ACE Examinations

Student Name: ______________________

2014

YEAR 12

YEARLY EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time - 5 minutes
- Working time - 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11-16

Total marks - 100

Section I
10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II
90 marks

- Attempt Questions 11-16
- Allow about 2 hour 45 minutes for this section

STUDENT NUMBER/NAME: ....

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Section I

10 marks
Attempt Questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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1 Which of the following is an expression for $\int \frac{1}{1+\sin x+\cos x}dx$?

Use the substitution $t=\tan\frac{x}{2}$.

(A) $\ln|t-1|+C$

(B) $\ln|t+1|+C$

(C) $\ln|t^2-1|+C$

(D) $\ln|t^2+1|+C$

2 What is the modulus and argument of $-1+i$?

(A) Modulus $\sqrt{2}$ and argument $\frac{\pi}{4}$

(B) Modulus $\sqrt{2}$ and argument $\frac{3\pi}{4}$

(C) Modulus 2 and argument $\frac{\pi}{4}$

(D) Modulus 2 and argument $\frac{3\pi}{4}$

3 Consider the equation $z^2+kz+(2-i)=0$. What is the complex number k, given that i is a root of the equation.

(A) $-1+i$

(B) $-1-i$

(C) $1+i$

(D) $1-i$

4 Which of the following polynomials is monic, of degree 3, with 5 as a single zero and -2 as a zero of multiplicity 2?

(A) $P(x)=x^3-2x^2+5x$

(B) $P(x)=x^3+x^2-16x+20$

(C) $P(x)=x^3-x^2-16x-20$

(D) $P(x)=x^3-8x^2+5x+50$

5 Use the substitution $u=\sin x$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}}dx$.

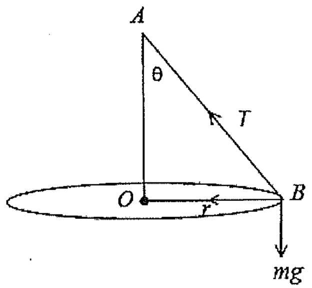
(A) $\ln(1+\sqrt{2})$

(B) $\ln(1+\sqrt{3})$

(C) $\frac{\pi}{2}$

(D) π

6 A particle at B is attached to a string AB that is fixed at A. The particle rotates in a horizontal circle with a radius of r. Let T be the tension in the string and $\angle BOA=\theta$.



Which of the following statements is correct?

