St George Girls High School

Year 11

End of Preliminary Course Examination

2011

Mathematics Extension 1

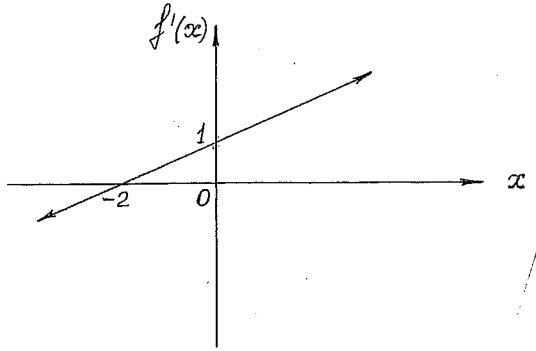
Time Allowed: 2 hours
(plus 5 minutes reading time)

Instructions

1. Attempt all 6 questions.
2. All necessary working must be shown.
3. Begin each question on a **new page**.
4. Marks will be deducted for careless work or poorly presented solutions.
5. All sketches are to be at least ⅓ page.

Question 1 (14 marks) – Start a New Page **Marks**

a)



The graph of the derivative function is shown above.

Consider the original function $y = f(x)$:

(i) State the values of x for which it is decreasing. 1

(ii) State the x-coordinate of the stationary point. 1

(iii) What is the nature of the above stationary point? 1

b) Classify the parabola $y = -2x^2 + 8x - 9$ as positive definite, negative definite or indefinite. Give reasons. 1

c) Find $\frac{dy}{dx}$ if $y = \frac{2-x}{(x^2+1)^2}$ 2

Question 1 (cont'd) **Marks**

d) The quadratic equation $2x^2 + x - 3 = 0$ has roots α and β. Without solving find the values of:

(i) $\alpha + \beta$ 1

(ii) $\alpha\beta$ 1

(iii) $\alpha^2 + \beta^2$ 2

e) Factor $a^4 - 81b^4$ 1

f) The point $P(2,-1)$ divides the interval AB externally in the ratio $2:5$. If A is the point $(3,-4)$, find the coordinates of point B. 3

Question 2 (14 marks) – Start a New Page **Marks**

a) (i) Graph the functions 2

$$y = |2x+1| \text{ and } y = |x-3|$$

on the same number plane and mark the two points of intersection of the graphs A and B.

(ii) Find the x-coordinates of A and B. 2

(iii) Hence state the solution of 1

$$|2x+1| \le |x-3|$$

b) If $f(x) = x\sqrt{x}$ find $f''(4)$ 3

c) For the parabola $y = 8x - x^2 - 17$ find:

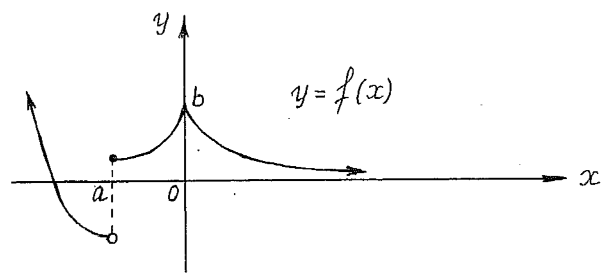
(i) the coordinates of the vertex 2

(ii) focal length 1

(iii) the focus 1

(iv) the directrix 1

d)



A function $f(x)$ is represented by the curve above. State any value(s) of x for which $f(x)$ is not differentiable. 1

Question 3 (14 marks) – Start a New Page **Marks**

a) $3x^2 + 7x + 4 \equiv A(x+2)^2 + B(x+2) + C$. 3

Find the values of A, B and C

b) (i) Write down the largest possible domain for which $f(x) = (x-2)^2 - 1$ has an inverse function. 1

(ii) Draw a careful sketch of $f(x)$ for this restricted domain. 1

(iii) Derive the inverse function $f^{-1}(x)$ for the domain stated above. 2

(iv) Sketch $f^{-1}(x)$ on the same number plane showing any points of intersection with $f(x)$ 1

c) Find the values of k for which the quadratic equation 3

$$x^2 - (k+1)x + (2k+7) = 0$$

has equal roots.

