Student Number:_______________

St Catherine's School
Waverley

Preliminary Examination 2011

Mathematics Extension 1

General Instructions

- Working time – 2 hrs
- Write using black or blue pen.
- Board approved calculators may be used.
- Write all answers in answer booklets.
- Start a new page for each question.
- Show all appropriate working.

Booklet 1 – Question 1, 2 and 3
Booklet 2 – Question 4 and 5
Booklet 3 – Question 6 and 7

Total Marks - 84

Attempt all questions 1 – 7

Question 1	/12
Question 2	/12
Question 3	/12
Question 4	/12
Question 5	/12
Question 6	/12
Question 7	/12
Total	**/84**

Question 1 (12 marks) **Start a NEW PAGE in writing booklet** **Marks**

(a) Solve the inequality $\frac{x+1}{x-1} \le 3$ **3**

(b) Solve $3^k = 17$, giving the answer correct to three significant figures. **2**

(c) Find $\lim_{x\to\infty} \frac{2x^4+x^3-5x-1}{3x^3+x-4x^4-3}$ **1**

(d) Prove that $\frac{\cos\theta(\sin\theta+\cos\theta)}{(1+\sin\theta)(1-\sin\theta)} = 1+\tan\theta$ **2**

(e) Give a quick (but neat) sketch of the polynomial $P(x)=x^3(x-2)(x+2)^2$ **2**
You do not need to use calculus or find any stationary/turning points or points of inflexion

(f) Find the exact distance between the parallel lines $2x+3y+6=0$ and $2x+3y-6=0$ **2**

Question 2 (12 marks) **Start a NEW PAGE in writing booklet** **Marks**

(a) Find the value of k if $x-3$ is a factor of $P(x)=x^3-3kx+18$ **2**

(b) Find the remainder when the polynomial $P(x)=x^3-4x$ is divided by $2x-5$ **2**

(c) Find, to the nearest degree, the acute angle between the lines $3x-y+5=0$ and $y=-2x+1$. **2**

(d) Find the coordinates of the points on the curve $y=x^3+3x^2$ where the tangent is parallel to the line $9x-y-5=0$ **3**

(e) (i) Find the coordinates of the point P which divides the interval AB *externally* in the ratio 5:2 given that $A(-3,5)$ and $B(1,2)$ **2**

(ii) In what ratio does B divide the interval AP? **1**

Question 3 (12 marks) **Start a NEW PAGE in writing booklet** **Marks**

(a) Solve the equation $\sin\theta-\sin 2\theta=0$ for $0°\le\theta\le 360°$ **2**

(b) If If $\tan\frac{\theta}{2}=t$, prove $\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}=t$ **3**

(c) For the curves $f(x)=x^2-3x$ and $g(x)=x^2-9$ find:

(i) their point(s) of intersection **1**

(ii) The *obtuse* angle between the curves at their point of intersection **3**

(d) (i) Find the domain and range of the function $y=\sqrt{9-x^2}$ **2**

(ii) Hence draw a sketch of the curve **1**

Question 4 (12 marks) **Start a NEW PAGE in writing booklet** Marks

(a) The sum of the first 12 terms of an arithmetic series is 186 and the 20^{th} term is 83. **3**
Find the sum of the first 40 terms.

(b) Show that $\cos(A-B)-\cos(A+B)=2\sin A\sin B$ **2**

(c) How many terms of the series $24+8+\frac{8}{3}+.......$ are needed to give a sum of $\frac{320}{9}$? **2**

(d) (i) If $\sqrt{3}\cos\theta+\sin\theta\equiv R\cos(\theta-\alpha)$ where $R>0$ and $0^0\le\alpha\le 360^0$, **3**
find the values of R and α

(ii) Hence solve $\sqrt{3}\cos\theta+\sin\theta=\sqrt{3}$ for $0\le x\le 360^\circ$ **2**