(A) $T\cos\theta-mg=ma$

(B) $T\sin\theta=mr\omega^2$

(C) $T=-mg$

(D) $T-mg=ma$

7 What is the eccentricity for the hyperbola $\frac{y^2}{225}-\frac{x^2}{64}=1$?

(A) $\frac{8}{17}$

(B) $\frac{15}{17}$

(C) $\frac{17}{15}$

(D) $\frac{17}{8}$

8 What is the value of $\frac{z_1}{z_2}$ when $z_1=4+i$ and $z_2=2+3i$?

(A) $\frac{-11+10i}{5}$

(B) $\frac{5-10i}{5}$

(C) $\frac{5-10i}{13}$

(D) $\frac{11-10i}{13}$

9 Which of the following is the expression for $\int \sin^3 x\,dx$?

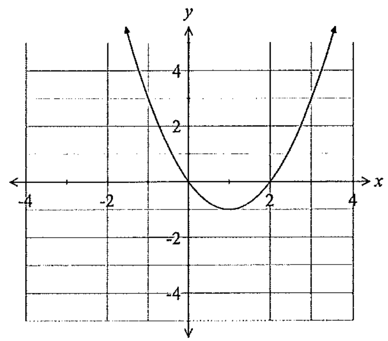
(A) $\frac{1}{3}\cos^3 x-\cos x+C$

(B) $\frac{1}{3}\cos^3 x+\cos x+C$

(C) $\frac{1}{3}\sin^3 x-\sin x+C$

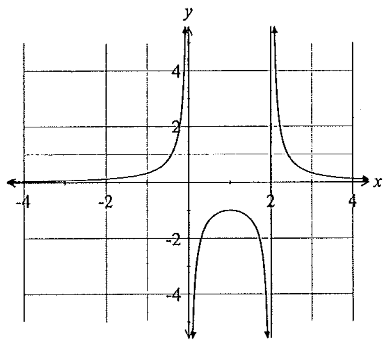
(D) $\frac{1}{3}\sin^3 x+\sin x+C$

10 The diagram below shows the graph of the function $y=f(x)$.

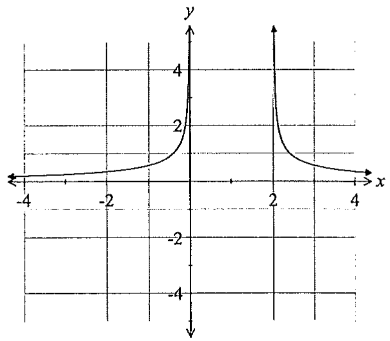


Which of the following is the graph of $y=\frac{1}{f(x)}$?

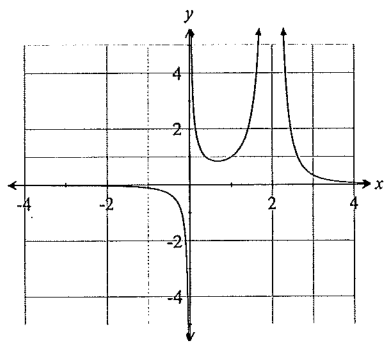
(A)



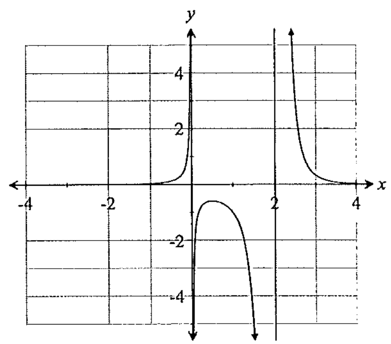
(B)



(C)



(D)



Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet.

All necessary working should be shown in every question.

Question 11 (15 marks) **Marks**

(a) Find all pairs of integers x and y that satisfy $(x+iy)^2 = 33+56i$. **2**

(b) On an Argand diagram mark the points P, Q representing the complex numbers $z_1 = 4+i$ and $z_2 = 1+4i$ respectively.

(i) Show how to construct the point R representing $z_1 + z_2$. **2**

(ii) What type of quadrilateral is $OPQR$ where O is the origin? Explain. **1**

(iii) Find the area of the quadrilateral is $OPQR$. **1**

(c) (i) Find real numbers a, b and c such that **2**

$$\frac{x}{(x^2+2)(x-3)} = \frac{ax+b}{x^2+2} + \frac{c}{x-3}$$

(ii) Hence evaluate in simplest form $\int \frac{x}{(x^2+2)(x-3)}dx$. **2**

(d) Derive the equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P(x_1, y_1)$. **3**

(e) The complex number z lies on the unit circle and $0 \le \arg z \le \frac{\pi}{2}$. **2**

Prove that $2\arg(z+1) = \arg z$.

Question 12 (15 marks) **Marks**

(a) The polynomial equation $x^3 + x^2 - x - 4 = 0$ has roots α, β and γ.

(i) Find the equation with 2α, 2β and 2γ as the roots. **2**

(ii) Find the equation with $\alpha - 2$, $\beta - 2$ and $\gamma - 2$ as the roots. **2**

(b) Tangents to the ellipse $16x^2 + 25y^2 = 400$ at $P(5\cos\alpha, 4\sin\alpha)$ and $Q(\cos\beta, 4\sin\beta)$ are at right angles to each other.

(i) Show that the gradient of the tangent at P is $-\frac{4\cos\alpha}{5\sin\alpha}$. **2**

(ii) Hence show that $25\tan\alpha\tan\beta = -16$. **2**

(c) A particle of mass m moves in a horizontal straight line. The particle is resisted by a constant force mk and a variable force mv^2, where k is a positive constant and v is the speed. Initially $v = u$ and $x = 0$.