d) Solve for x: 3

$$3^{2x+1} - 12 \times 3^x + 9 = 0$$

Question 4 (14 marks) – Start a New Page **Marks**

a) Consider the parabola $x^2 = 4ay$

(i) Show that the equation of the tangent to $x^2 = 4ay$ at point $P(2ap, ap^2)$ is $y = px - ap^2$ 2

(ii) Show that the coordinates of point A, where the tangent crosses the line $y = -3a$ are $\left(\frac{ap^2-3a}{p}, -3a\right)$ 1

(iii) Find the coordinates of the midpoint M of the interval between A and the focus, S, of the parabola. 2

(iv) Find the equation of the locus of M. 1

b) Prove by mathematical induction that 4

$$2 + 5 + 9 + \cdots + (2^{n-1} + 2n - 1) = 2^n + n^2 - 1$$

for all integers $n \geq 1$

c) If $x = 2 + \sqrt{3}$

(i) Show that $\frac{1}{x} + x = 4$ 2

(ii) Hence evaluate $\frac{1}{x^2} + x^2$ 2

Question 5 (14 marks) – Start a New Page **Marks**

a) Solve the equation $2\cos^2 x - \sin x - 1 = 0$ for $0° \le x \le 360°$ 3

b) Consider the function $y = \frac{x^2}{x-1}$

(i) State its natural domain and therefore determine any vertical asymptotes 1

(ii) Show that $\frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$ and state any oblique asymptotes. 3

(iii) Find any stationary points and determine their nature. 3

(iv) Explain why the curve does not have any points of inflexion. 1

(v) Carefully sketch the curve showing all important features. 3

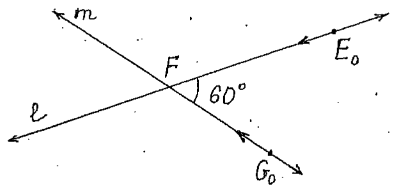
Question 6 (14 marks) – Start a New Page **Marks**

a) Solve for x: $\frac{3x}{x-4} \le 1$ 3

b) Find the Cartesian equation of the curve given by the following pair of parametric equations: 3

$$x = 1 + 2\tan\theta$$
$$y = 3\sec\theta - 4$$

c) Two streets l and m intersect at point F at an angle of $60°$, as shown.



Initially, Emma is at point E_0, 10 km from F and walks towards F at 5 km/h, Garry is at G_0, 8 km from F and walks towards F at 6 km/h towards F.

Let the corresponding positions of Emma and Garry after t hours be E and G. Let the distance between E and G be d.

(i) Copy and complete the diagram. 1

(ii) Show that after t hours $d = \sqrt{31t^2 - 96t + 84}$ 2

(iii) Find the time to the nearest minute when the distance between Emma and Gary is the shortest. 3

(iv) Find this distance to the nearest metre. 1

(v) Show that when they are at their closest distance, Garry will have passed the point F. 1

End of Paper

Question 1

(a) (i) $x < -2$

(ii) $x = -2$

(iii) minimum t.p.

(b) $y = -2x^2 + 8x - 9$

$a < 0$

$\Delta = 8^2 - 4 \times (-2) \times (-9)$

$= -8 < 0$

∴ Parabola is negative definite.

(c) $\frac{dy}{dx} = \frac{-1 \times (x^2+1)^2 - (2-x) \times 2(x^2+1) \times 2x}{(x^2+1)^4}$

$= \frac{(x^2+1)(-(x^2+1) - 4x(2-x))}{(x^2+1)^4}$

$= \frac{(-x^2 - 1 - 8x + 4x^2)}{(x^2+1)^3}$

$= \frac{3x^2 - 8x - 1}{(x^2+1)^3}$

(d) (i) $\alpha + \beta = -\frac{b}{a} = -\frac{1}{2}$

(ii) $\alpha\beta = \frac{c}{a} = \frac{3}{2}$

(iii) $\alpha^2 + \beta^2 = (\alpha^2 + 2\alpha\beta + \beta^2) - 2\alpha\beta$

$= (\alpha + \beta)^2 - 2\alpha\beta$

$= \frac{1}{4} - 2 \times \frac{3}{2} = \frac{1}{4} - 3$

$= -2\frac{3}{4}$

(e) $a^4 - 81b^4 = (a^4 - (3b)^4)$

$= (a - 3b)(a^3 + 3a^2b + 9ab^2 + 27b^3)$

(f) Externally $2:5$ → $-2:5$. Let B be (x_2, y_2)