Question 5 (12 marks) **Start a NEW PAGE in writing booklet** Marks

(a) The polynomial $P(x)=x^4-3x^3+ax^2+bx-6$ leaves a remainder of 8 when divided by $(x+1)$. It is also given that $(x-3)$ is a factor of $P(x)$ **3**
Find the values of a and b

(b) If α, β and γ are the roots of $2x^3+x^2-5x+7=0$ find

(i) $\alpha^{-1}+\beta^{-1}+\gamma^{-1}$ **2**

(ii) $\alpha^2+\beta^2+\gamma^2$ **2**

(c) Consider the polynomial $P(x)=x^3-x^2-10x-8$

(i) Express $P(x)$ as a product of three linear factors **3**

(ii) Hence sketch $P(x)$, clearly indicating all intercepts on the axes **1**

(iii) Solve by inspection of the graph where, $x^3-x^2-10x-8\ge 0$ **1**

Question 6 (12 marks) **Start a NEW PAGE in writing booklet** **Marks**

(a) For the curve $y = x^3 - 3x^2 - 9x + 4$, Find:

(i) the stationary points and determine their nature. 3

(ii) any point of inflection. 1

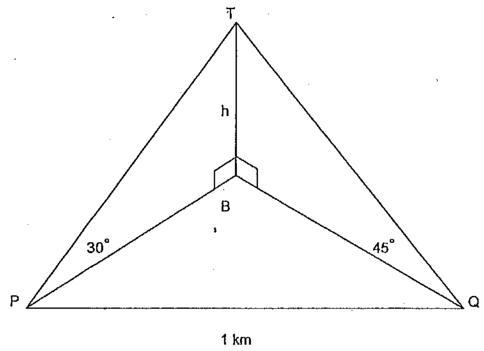
(iii) Sketch the curve for the domain $-2 \le x \le 4$. 2

(b) If $\tan A = \frac{1}{2}$ and $\tan B = \frac{2}{3}$ and A and B are acute angles, find the exact value of $\sin(A+B)$ 3

(c) Find the general solution to the equation $2\cos^2 x - \sqrt{3}\cos x = 0$. 3

Question 7 (12 marks) **Start a NEW PAGE in writing booklet booklet** **Marks**

(a) The angle of elevation from a boat at P to a point T at the top of a vertical cliff is measured to be $30°$. The boat sails 1 km to a second point Q, from which the angle of elevation to T is measured to be $45°$. Let B be the point at the base of the cliff directly below T and let $h = BT$ be the height of the cliff in metres.
The bearings of B from P and Q are $050°$ and $310°$ respectively.



(i) Show that $\angle PBQ = 100°$ 1

(ii) Show that $PB = h\cot 30°$ and $QB = h\cot 45°$ 1

(iii) Hence show that 2

$$h^2 = \frac{1000^2}{\cot^2 30° + \cot^2 45° - 2\cot 30° \cot 45° \cos 100°}$$

(iv) Find the height of the cliff, correct to the nearest metre. 1

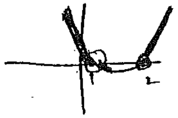
(b) Given that $f(x) = \frac{x}{4 - x^2}$:

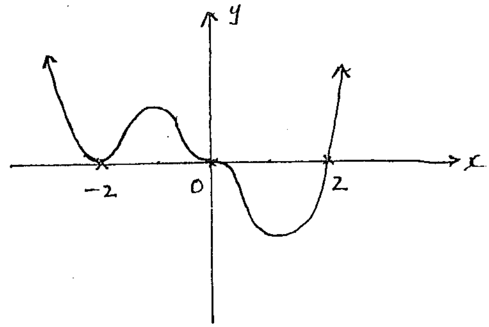
(i) Show that $f(x)$ is odd. 1

(ii) Show that $f(x)$ has no stationary points 2

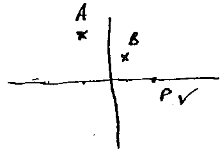
(iii) Find any horizontal or vertical asymptotes 2