(i) Show that the distance travelled is $-\frac{1}{2}\ln\left(\frac{k+v^2}{k+u^2}\right)$. **2**

(ii) Show that the time taken for the particle to be brought to rest is **3**

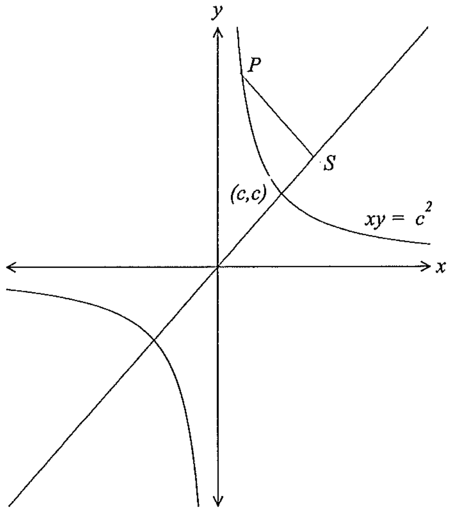
$$t = \frac{1}{\sqrt{k}}\tan^{-1}\left(\frac{u}{\sqrt{k}}\right)$$

(d) Let $z = 3-2i$ be a root of the polynomial $z^2 + bz + c = 0$ where b and c are real numbers. Find the value of b and c. **2**

Question 13 (15 marks) **Marks**

(a) The region $\{(x,y): 0 \le x \le 1, x \le y \le \sqrt{x}\}$ is rotated about the x-axis to form a solid. Use the method of slicing to obtain the volume of this solid. **3**

(b) The point $P\left(cp, \frac{c}{p}\right)$ with $p > 0$ lies on the rectangular hyperbola $xy = c^2$ with focus S. The point T divides the interval PS in the ratio 1:2.



(i) Determine the coordinates of T. **2**

(ii) Find the equation of the locus of T **2**

(c) Use the substitution $u = e^x$ to evaluate $\int \frac{e^x + e^{2x}}{1 + e^{2x}} dx$. **4**

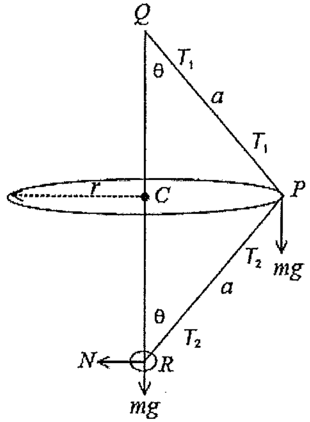
(d) Let $f(x) = \frac{1-x}{x}$. Draw separate one-third page sketches of these functions.

(i) $y = f(|x|)$ **2**

(iii) $y = e^{f(x)}$ **2**

Question 14 (15 marks) **Marks**

(a) A mass m at P is freely joined to two equal light rods PQ and PR of length a. The end Q of PQ is pivoted to a fixed point Q and the end R of PR is freely joined to a ring of mass m that slides on a smooth vertical pole. P rotates in a horizontal circle with uniform angular velocity ω. T_1 and T_2 are tensions in the rods and N is the normal reaction of QR on the ring R. **4**



Find the inclination of the rods PQ and PR to the vertical.

(b) Sketch the graph of $y = x^2 \log_2(x+2)$ **2**

(c) Let $I_n = \int x(\ln x)^n dx$ for $n = 0, 1, 2, 3, \ldots$ **3**

Show that $I_n = \frac{x^2}{2}(\ln x)^n - \frac{n}{2} I_{n-1}$ for $n \ge 1$

(d) The diameter AB of a circle is produced to E. EC is a tangent touching the circle at C, and the perpendicular to AE at E meets AC produced at D. Show that ΔCDE is isosceles. **3**

(e) Using calculus, show that $x \ge \ln(1+x)$ for $x \ge -1$. **3**

Question 15 (15 marks) **Marks**

(a) (i) For positive real numbers a, b show that $a^2 + b^2 \geq 2ab$ **1**

(ii) Hence show for positive real numbers a, b, c and d that **2**

$$3(a^2 + b^2 + c^2 + d^2) \geq 2(ab + ac + ad + bc + bd + cd)$$

(iii) Hence show for positive real numbers a, b, c and d and if **2**

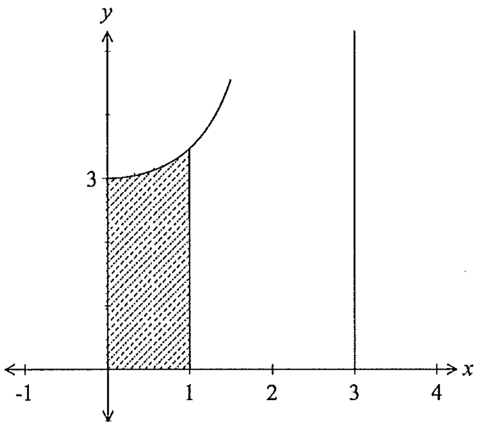
$a + b + c + d = 1$ then $ab + ac + ad + bc + bd + cd \leq \frac{3}{8}$.