$2 = \frac{5 \cdot 3 + (-2)x_2}{3}$

$15 - 2x_2 = 6$

$2x_2 = 9$

$x_2 = \frac{9}{2}$

$x = \frac{\ell x_1 + k x_2}{k + \ell}$

$y = \frac{\ell y_1 + k y_2}{k + \ell}$

$x = 2,\ y = -1$

$k = -2,\ \ell = 5$

$-1 = \frac{5 \times (-4) + (-2)y_2}{3}$

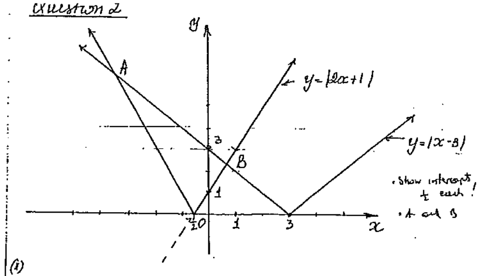
$-20 - 2y_2 = -3$

$2y_2 = -17$

$y_2 = -\frac{17}{2}$

∴ B has coordinates $\left(\frac{9}{2}, -\frac{17}{2}\right)$

Question 2



(a) (i) A is where $y = -2x - 1$ intersects with $y = -x + 3$

$-2x - 1 = -x + 3$

$x = -4$

B is where $y = 2x + 1$ intersects with $y = -x + 3$

$2x + 1 = -x + 3$

$3x = 2$

$x = \frac{2}{3}$

(ii) $-4 \le x \le \frac{2}{3}$

(b) $f(x) = x\sqrt{x} = x^{3/2}$

$f'(x) = \frac{3}{2} x^{\frac{1}{2}} = \frac{3\sqrt{x}}{2}$

$f''(x) = \frac{3}{4} x^{-\frac{1}{2}} = \frac{3}{4\sqrt{x}}$

$f''(4) = \frac{3}{4\sqrt{4}} = \frac{3}{8}$

(c) $y = 8x - x^2 - 17$

$= -x^2 + 8x - 17$

$= -(x^2 - 8x + 17)$

$= -(x^2 - 8x + 16) - 1$

$y + 1 = -(x-4)^2$

$(x-4)^2 = -(y+1)$

$(x-4)^2 = -4 \times \frac{1}{4}(y+1)$

(i) Vertex has coordinates $(4, -1)$

(ii) focal length is $\frac{1}{4}$

(iii) focus is below vertex ⇒ $(4, -1\frac{1}{4})$

(iv) directrix is above vertex ⇒ $y = -\frac{3}{4}$

(d) $x = a$ and $x = 0$

Question 3

(a) $3x^2 + 7x + 4 = A(x+2)^2 + B(x+2) + C$

$= A(x^2 + 4x + 4) + Bx + 2B + C$

$= Ax^2 + (4A + B)x + 4A + 2B + C$

Equating coefficients:

(1) $A = 3$

(2) $4A + B = 7$

$12 + B = 7$

$B = -5$

(3) $4A + 2B + C = 4$

$12 - 10 + C = 4$

$C = 4 - 12 + 10$

$C = 2$

(b) $f(x) = (x-2)^2 - 1$

Vertex is $(2, -1)$

Restricted domain $x \ge 2$, x is real.

Let $y = (x-2)^2 - 1$, $x \ge 2$ $(y \ge -1)$

Inverse:

$x = (y-2)^2 - 1$, $y \ge 2$, $x \ge -1$

$x + 1 = (y-2)^2$

$y - 2 = \pm\sqrt{x+1}$

$y = 2 + \sqrt{x+1}$, since $y \ge 2$

Domain of inverse is $x \ge -1$

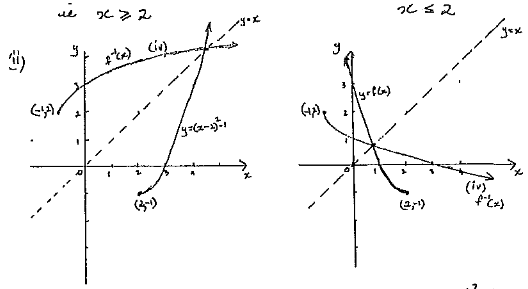
* Show intercepts.

