ACE Examinations

Student Name: ____________________

2014

YEAR 12

HALF YEARLY EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time - 5 minutes
- Working time - 1 hour
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 6-7

Total marks - 35

Section I
5 marks

- Attempt Questions 1-5
- Allow about 8 minutes for this section

Section II
30 marks

- Attempt Questions 6-7
- Allow about 52 minutes for this section

STUDENT NUMBER/NAME: ....

STANDARD INTEGRALS

$\int x^n dx$	$= \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$
$\int \frac{1}{x} dx$	$= \ln x, \quad x > 0$
$\int e^{ax} dx$	$= \frac{1}{a}e^{ax}, \quad a \neq 0$
$\int \cos ax \, dx$	$= \frac{1}{a}\sin ax, \quad a \neq 0$
$\int \sin ax \, dx$	$= -\frac{1}{a}\cos ax, \quad a \neq 0$
$\int \sec^2 ax \, dx$	$= \frac{1}{a}\tan ax, \quad a \neq 0$
$\int \sec ax \tan ax \, dx$	$= \frac{1}{a}\sec ax, \quad a \neq 0$
$\int \frac{1}{a^2+x^2} dx$	$= \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$
$\int \frac{1}{\sqrt{a^2-x^2}} dx$	$= \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$
$\int \frac{1}{\sqrt{x^2-a^2}} dx$	$= \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$
$\int \frac{1}{\sqrt{x^2+a^2}} dx$	$= \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE: $\ln x = \log_e x, \quad x > 0$

Section I

5 marks
Attempt Questions 1 - 5
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1-5

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1 Which of the following is an expression for $\int x\sqrt{4-x^2}\,dx$?

Use the substitution $u=\sqrt{4-x^2}$.

(A) $-\dfrac{\left(4-x^2\right)^3}{3}+C$

(B) $\dfrac{\left(4-x^2\right)^3}{3}+C$

(C) $-\dfrac{\left(4-x^2\right)^{\frac{3}{2}}}{3}+C$

(D) $\dfrac{\left(4-x^2\right)^{\frac{3}{2}}}{3}+C$

2 A piece of hot aluminium is placed in a room with a surrounding air temperature of 15°C and allowed to cool. It loses heat according to Newton's law of cooling, $\dfrac{dT}{dt}=-k(T-A)$ where T is the temperature of the aluminium in degrees Celsius at time t minutes, A is the surrounding air temperature and k is a positive constant. After 5 minutes the temperature of the aluminium is 75°C, and after a further 3 minutes it is 45°C. What is the value of k in the above equation?

(A) $k=3\log_e 0.5$

(B) $k=3\log_e 2$

(C) $k=\dfrac{\log_e 0.5}{3}$

(D) $k=\dfrac{\log_e 2}{3}$

3 What is the exact value of the definite integral $\int_{\frac{\pi}{2}}^{\pi} 2(\sin^2 x+x)\,dx$?

(A) $\dfrac{3\pi^2+\pi+2}{4}$

(B) $\dfrac{3\pi^2+\pi}{4}$

(C) $\dfrac{3\pi^2+2\pi+2}{4}$

(D) $\dfrac{3\pi^2+2\pi}{4}$

4 The function $f(x)=\log_e x-\sin x$ has a real root between 2 and 3.
Let $x=2$ be a first approximation to the root.
What is the second approximation to the root using Newton's method?

(A) 2.235

(B) 2.236

(C) 2.663

(D) 2.664

5 What is the indefinite integral for $\int\left(\cos^2 x+2\sec^2 x\right)dx$?

(A) $\dfrac{1}{2}x+\dfrac{1}{4}\sin 2x+\tan x+C$

(B) $\dfrac{1}{2}x-\dfrac{1}{4}\sin 2x+\tan x+C$

(C) $\dfrac{1}{2}x+\dfrac{1}{4}\sin 2x+2\tan x+C$

(D) $\dfrac{1}{2}x-\dfrac{1}{4}\sin 2x+2\tan x+C$

Section II

30 marks
Attempt Questions 6' 7
Allow about 52 minutes for this section

Answer each question in the appropriate writing booklet.

All necessary working should be shown in every question.