(b) (i) The polynomial $P(x)$ has a double zero at $x = \alpha$. Prove that $P'(x)$ has a root at $x = \alpha$. **2**

(ii) The polynomial $P(x) = x^4 + ax^3 + bx + 21$ has a double root at $x = 1$. Find the values of a and b. **2**

(iii) Factorise the polynomial $P(x) = x^4 + ax^3 + bx + 21$ over the field of real numbers. **2**

(c) The region between the curve $y = \frac{6}{\sqrt{4 - x^2}}$, the x-axis, $x = 0$ and $x = 1$, is rotated about the line $x = 3$. **4**



Show that the volume of the solid is given by $V = 12\pi \int_0^1 \frac{3 - x}{\sqrt{4 - x^2}} dx$.

Question 16 (15 marks) **Marks**

(a) Use integration by parts and the table of standard integrals to show that **3**

$$\int \sqrt{x^2 + 8}\,dx = \frac{1}{2} x\sqrt{x^2 + 8} + 4\ln\left(x + \sqrt{x^2 + 8}\right) + c$$

(b) A five digit number is formed from the numbers 1, 2, 3, 4 and 5 without repetition

(i) How many numbers are greater than 45321? **1**

(ii) How many of these numbers are less than 45321? **1**

(c) Given $z = r(\cos\theta + i\sin\theta)$ where $z \neq 0$.

(i) Use De Moivre's theorem to show that $z^n + \frac{1}{z^n} = 2\cos n\theta$ for positive integers $n \geq 1$. **2**

(ii) Expand $\left(z + \frac{1}{z}\right)^5$ show that $\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$. **3**

(iii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^5\theta\, d\theta$. **2**

(d) If $T_1 = 8$, $T_2 = 20$ and $T_n = 4T_{n-1} - 4T_{n-2}$ for $n \geq 3$ show that **3**