(iv) Neatly sketch the curve 2

Qn	Solution	Marks	Comments: Criteria
Q1	a). $\frac{x+1}{x-1} \le 3 \qquad x \ne 1$ $(x-1)(x+1) \le 3(x-1)^2$ $x^2-1 \le 3x^2-6x+3$ $0 \le 2x^2-6x+4$ $0 \le x^2-3x+2$ $0 \le (x-2)(x-1)$  $\therefore x<1$ or $x \ge 2$	1 1 1	
	b). $3^k = 17$ $\log_{10} 3^k = \log_{10} 17$ $k \log_{10} 3 = \log_{10} 17$ $k = \frac{\log_{10} 17}{\log_{10} 3}$ $\therefore k = 2.58$ (3 sig figs)	1 1	
	c). $\lim\limits_{x \to \infty} \frac{2x^4+x^3-5x+1}{3x^3+x-4x^4-3}$ $= \lim \frac{2+\frac{1}{x}-\frac{5}{x^3}-\frac{1}{x^4}}{\frac{3}{x}+\frac{1}{x^3}-4-\frac{3}{x^4}}$ $= -\frac{1}{2}$	1	
	d). $LHS = \frac{\cos\theta(\sin\theta+\cos\theta)}{(1+\sin\theta)(1-\sin\theta)}$ $= \frac{\cos\theta(\sin\theta+\cos\theta)}{1-\sin^2\theta}$ $= \frac{\sin\theta+\cos\theta}{\cos\theta}$ $= \frac{\sin\theta}{\cos\theta} + 1$ $= 1+\tan\theta$ $= RHS$	1 1	

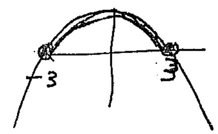
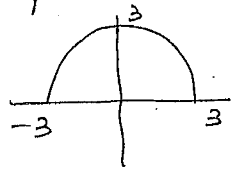
Qn	Solution	Marks	Comments: Criteria
1 (e)		2	
	f). $2x+3y+6=0$ —(1) $\quad 2x+3y-6=0$ —(2) $(0,-2)$ lies on 1 perp distance from $(0,-2)$ to (2) is $d = \left\lvert \frac{0-6-6}{\sqrt{4+9}} \right\rvert$ $= \left\lvert \frac{-12}{\sqrt{13}} \right\rvert$ $= \frac{12}{\sqrt{13}} \times \frac{\sqrt{13}}{\sqrt{13}}$ $= \frac{12\sqrt{13}}{13}$	1 1	

Qn	Solution	Marks	Comments: Criteria
2 a)	If $x-3$ is a factor of $P(x)$ then $P(3)=0$	1	
	$\therefore 3^3 - 9k + 18 = 0$ $45 = 9k$ $5 = k$	1	
b)	$P(x) = x^3 - 4x \quad \div 2x-5$ Now $P\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^3 - 4\left(\frac{5}{2}\right)$	1	
	$= \frac{125}{8} - 10$ $= 15\frac{5}{8} - 10$ $= 5\frac{5}{8}$	1	
	OR $x^3-4x = (2x-5)\left(\frac{x^2}{2} + \frac{5x}{4} + \frac{9}{8}\right) + \frac{45}{8}$ $-\frac{5x^2}{2} - \frac{25x}{4}$	1	
	$\therefore$ remainder $= \frac{45}{8}$ $= 5\frac{5}{8}$	1	
c).	$3x - y + 5 = 0 \qquad m = 3$ $y = -2x + 1 \qquad m = -2$ $\tan\theta = \left\lvert \frac{3-(-2)}{1+3(-2)} \right\rvert$	1	
	$= \left\lvert \frac{5}{-5} \right\rvert$ $= 1$ $\therefore \theta = 45^\circ$	1	
d).	$y = x^3 + 3x^2 \qquad 9x - y - 5 = 0$, $m = 9$	½	
	$y' = 3x^2 + 6x$	½	
	$\therefore 3x^2 + 6x = 9$ $x^2 + 2x - 3 = 0$	1	
	$(x+3)(x-1) = 0$ $x = -3, 1$ $\therefore y = 0, 4$ $\therefore$ points are $(-3,0)$ & $(1,4)$	1	