Question 6 (15 marks) **Marks**

(a) Use the substitution $u = x+1$ to evaluate $\int_0^3 \frac{x-2}{\sqrt{x+1}}dx$. **3**

(b) Use the principle of mathematical induction to prove that for all positive integers n: **3**

$$1+5+9+...+4n-3=2n^2-n$$

(c) Evaluate $\int_0^{\frac{\pi}{2}} \sin^2 3x dx$. **2**

(d) Solve the equation $2e^{-x}-x=0$, correct to two decimal places, given $x=0.8$ is an approximate root. **2**

(e) (i) Prove $\frac{\sec^2 x}{\tan x}=\frac{\operatorname{cosec} x}{\cos x}$. **1**

(ii) Use the substitution $u=\tan x$ to find the exact value of this integral **2**

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos ecx}{\cos x}dx$$

(f) Use the substitution $u=\log_e x$ to evaluate $\int \frac{1}{x(\log_e x)^2}dx$. **2**

Question 7 (15 marks) **Marks**

(a) Let 3.5 be a first approximation to the root of the equation $x^3=50$. Apply Newton's method once to obtain a better approximation to the root. Answer correct to 3 significant figures. **2**

(b) Prove by mathematical induction that n^3+2n is divisible by 3 for all positive integers n. **3**

(c) Find the exact value of $\int_{-1}^{1}\sqrt{4-x^2}dx$, using the substitution $x=2\sin\theta$. **3**

(d) Newton's law of cooling states that when an object at temperature T°C is placed in an environment at temperature T_0°C, the rate of the temperature loss is given by the equation

$$\frac{dT}{dt}=-k(T-T_0)$$

where t is the time in minutes and k is a positive constant.

(i) Verify that $T=T_0+Ae^{-kt}$ satisfies the above equation. **1**

(ii) A cup of hot tea is cooled from 100°C to 70°C after 10 minutes in a room where the temperature is 20°C. How much longer would it take for the cup of tea to cool down to 40°C? **3**

(e) Use the principle of mathematical induction to prove that for all positive integers n: **3**

$$4^n > 2n+1$$

End of paper

ACE Examination 2014

HSC Mathematics Extension 1 Half Yearly Examination

Worked solutions and marking guidelines

Section I

	Solution	Criteria
1	$u=\sqrt{4-x^2}$ $\frac{du}{dx}=\frac{1}{2}(4-x^2)^{-\frac{1}{2}}-2x$ $du=\frac{-x}{(4-x^2)^{\frac{1}{2}}}dx$ $-(4-x^2)^{\frac{1}{2}}du=xdx$ or $-udu=xdx$ $\int x\sqrt{4-x^2}\,dx=\int u\times -udu$ $=\int -u^2du$ $=-\frac{u^3}{3}+C=-\frac{(4-x^2)^{\frac{3}{2}}}{3}+C$	1 Mark: C
2	$T=A+Be^{-kt}$ satisfies the equation $\frac{dT}{dt}=-k(T-A)$ $A=15$ (surrounding air temperature). Also $t=5$ and $T=75$ $T=15+Be^{-kt}$ $75=15+Be^{-k\times 5}$ $Be^{-5k}=60$ ....(1) Also $t=8$ and $T=45$ $T=15+Be^{-kt}$ $45=15+Be^{-k\times 8}$ $Be^{-8k}=30$..........(2) Eqn (1) ÷ (2) $\frac{Be^{-5k}}{Be^{-8k}}=\frac{60}{30}$ $e^{3k}=2$ $3k=\log_e 2$ $k=\frac{\log_e 2}{3}$	1 Mark: D

	Solution	Criteria
3	$\int_{\frac{\pi}{2}}^{\pi}2(\sin^2 x+x)dx=2\int_{\frac{\pi}{2}}^{\pi}[\frac{1}{2}(1-\cos 2x)+x]dx$ $=2\left[\frac{1}{2}x-\frac{1}{4}\sin 2x+\frac{x^2}{2}\right]_{\frac{\pi}{2}}^{\pi}$ $=2\left(\frac{\pi}{2}-\frac{\sin 2\pi}{4}+\frac{\pi^2}{2}\right)-2\left(\frac{\pi}{4}-\frac{\sin\pi}{4}+\frac{\pi^2}{8}\right)$ $=2\left[\frac{\pi}{2}+\frac{\pi^2}{2}-\frac{\pi}{4}-\frac{\pi^2}{8}\right]$ $=\frac{3\pi^2}{4}+\frac{\pi}{2}=\frac{3\pi^2+2\pi}{4}$	1 Mark: D
4	$f(x)=\log_e x-\sin x$ $f'(x)=\frac{1}{x}-\cos x$ $x_1=x_0-\frac{f(x_0)}{f'(x_0)}$ $=2-\frac{\log_e 2-\sin 2}{\frac{1}{2}-\cos 2}$ $=2.23593406...\approx 2.236$	1 Mark: B
5	$\int\left(\cos^2 x+2\sec^2 x\right)dx=\int\left(\frac{1}{2}(1+\cos 2x)+2\sec^2 x\right)dx$ $=\frac{1}{2}x+\frac{1}{4}\sin 2x+2\tan x+C$	1 Mark: C

Section II

6(a)	$u=x+1$ $x=3$ then $u=4$ $du=dx$ $x=0$ then $u=1$ $\int_0^3\frac{x-2}{\sqrt{x+1}}dx=\int_1^4\frac{u-3}{\sqrt{u}}du$ $=\int_1^4(u^{\frac{1}{2}}-3u^{-\frac{1}{2}})du$ $=\left[\frac{2u^{\frac{3}{2}}}{3}-6u^{\frac{1}{2}}\right]_1^4$ $=\frac{2}{3}(8-1)-6(2-1)$ $=-\frac{4}{3}$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution 1 Mark: Calculates $\frac{du}{dx}$ and changes the limits.