Vertex $(2, -1)$

(i) Largest domain for which there is an inverse function is for when increasing OR for decreasing

ie $x \ge 2$ $x \le 2$

(ii)



(iii) $y = (x-2)^2 - 1$ ⇒ domain ⇔ range ⇒ $x = (y-2)^2 - 1$

change subject of relation

$x + 1 = (y-2)^2$

$\pm\sqrt{x+1} = y - 2$

Thus, since $x \ge 2$ for $y = f(x)$ then, $y \ge 2$ for $y = f^{-1}(x)$

Thus, $f^{-1}(x) = \sqrt{x+1} + 2$

$= 2 + \sqrt{x+1}$

Thus, since $x \le 2$ for $y = f(x)$ then, $y \le 2$ for $y = f^{-1}(x)$

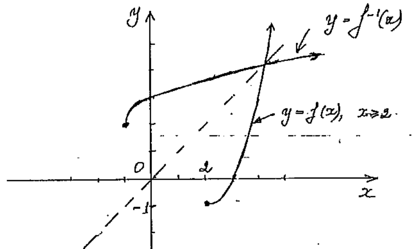
Thus, $f^{-1}(x) = -\sqrt{x+1} + 2$

$= 2 - \sqrt{x+1}$

(iv) above

Note: Question says to show points of intersection not find but if you want to!

$\begin{cases} y = x \\ y = (x-2)^2 - 1 \end{cases}$ will give the solution for each alternative



(c) $x^2 - (k+1)x + 2k + 7 = 0$

Equal roots = One double root

Let α be the root:

① $2\alpha = k + 1$

$\alpha = \frac{k+1}{2}$

② $\alpha^2 = 2k + 7$

$\left(\frac{k+1}{2}\right)^2 = 2k + 7$ (from ①)

$k^2 + 2k + 1 = 4(2k + 7)$

$k^2 + 2k + 1 = 8k + 28$

$k^2 - 6k - 27 = 0$

$(k-9)(k+3) = 0$

∴ $k = 9$ or $k = -3$

(d) $3^{2x+1} - 12 \times 3^x + 9 = 0$

$3 \times 3^{2x} - 12 \times 3^x + 9 = 0$

Let $3^x = y$

$3y^2 - 12y + 9 = 0$

$y^2 - 4y + 3 = 0$

$(y-3)(y-1) = 0$

$y = 3$ or $y = 1$

$3^x = 3$ or $3^x = 1$

$3^x = 3^1$ or $3^x = 3^0$

$x = 1$ or $x = 0$

Question 7.

(a) $x^2 = 4ay$

(i) $y = \frac{x^2}{4a}$ $\quad \frac{dy}{dx} = \frac{1}{4a} \times 2x = \frac{x}{2a}$

At $P(2ap, ap^2)$.

$\frac{dy}{dx} = \frac{2ap}{2a} = p$.

Eq. of tangent:

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$ as required.

(ii) tangent crosses $y = -3a$ when

$px - ap^2 = -3a$

$px = ap^2 - 3a$

$x = \frac{ap^2 - 3a}{p}$

and $y = -3a$ is a horizontal line, ∴ y-coordinate is $-3a$.

∴ point A has coordinates $\left(\frac{ap^2-3a}{p}, -3a\right)$

(iii) Focus has coordinates $(0, a)$

M has coordinates $x_m = \frac{\frac{ap^2-3a}{p} + 0}{2} = \frac{ap^2-3a}{2p}$

$y_m = \frac{-3a + a}{2} = -a$

$= 2 \times 2^k + k^2 + 2k + 2 - 2$

$= 2^{k+1} + k^2 + 2k$

$= RHS$

④. The statement is true for $n = k+1$ if it is true for $n = k$. Since it is true for $n = 1$, it will also be true for $n = 2, 3, \ldots$ and by the principle of mathematical induction it is true for all integers $n \geq 1$.