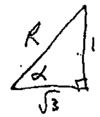
Qn	Solution	Marks	Comments: Criteria
2 e)	(i) $5 : -2$ $A(-3,5) \qquad B(1,2)$ XP	1 XP	
	$\therefore P\left(\frac{6+5}{3}, \frac{-10+10}{3}\right)$ $\therefore P\left(\frac{11}{3}, 0\right)$	1	
			
	(ii) $3 : 2$	1	

Qn	Solution	Marks	Comments: Criteria
Q3a)	$\sin\theta - \sin 2\theta = 0 \quad 0^\circ \le \theta \le 360^\circ$		
	$\sin\theta - 2\sin\theta\cos\theta = 0$		
	$\sin\theta(1-2\cos\theta) = 0$		
	$\therefore \sin\theta = 0$ or $\cos\theta = \frac{1}{2}$	1	
	$\therefore \theta = 0^\circ, 180^\circ, 360^\circ, 60^\circ, 300^\circ$	1	
b).	$t = \tan\frac{\theta}{2} \qquad \sin\theta = \frac{2t}{1+t^2}$	1	
	$\cos\theta = \frac{1-t^2}{1+t^2}$		
	$\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}$	1	
	$= \frac{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}$		
	$= \frac{1+t^2+2t-1+t^2}{1+t^2+2t+1-t^2} =$		
	$= \frac{2t^2+2t}{2+2t} =$	1	
	$= \frac{2t\cancel{(1+t)}}{2\cancel{(1+t)}} = t$		
c)	$f(x) = x^2-3x \qquad g(x) = x^2-9$		
	(i) for intersection $x^2-3x = x^2-9$		
	$x = 3$		
	$y = 0$		
	$\therefore$ point of intersection $(3,0)$	1	
	(ii) $f'(x) = 2x-3 \qquad g'(x) = 2x$	1	
	at $(3,0)$ $f'(x) = 3 = m_1 \qquad g'(x) = 6 = m_2$	1	
	now $\tan\theta = \frac{3-6}{1+18}$		
	$= -\frac{3}{19} \qquad \therefore \theta = \tan^{-1}\left(\frac{-3}{19}\right)$		
	$= \cancel{98^\circ 58'}$	1	