6(b)	Step 1: To prove the statement true for $n=1$ $\text{LHS}=1 \quad \text{RHS}=2(1)^2-1=1$ Result is true for $n=1$ Step 2: Assume the result true for $n=k$ $1+5+9+...+4k-3=2k^2-k$ To prove the result is true for $n=k+1$ $1+5+9+...+4k-3+4(k+1)-3=2(k+1)^2-(k+1)$ $\text{LHS}=1+5+9+...+4k-3+4(k+1)-3$ $=2k^2-k+4(k+1)-3$ $=2k^2+4k+2-k-1$ $=2(k^2+2k+1)-(k+1)$ $=2(k+1)^2-(k+1)$ $=\text{RHS}$ Result is true for $n=k+1$ if true for $n=k$ Step 3: Result true by principle of mathematical induction.	3 Marks: Correct answer. 2 Marks: Proves the result true for $n=1$ and attempts to use the result of $n=k$ to prove the result for $n=k+1$. 1 Mark: Proves the result true for $n=1$.
6(c)	$\int_0^{\frac{\pi}{2}}\sin^2 3x\,dx=\int_0^{\frac{\pi}{2}}\frac{1}{2}(1-\cos 6x)dx$ $=\frac{1}{2}\left[x-\frac{1}{6}\sin 6x\right]_0^{\frac{\pi}{2}}$ $=\frac{1}{2}\left[\left(\frac{\pi}{2}-\frac{1}{6}\sin 3\pi\right)-\left(0-\frac{1}{6}\sin 0\right)\right]$ $=\frac{\pi}{4}$	2 Marks: Correct answer. 1 Mark: Applies double angle trig identity.
6(d)	$f(x)=2e^{-x}-x$ and $f'(x)=-2e^{-x}-1$ $x_1=x_0-\frac{f(x_0)}{f'(x_0)}$ $=0.8-\frac{2e^{-0.8}-0.8}{-2e^{-0.8}-1}$ $=0.85196192...$ ≈ 0.85	2 Marks: Correct answer. 1 Mark: Shows some understanding of Newton's method

6(e) (i)	$\text{LHS}=\frac{\sec^2 x}{\tan x}$ $=\frac{1}{\cos^2 x}\div\tan x$ $=\frac{1}{\cos^2 x}\times\frac{\cos x}{\sin x}$ $=\frac{1}{\cos x}\times\text{cosec}\,x$ $=\frac{\text{cosec}\,x}{\cos x}=\text{RHS}$	1 Mark: Correct answer.
6(e) (ii)	$u=\tan x \qquad u=\tan\frac{\pi}{3}=\sqrt{3} \qquad u=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$ $du=\sec^2 x dx$ $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\cos ecx}{\cos x}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sec^2 x}{\tan x}dx$ $=\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\frac{1}{u}du$ $=\left[\log_e u\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$ $=\log_e\sqrt{3}-\log_e\frac{1}{\sqrt{3}}$ $=\log_e 3$	2 Marks: Correct answer. 1 Mark: Recognises the use of part (i) or makes progress in the substitution.
6(f)	$u=\log_e x$ and $du=\frac{1}{x}dx$ $\int\frac{1}{x(\log_e x)^2}dx=\int\frac{1}{u^2}du$ $=-u^{-1}+C$ $=-\frac{1}{\log_e x}+C$	2 Marks: Correct answer. 1 Mark: Sets up the integration and shows some understanding.
7(a)	$f(x)=x^3-50$ and $f'(x)=3x^2$ $x_1=x_0-\frac{f(x_0)}{f'(x_0)}$ $=3.5-\frac{3.5^3-50}{3\times 3.5^2}$ $=3.693877551...$ ≈ 3.69	2 Marks: Correct answer. 1 Mark: Shows some understanding of Newton's method