(c) If $x = 2 + \sqrt{3}$

$\frac{1}{x} + x = \frac{1}{2+\sqrt{3}} + 2 + \sqrt{3}$

$= \frac{1 + (2+\sqrt{3})^2}{2+\sqrt{3}}$

$= \frac{1 + 4 + 4\sqrt{3} + 3}{2+\sqrt{3}}$

$= \frac{4\sqrt{3} + 8}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}$

$= \frac{8\sqrt{3} - 12 + 16 - 8\sqrt{3}}{4-3}$

$= \frac{4}{1} = 4$, as required

(ii) $\left(\frac{1}{x} + x\right)^2 = \frac{1}{x^2} + 2\frac{1}{x} \times x + x^2$

$= \frac{1}{x^2} + 2 + x^2$

Question 5

(a) $2\cos^2 x - \sin x - 1 = 0$ $\quad 0° \leq x \leq 360°$.

$2(1 - \sin^2 x) - \sin x - 1 = 0$.

$2 - 2\sin^2 x - \sin x - 1 = 0$.

$-2\sin^2 x - \sin x + 1 = 0$.

(×−1)

$2\sin^2 x + \sin x - 1 = 0$.

$2\sin^2 x + 2\sin x - \sin x - 1 = 0$

$2\sin x(\sin x + 1) - 1(\sin x + 1) = 0$.

$(2\sin x - 1)(\sin x + 1) = 0$

$2\sin x = 1$ or $\sin x = -1$

$\sin x = \frac{1}{2}$ $\quad x = 270°$

$x = 30°, 180° - 30°$

$= 30°, 150°$

∴ $x = 30°, 150°, 270°$.

When $x = 0$ $\quad \frac{d^2y}{dx^2} = \frac{3}{-1} = -3$ ⇒ ∩

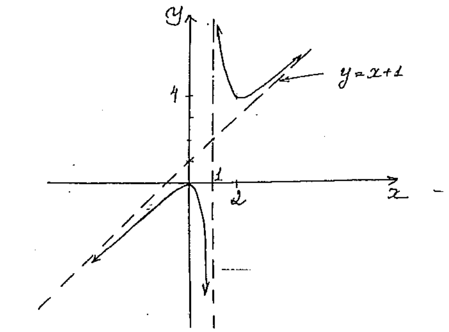
When $x = 0$ $y = 0$ there is a maximum turning point

When $x = 2$ $y = 4$

$\frac{d^2y}{dx^2} = \frac{3}{1} = 3 > 0$ ⇒ ∪,

there is a minimum turning point at $x = 2$ $y = 4$

(iv) $\frac{d^2y}{dx^2} = \frac{3}{(x-1)^3} \neq 0$, ∴ no inflexion points.



(iv) y-coordinate of point M does not depend on the x-coordinate.

∴ The locus of M is $y = -a$.

(b) ① For $n = 1$

$LHS = 2^{1-1} + 2 \times 1 - 1$

$= 1 + 2 - 1$

$= 2$

$RHS = 2^1 + 1^2 - 1$

$= 2 + 1 - 1$

$= 2$

$= LHS$

∴ The statement is true for $n = 1$

② Let $k \geq 1$ be an integer for which the statement is true, that is:

$2 + 5 + 9 + \ldots + (2^{k-1} + 2k - 1) = 2^k + k^2 - 1$

③ We are required to prove that then it will be true for $k+1$, that is

$2 + 5 + 9 + \ldots + (2^{k-1} + 2k - 1) + (2^k + 2(k+1) - 1) = 2^{(k+1)} + (k+1)^2 - 1$

or

$2 + 5 + 9 + \ldots + (2^{k-1} + 2k - 1) + (2^k + 2(k+1) - 1) = 2^{k+1} + k^2 + 2k$

$LHS = 2 + 5 + 9 + \ldots + (2^{k-1} + 2k - 1) + (2^k + 2(k+1) - 1)$

$= 2^k + k^2 - 1 + 2^k + 2(k+1) - 1$, by inductive hypothesis

But $\left(\frac{1}{x} + x\right)^2 = 4^2 = 16$

$\frac{1}{x^2} + 2 + x^2 = 16$

$\frac{1}{x^2} + x^2 = 16 - 2$

$\frac{1}{x^2} + x^2 = 14$

(b) (i) $x \neq 1$, x is real

Vertical asymptote $x = 1$.