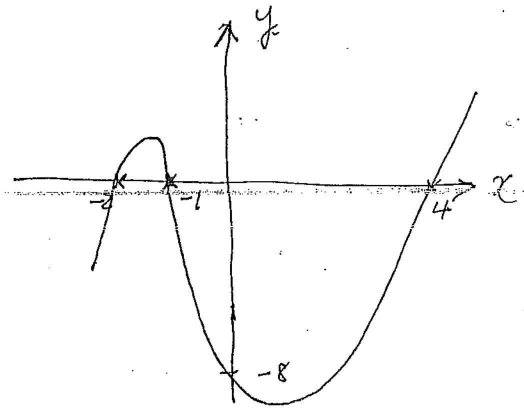
$171^\circ 2'$

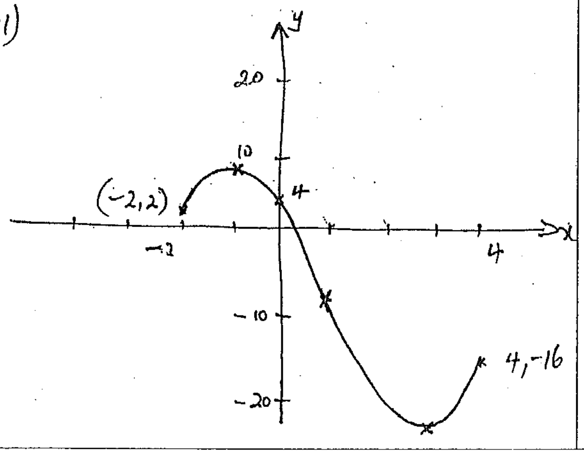
Qn	Solution	Marks	Comments: Criteria
Q3d).(i)	$y = \sqrt{9-x^2}$		
	$9-x^2 \ge 0$		
	$(3-x)(3+x) \ge 0$ 		
	D: $-3 \le x \le 3$	1	
	R: $0 \le y \le 3$	1	
	OR		
	recognise $y = \sqrt{9-x^2}$ as upper semi circle of circle $x^2+y^2=9$		
		1	
	Clearly D: $-3 \le x \le 3$		
	R: $0 \le y \le 3$		

Qn	Solution	Marks	Comments: Criteria
4a)	$S_n = \frac{n}{2}(2a+(n-1)d)$ $S_n = 186$ when $n=12$ $\therefore 6(2a+11d)=186$ $2a+11d = 31$ —— (1) $T_n = a+(n-1)d.$ $T_n = 83$ when $n=20$ $\therefore a+19d = 83$ —— (2) $2\times(2)-(1)$ $27d = 135$ $d = 5$ $\therefore a = -12$ Now $S_{40} = \frac{40}{2}[-24+39\times 5]$ $= 20[171]$ $= 3420$	1 1 1	
b)	$\cos(A-B) = \cos A\cos B + \sin A\sin B$ —— (1) $\cos(A+B) = \cos A\cos B - \sin A\sin B$ —— (2) (1)−(2) $= 2\sin A\sin B$ $\therefore \cos(A-B) - \cos(A+B) = 2\sin A\sin B$	1 1	
c)	$24+8+\frac{8}{3}+\dots$ G.P. $a=24$ $r=\frac{1}{3}$ $S_n = \frac{24[1-(\frac{1}{3})^n]}{1-\frac{1}{3}} = \frac{320}{9}$ $\therefore \frac{24(1-\frac{1}{3^n})}{\frac{2}{3}} = \frac{320}{9}$ $36(1-\frac{1}{3^n}) = \frac{320}{9}$ $1-\frac{1}{3^n} = \frac{320}{324}$ $\frac{1}{3^n} = \frac{4}{324} = \frac{1}{81}$ $\therefore n=4$	1 1	

Qn	Solution	Marks	Comments: Criteria
4d)	(i) $\sqrt{3}\cos\theta + \sin\theta = R\cos(\theta-\alpha)$ $\sqrt{3}\cos\theta + \sin\theta = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$ $\therefore R\cos\alpha = \sqrt{3}$ and $R\sin\alpha = 1$ $\cos\alpha = \frac{\sqrt{3}}{R}$ $\sin\alpha = \frac{1}{R}$  $\therefore \tan\alpha = \frac{1}{\sqrt{3}}$ $R^2 = 1+3$ $\alpha = 30°$ $\therefore R = 2.$ $\therefore \sqrt{3}\cos\theta + \sin\theta = 2\cos(\theta - 30°).$ (ii) $\sqrt{3}\cos\theta + \sin\theta = \sqrt{3}$ $\therefore 2\cos(\theta-30°) = \sqrt{3}$ $\cos(\theta-30°) = \frac{\sqrt{3}}{2}$ $\therefore \theta - 30° = \cos^{-1}\frac{\sqrt{3}}{2}$ $\theta - 30° = -30°, 30°, 330°$ $\theta = 0°, 60°, 360°$	1 1 1 1 1	