7(b)	Step 1: To prove the statement true for $n=1$ $1^3+2\times1=3$ (Divisible by 3) Result is true for $n=1$ Step 2: Assume the result true for $n=k$ $k^3+2k=3P$ where P is an integer (1) To prove the result is true for $n=k+1$ $(k+1)^3+2(k+1)=3Q$ where Q is an integer. $\text{LHS}=(k+1)^3+2(k+1)$ $=k^3+3k^2+3k+1+2k+2$ $=3k^2+3k+3+k^3+2k$ $=3(k^2+k+1)+(k^3+2k)$ $=3(k^2+k+1)+(3P)$ from (1) $=3\times(k^2+k+1+P)$ $=3Q$ $=$ RHS Q is an integer as P and k are integers. Result is true for $n=k+1$ if true for $n=k$ Step 3: Result true by principle of mathematical induction.	3 Marks: Correct answer. 2 Marks: Proves the result true for $n=1$ and attempts to use the result of $n=k$ to prove the result for $n=k+1$. 1 Mark: Proves the result true for $n=1$.
7(c)	$x=2\sin\theta,\quad \frac{dx}{d\theta}=2\cos\theta,\quad dx=2\cos\theta d\theta$ When $x=1$ $\quad 1=2\sin\theta$ $\sin\theta=\frac{1}{2}$ or $\theta=\frac{\pi}{6}$ When $x=-1$ $\quad -1=2\sin\theta$ $\sin\theta=-\frac{1}{2}$ or $\theta=-\frac{\pi}{6}$ $\int_{-1}^{1}\sqrt{4-x^2}dx=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\sqrt{4-(2\sin\theta)^2}\times2\cos\theta d\theta$ $=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}2\sqrt{1-\sin^2\theta}\times2\cos\theta d\theta$ $=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}2\cos\theta\times2\cos\theta d\theta$ $=4\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\cos^2\theta d\theta$ $=4\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\frac{1}{2}(1+\cos2\theta)d\theta$ $=2\left[x+\frac{1}{2}\sin2\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$ $=2\left[\left(\frac{\pi}{6}+\frac{1}{2}\sin\frac{\pi}{3}\right)-\left(-\frac{\pi}{6}+\frac{1}{2}\sin-\frac{\pi}{3}\right)\right]$ $=2\left[\frac{\pi}{6}+\frac{1}{2}\times\frac{\sqrt3}{2}+\frac{\pi}{6}-\frac{1}{2}\times-\frac{\sqrt3}{2}\right]$ $=2\left[\frac{\pi}{3}+\frac{\sqrt3}{2}\right]$ $=\frac{2\pi}{3}+\sqrt3$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution 1 Mark: Sets up the integration by finding dx and adjusting the limits.
7(d) (i)	$T=T_0+Ae^{-kt}$ or $Ae^{-kt}=T-T_0$ $\frac{dT}{dt}=-kAe^{-kt}$ $=-k(T-T_0)$	1 Mark: Correct answer.
7(d) (ii)	Initially $t=0$ and $T=100$, $T_0=20$ $T=T_0+Ae^{-kt}$ $100=20+Ae^{-k\times0}$ $A=80$ Also $t=10$ and $T=70$ $70=20+80e^{-k\times10}$ $e^{-10k}=\frac{50}{80}=\frac{5}{8}$ $-10k=\log_e\frac{5}{8}$ $k=-\frac{1}{10}\log_e\frac{5}{8}$ $=\frac{1}{10}\log_e\frac{8}{5}$ $=0.04700036...$	3 Marks: Correct answer. 2 Marks: Finds the value of A and an expression for k. 1 Mark: Finds the value of A.

HSC Mathematics Extension 1

	We need to find t when $T = 40$ $40 = 20 + 80e^{-kt}$ $e^{-kt} = \frac{20}{80} = \frac{1}{4}$ $-kt = \log_e \frac{1}{4}$ $t = -\frac{1}{k}\log_e 0.25$ $= -10\frac{\log_e 0.25}{\log_e \frac{8}{5}}$ $= 29.49539695..$ ≈ 29.5 min Time taken to cool down from 70°C to 40°C $= 29.5 - 10$ $= 19.5$ min	
7(e)	Step 1: To prove the statement true for $n = 1$ $4^1 > 2\times 1 + 1$ Result is true for $n = 1$ Step 2: Assume the result true for $n = k$ $4^k > 2k + 1$ To prove the result is true for $n = k + 1$ $4^{k+1} > 2(k+1) + 1$ $> 2k + 3$ $\text{LHS} = 4^{k+1}$ $= 4\times 4^k$ $> 4\times(2k+1)$ from (1) $> 8k + 4$ $> 2k + 3$ (k is a positive integer) $=$ RHS Result is true for $n = k + 1$ if true for $n = k$ Step 3: Result true by principle of mathematical induction.	3 Marks: Correct answer. 2 Marks: Proves the result true for $n = 1$ and attempts to use the result of $n = k$ to prove the result for $n = k + 1$. 1 Mark: Proves the result true for $n = 1$.