 0$

$v = 600$ ✓

b) $T_k = {}^{12}C_k (2x)^{12-k} y^k$

$= {}^{12}C_k 2^{12} x^{12-k} y^k$

$T_{k+1} = {}^{12}C_{k+1} (2x)^{12-(k+1)} y^{k+1}$

$= {}^{12}C_{k+1} (2x)^{11-k} y^{k+1}$

$\frac{T_{k+1}}{T_k} = \frac{{}^{12}C_{k+1}(2x)^{11-k} y^{k+1}}{{}^{12}C_k (2x)^{12-k} y^k}$

$= \frac{\frac{12!}{(k+1)!(11-k)!}(2x)^{11-k} y^{k+1}}{\frac{12!}{k!(12-k)!}(2x)^{12-k} y^k}$ ✓

$= \frac{(12-k)y}{(k+1)2x}$

$\frac{T_{k+1}}{T_k} \geqslant 1$ when $x=4 \quad y=5$

$\frac{(12-k)5}{(k+1)8} \geqslant 1$

$60 - 5k \geqslant 8k + 8$

$52 \geqslant 13k$ ✓

$k \leqslant 4$

if $k = 4 \quad \frac{T_5}{T_4} = 1$ so $T_4 = T_5$

if $k > 4 \quad \frac{T_6}{T_5} < 1$ ✓

if $k < 4 \quad \frac{T_4}{T_3} > 1$

Q6

c) i) $2\cos 3x \sin 4x + 2\cos 3x - \sin 4x - 1 = 0$

$2\cos 3x(\sin 4x + 1) - 1(\sin 4x + 1) = 0$ ✓

$(2\cos 3x - 1)(\sin 4x + 1) = 0$

$2\cos 3x = 1$

$\cos 3x = \frac{1}{2}$

$3x = 2n\pi \pm \cos^{-1}(\frac{1}{2})$

$= 2n\pi \pm \frac{\pi}{3}$

$x = \frac{2n\pi}{3} \pm \frac{\pi}{9}$ ✓

for some integer n

$\sin 4x = -1$

$4x = \sin^{-1}(-1) + 2n\pi$

$= -\frac{\pi}{2} + 2n\pi$ for some integer n

$x = -\frac{\pi}{8} + \frac{n\pi}{2}$

or ✓

$4x = (\pi - \sin^{-1}(-1)) + 2n\pi$

$= \frac{3\pi}{2} + 2n\pi$

$x = \frac{3\pi}{8} + \frac{n\pi}{2}$

Alternatively.

$4x = m\pi + (-1)^m \sin^{-1}(-1)$

$= m\pi + (-1)^m(-\frac{\pi}{2})$

$x = \frac{m\pi}{4} + (-1)^m(-\frac{\pi}{8})$

for some integer m.

ii) For $0 \leq x \leq \pi$

$x = \frac{2n\pi}{3} \pm \frac{\pi}{9} \quad n = 0, 1$

$= \frac{\pi}{9}, \frac{2\pi}{3} \pm \frac{\pi}{9}$

$= \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$

$x = \frac{m\pi}{4} + (-1)^n(-\frac{\pi}{8}) \quad (m = 0, 1, 2, \ldots)$

$= 0 - \frac{\pi}{8}, \frac{\pi}{4} + \frac{\pi}{8}, \frac{2\pi}{4} - \frac{\pi}{8}$

$\frac{3\pi}{4} + \frac{\pi}{8}, \pi - \frac{\pi}{8}$

$= \frac{3\pi}{8}, \cancel{\frac{3\pi}{8}}, \frac{7\pi}{8}$ ✓

Q7

a) $(1+x)^{2n} = (1+x)^n(1+x)^n$

LHS coefficient of x^n is $\binom{2n}{n}$

RHS $= (1+x)^n(1+x)^n$ ✓

$= \left[\binom{n}{0}x^0 + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{r}x^r + \dots + \binom{n}{n}x^n\right] \times$

$\left[\binom{n}{n}x^n + \binom{n}{n-1}x^{n-1} + \binom{n}{n-2}x^{n-2} + \dots + \binom{n}{n-r}x^{n-r} + \dots + \binom{n}{0}x^0\right]$

Coefficient of x^n on RHS

$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \dots + \binom{n}{n-r}\binom{n}{r} + \dots + \binom{n}{n}\binom{n}{0}$

$\binom{n}{0} = \binom{n}{n}$ $\binom{n}{1} = \binom{n}{n-1}$ and $\binom{n}{n-r}\binom{n}{r}$ for all $r \le n$

$\therefore$ Coefficients of x^n on RHS is $\sum_{k=0}^{n}\binom{n}{k}^2$ ✓

Equating coefficients of x^n on LHS & RHS

$$\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2$$

Q7

b) $x = Vt\cos\alpha$ $y = Vt\sin\alpha - 5t^2$

$t = \dfrac{x}{V\cos\alpha}$

So $y = \dfrac{\cancel{V}x\sin\alpha}{\cancel{V}\cos\alpha} - \dfrac{5x^2}{V^2\cos^2\alpha}$ ✓