Qn	Solution	Marks	Comments: Criteria
5a)	$P(x) = x^4 - 3x^3 + ax^2 + bx - 6$		
	$P(-1) = 1 + 3 + a - b - 6 = 8$		
	$a - b = 10$ — ①	1	
	$P(3) = 81 - 81 + 9a + 3b - 6 = 0$		
	$9a + 3b = 6$		
	$3a + b = 2$ — ②	1	
	① + ② $\quad 2a = 13$		
	$4a = 12$		
	$a = 3$		
	$b = -7$	1	
b)	$2x^3 + x^2 - 5x + 7 = 0.$		
	(i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$	1	
	$= \frac{-5/2}{-7/2}$		
	$= \frac{5}{7}$	1	
	(ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$	1	
	$= \left(-\frac{1}{2}\right)^2 - 2\left(-\frac{5}{2}\right)$		
	$= \frac{1}{4} + 5$		
	$= 5\frac{1}{4}$	1	

Qn	Solution	Marks	Comments: Criteria
5a	$P(x) = x^3 - x^2 - 10x - 8$		
(i)	$P(-1) = 0$ $\therefore$ $x+1$ is a factor	1	
	now $x^3 - x^2 - 10x - 8 = (x+1)(x^2 - 2x - 8)$	1	
	$= (x+1)(x-4)(x+2)$	1	
(ii)		1	
	$f'(x) = 3x^2 - 2x - 10$		
	$f'(0) = -10$		
(iii)	$x^3 - x^2 - 10x - 8 \geq 0$ for	1	
	$-2 \leq x \leq -1$ or $x \geq 4$		

Qn	Solution	Marks	Comments: Criteria
Q6a)	$y = x^3 - 3x^2 - 9x + 4$		
(i)	$y' = 3x^2 - 6x - 9 = 0$ for st pts.		
	$x^2 - 2x - 3 = 0$		
	$(x-3)(x+1) = 0$		
	$x = 3, -1$		
	$y = -23, 9.$	1	
	$\therefore$ st pts at $(3,-23)$ and $(-1,9)$		
	$y'' = 6x - 6$		
	at $(3,-23)$ $y'' > 0$ $\therefore$ min turning pt.	1	
	at $(-1, 9)$ $y'' < 0$ $\therefore$ max turning pt.	1	
(ii)	$y'' = 6x - 6 = 0$ for possible inflexion		
	$x = 1$		
	$y = -7.$		
	possible inflexion at $(1,-7)$		
	for $x = 1 - \varepsilon$ $y'' < 0$		
	$x = 1 + \varepsilon$ $y'' > 0$		
	$\therefore$ inflexion at $(1,-7)$	1	
(iii)		2	$-\frac{1}{2}$ for each of end points y intercept inflexion turning points

Qn	Solution	Marks	Comments: Criteria
6b)	$\tan A = \frac{1}{2}$ $\therefore \sin A = \frac{1}{\sqrt{5}}$, $\cos A = \frac{2}{\sqrt{5}}$	1	
	$\tan B = \frac{2}{3}$ $\therefore \sin B = \frac{2}{\sqrt{13}}$, $\cos B = \frac{3}{\sqrt{13}}$	1	
	Now $\sin(A+B) = \sin A \cos B + \cos A \sin B$		
	$= \frac{1}{\sqrt{5}} \cdot \frac{3}{\sqrt{13}} + \frac{2}{\sqrt{5}} \cdot \frac{2}{\sqrt{13}}$		
	$= \frac{3}{\sqrt{65}} + \frac{4}{\sqrt{65}}$		
	$= \frac{7}{\sqrt{65}}$ or $\frac{7\sqrt{65}}{65}$	1	
c)	$2\cos^2 x - \sqrt{3}\cos x = 0$		
	$\cos x\,(2\cos x - \sqrt{3}) = 0$		
	$\cos x = 0, \frac{\sqrt{3}}{2}$	1	
	$x = 2n \times 180^\circ \pm 90^\circ$ $n = 0, \pm 1, \pm 2, \ldots$	1	
	OR $x = 2n \times 180^\circ \pm 30^\circ$ $n = 0, \pm 1, \pm 2, \ldots$	1	

Qn	Solution	Marks	Comments: Criteria

7 b) $f(x) = \dfrac{x}{4-x^2}$

(i) $f(-x) = \dfrac{-x}{4-(-x)^2}$

$= -\dfrac{x}{4-x^2}$

$= -f(x)$ — 1

$\therefore f(x)$ is odd.