(ii)

$= x + 1 + \frac{1}{x-1}$

$= \frac{(x+1)(x-1) + 1}{x-1}$

$= \frac{x^2 - 1 + 1}{x-1}$

$= \frac{x^2}{x-1} = LHS$

∴ Oblique asymptote is $y = x + 1$.

(iii) $\frac{dy}{dx} = \frac{2x(x-1) - x^2 \times 1}{(x-1)^2}$

$= \frac{2x^2 - 2x - x^2}{(x-1)^2}$

$= \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$

$\frac{dy}{dx} = 0$ when $x = 0$ or $x = 2$

$\frac{d^2y}{dx^2} = \frac{(x-1)^2(2x-2) - (x^2-2x) \times 2(x-1)}{(x-1)^4}$

$= \frac{(x-1)((x-1)(2x-2) - 2(x^2-2x))}{(x-1)^4}$

$= \frac{2x^2 - 2x - 2x + 2 - 2x^2 + 4x}{(x-1)^3}$

$= \frac{2}{(x-1)^3}$

Question 6

(a) $\frac{3x}{x-4} \times (x-4)^2 \leq 1 \times (x-4)^2$ $[x \neq 4]$

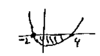
$3x(x-4) \leq (x-4)^2$ (−1)

$3x^2 - 12x - x^2 + 8x - 16 \leq 0$

$2x^2 - 4x - 16 \leq 0$

$x^2 - 2x - 8 \leq 0$ — ①

$(x-4)(x+2) \leq 0$



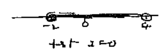
∴ $-2 \leq x < 4$ — ①

$\frac{3x}{x-4} - 1 \leq 0$ $\quad x \neq 4$

$\frac{3x - (x-4)}{x-4} \leq 0$

$\frac{2(x+2)}{x-4} \leq 0$

CV −2 and 4



test $x = 0$

$\frac{4}{-4} = -1 \leq 0$ ✓

$-2 \leq x < 4$

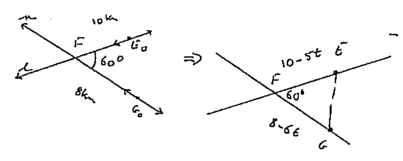
(b) $\frac{x-1}{2} = \tan\theta$ --- (I) — ①

$\frac{y+4}{3} = \sec\theta$ --- (II) — ①

Now $\tan^2\theta + 1 = \sec^2\theta$

∴ $\left(\frac{x-1}{2}\right)^2 + 1 = \left(\frac{y+4}{3}\right)^2$ — ①

(c) (i)



— ①

(ii) let $EG = d$

then $d^2 = (8-6t)^2 + (10-5t)^2 - 2(8-6t)(10-5t)\cos 60°$ — ①

$d^2 = [64 - 96t + 36t^2] + [100 - 100t + 25t^2] - [80 - 100t + 30t^2]$ — ①

$\ldots - 96t + 84$

$\frac{d}{dt} d = \frac{1}{2}(31t^2 - 96t + 84)^{-\frac{1}{2}} \cdot (62t - 96)$

$= \frac{62t - 96}{2\sqrt{31t^2 - 96t + 84}}$ — ①

Test for nature

t	0	$\frac{48}{31}$	2
$\frac{d}{dt}d$	−	0	+

−ve, +ve: Local min

let $\frac{d}{dt} d = 0$ then $62t - 96 = 0$

$t = \frac{48}{31}$

∴ Minimum distance at $t = \frac{48}{31}$ min

[1 hour 9 33 min]

iv) at $t = \frac{48}{31}$ then $d = \sqrt{31 \times (\frac{48}{31})^2 - 96 \times \frac{48}{31} + 84}$

$= 3111$ m

v) at $t = \frac{48}{31}$

Gary has travelled $6 \times \frac{48}{31}$ km $\doteq 9.29$ km — ①

which is greater than the initial 8km from F,

∴ Gary has travelled beyond F.