$= x\tan\alpha - \dfrac{5x^2\sec^2\alpha}{V^2}$

when $x = p$, $y = h$

$h = p\tan\alpha - \dfrac{5p^2\sec^2\alpha}{V^2}$

$h - p\tan\alpha = \dfrac{-5p^2\sec^2\alpha}{V^2}$ ✓

$V^2 = \dfrac{-5p^2(1+\tan^2\alpha)}{h - p\tan\alpha}$

ii) Also

$V^2 = \dfrac{-5q^2(1+\tan^2\alpha)}{h - q\tan\alpha}$

$\dfrac{\cancel{-5}p^2(1+\tan^2\alpha)}{h - p\tan\alpha} = \dfrac{\cancel{-5}q^2(1+\tan^2\alpha)}{h - q\tan\alpha}$ ✓

$p^2\cancel{(1+\tan^2\alpha)}(h - q\tan\alpha) = q^2\cancel{(1+\tan^2\alpha)}(h - p\tan\alpha)$

$p^2h - p^2q\tan\alpha = q^2h - pq^2\tan\alpha$

$p^2h - q^2h = p^2q\tan\alpha - pq^2\tan\alpha$

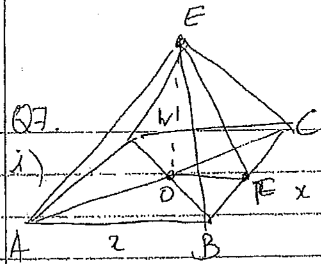
$h(p^2 - q^2) = pq\tan\alpha(p-q)$ ✓

$\tan\alpha = \dfrac{h\cancel{(p-q)}(p+q)}{pq\cancel{(p-q)}}$

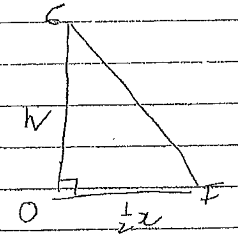
$= \dfrac{h(p+q)}{pq}$ as required.

Q7.

c) i)



Let the height of pyramid be h.

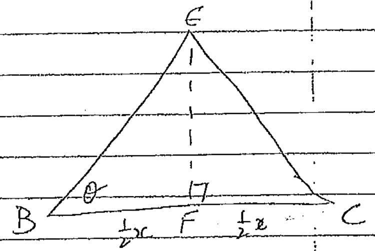


$V = \frac{1}{3}x^2 h$

$3V = x^2 h$

$h = \frac{3V}{x^2}$

$EF^2 = h^2 + \frac{1}{4}x^2$ ✓



Area of $\Delta EBC = \frac{1}{2} x \times EF$

$\frac{S}{4} = \frac{1}{2}x \times \sqrt{h^2 + \frac{x^2}{4}}$

$\frac{S^2}{16} = \frac{x^2}{4} \times \left(h^2 + \frac{x^2}{4}\right)$

$S^2 = 4x^2 \left[\left(\frac{3V}{x^2}\right)^2 + \frac{x^2}{4}\right]$

$= 4x^2 \times \left(\frac{9V^2}{x^4} + \frac{x^2}{4}\right)$ ✓

$= \frac{36V^2}{x^2} + x^4$ as required.

Q7

c ii) $S^2 = x^4 + 36V^2 x^{-2}$

$\frac{d(S^2)}{dx} = 4x^3 - 72V^2 x^{-3}$

$\frac{d(S^2)}{dx} = 0$ at stationary point.

$4x^3 - 72V^2 x^{-3} = 0$

$x^3 - 18V^2 x^{-3} = 0$

$x^6 = 18V^2$ ✓

$x^3 = \sqrt{18}\ V^2$

$= 3\sqrt{2}\ V$

$\frac{d^2 S^2}{dx^2} = 12x^2 + 216V^2 x^{-4}$

since $x^2 > 0$ $\frac{1}{x^4} > 0$ $V^2 > 0$

$\frac{d^2S^2}{dx^2} > 0$ for all x. ✓

$x^3 = 3\sqrt{2}\,V$ gives a minimum value for S.

iii) In ΔEBF $\tan\theta = \frac{EF}{\frac{1}{2}x}$

$\tan\theta = \frac{\sqrt{h^2 + \frac{x^2}{4}}}{\frac{1}{2}x}$

$h = \frac{3V}{x^2}$ and $V = \frac{x^3}{3\sqrt{2}}$

$h = \frac{\cancel{3}}{\cancel{x^2}} \times \frac{x^{\cancel{3}}}{\cancel{3}\sqrt{2}}$

$= \frac{x}{\sqrt{2}}$

$= \frac{\sqrt{\frac{x^2}{2} + \frac{x^2}{4}}}{\frac{1}{2}x}$ ✓

$= \sqrt{\frac{2x^2 + x^2}{4}} \div \frac{x}{2}$

$= \frac{\sqrt{3}\,\cancel{x}}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{x}}$ ✓

$= \sqrt{3}$

$\theta = 60^0$ $\therefore \Delta EBC$ is equilateral.