(ii) $f'(x) = \dfrac{(4-x^2).1 - x.-2x}{(4-x^2)^2}$ — 1

$= \dfrac{4+x^2}{(4-x^2)^2} \neq 0$ for any x

$\therefore f(x)$ has no stationary points — 1

(iii) vertical asymptotes: $x = 2$ and $x = -2$ — 1

other asymptotes

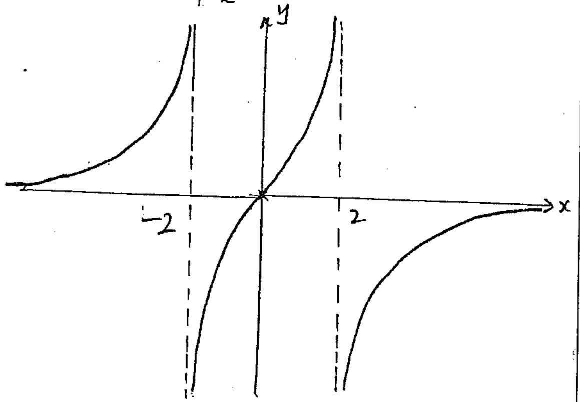
$\lim\limits_{x \to \pm\infty} \dfrac{x}{4-x^2} = 0$

as $4 - x^2 \to -\infty$ at a faster rate than $x \to \pm\infty$ — 1

$\therefore \lim\limits_{x \to \infty} \dfrac{x}{4-x^2} = 0^-$

$\lim\limits_{x \to -\infty} \dfrac{x}{4-x^2} = 0^+$

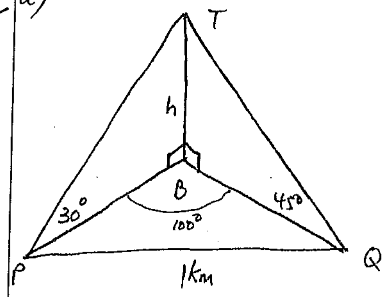
(iv)

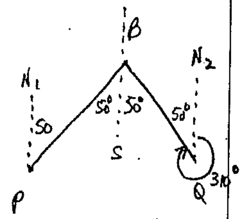


— 2

Qn	Solution	Marks	Comments: Criteria

Q7 a)





(i) $\angle N_1PB = \angle PBS = 50°$

$\angle N_2QB = \angle QBS = 50°$ — 1

Now $\angle PBQ = \angle PBS + \angle QBS$

$= 100°$

(ii) $PB = \dfrac{h}{\tan 30°}$ $\quad QB = \dfrac{h}{\tan 45°}$ — 1

$= h\cot 30°$ $\quad = h\cot 45°$

(iii) In $\triangle PBQ$:

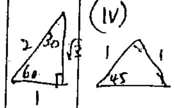
$1000^2 = \dfrac{h^2}{\tan^2 30} + \dfrac{h^2}{\tan^2 45} - 2.\dfrac{h}{\tan 30}.\dfrac{h}{\tan 45}\cos 100°$ — 1

$\therefore 1000^2 = h^2\cot^2 30° + h^2\cot^2 45 - 2h^2\cos 100°\cot 30°\cot 45°$

$1000^2 = h^2(\cot^2 30° + \cot^2 45° - 2\cot 30°\cot 45°\cos 100°)$

$\therefore h^2 = \dfrac{1000^2}{\cot^2 30° + \cot^2 45° - 2\cot 30°\cot 45°\cos 100°}$ — 1

(iv)



$h^2 = \dfrac{1000^2}{(3 + 1 - 2\sqrt{3}.\cos 100°)}$

$= \dfrac{1000^2}{4 - 2\sqrt{3}\cos 100°}$

$\therefore h = 466$ m (nearest metre) — 1