$$T_n = (n + 3)2^n \text{ for } n \geq 1$$

End of paper

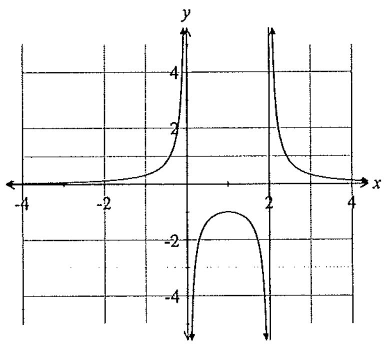
ACE Examination 2014

HSC Mathematics Extension 2 Yearly Examination

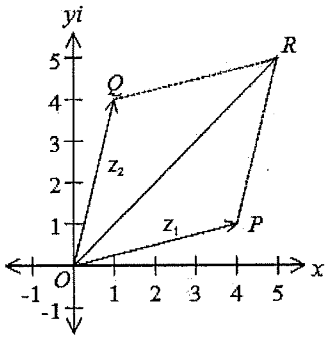
Worked solutions and marking guidelines

Section I

	Solution	Criteria
1	$t=\tan\frac{x}{2}$ $dt=\frac{1}{2}\sec^2\frac{x}{2}dx$ $dt=\frac{1}{2}(1+t^2)dx$ or $dx=\frac{2}{1+t^2}dt$ $1+\sin x+\cos x=\frac{1+t^2+2t+1-t^2}{1+t^2}=\frac{2(t+1)}{1+t^2}$ $\int\frac{1}{1+\sin x+\cos x}dx=\int\frac{1+t^2}{2(t+1)}\times\frac{2}{1+t^2}dt$ $=\int\frac{1}{t+1}dt=\ln\lvert t+1\rvert+C$	1 Mark: B
3	$\tan\theta=\frac{-1}{1}$ $\theta=\frac{3\pi}{4}$ $r^2=x^2+y^2$ $=(-1)^2+(1)^2$ $r=\sqrt{2}$ Modulus $\sqrt{2}$ and argument $\frac{3\pi}{4}$	1 Mark: B
2	Sum of roots $=-\frac{b}{a}$ $\alpha+i=-k$ (1) Product of roots $=\frac{c}{a}$ $\alpha i=2-i$ or $\alpha=\frac{2}{i}-1$ (2) sub eqn(2) into eqn(1) $\frac{2}{i}-1+i=-k$ $k=-(-2i-1+i)=1+i$	1 Mark: C
4	The polynomial has $(x-5)$ and $(x+2)^2$ as factors. $P(x)=(x-5)(x+2)^2$ $=(x-5)(x^2+4x+4)$ $=x^3-x^2-16x-20$	1 Mark: C

5	Let $u=\sin x$ then $du=\cos x dx$ Now $x=0$ then $u=0$, $x=\frac{\pi}{2}$ then $u=1$ $\int_0^{\frac{\pi}{2}}\frac{\cos x}{\sqrt{1+\sin^2 x}}dx=\int_0^1\frac{1}{\sqrt{1+u^2}}du$ $=\left[\ln\left(u+\sqrt{u^2+1}\right)\right]_0^1=\ln(1+\sqrt{2})$	1 Mark: A
6	Resolving the forces vertically and horizontally at P $T\cos\theta-mg=0$ $T\sin\theta=mr\omega^2$ Statement (B) is correct.	1 Mark: B
7	$\frac{y^2}{225}-\frac{x^2}{64}=1$, $a^2=64$ and $b^2=225$ $a^2=b^2(e^2-1)$ $64=225\times(e^2-1)$ $e=\sqrt{\frac{64}{225}+1}=\sqrt{\frac{289}{225}}=\frac{17}{15}$	1 Mark: C
8	$\frac{z_1}{z_2}=\frac{4+i}{2+3i}\times\frac{2-3i}{2-3i}$ $=\frac{11-10i}{13}$	1 Mark: D
9	$\int\sin^3 x dx=\int\sin x(\sin^2 x)dx$ $=\int\sin x(1-\cos^2 x)dx$ $=\int\sin x dx-\int\sin x\cos^2 x dx$ $=-\cos x+\frac{1}{3}\cos^3 x+C=\frac{1}{3}\cos^3 x-\cos x+C$	1 Mark: A
10		1 Mark: A

Section II

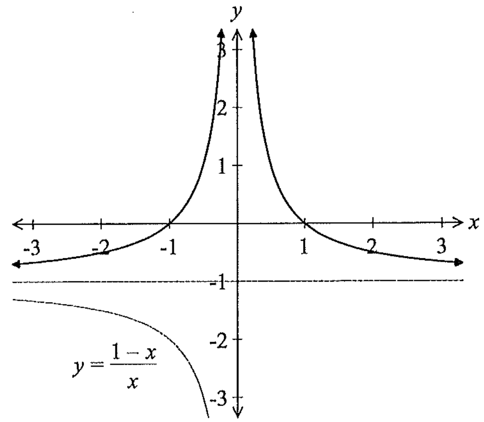
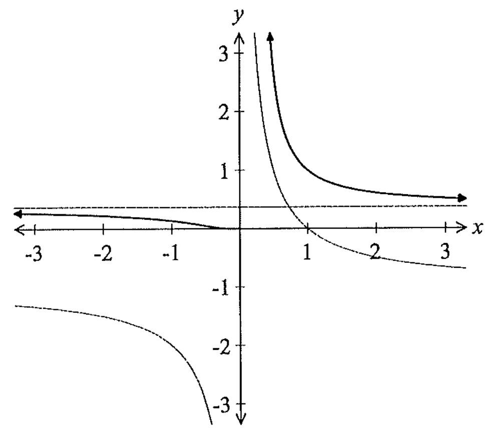
	Solution	Criteria
11(a)	$(x+iy)^2=33+56i$ $x^2-y^2+2xyi=33+56i$ $x^2-y^2=33$ (1) $2xy=56$ (2) From eqn (2) $y=\frac{28}{x}$ and sub this into eqn (1) $x^2-\left(\frac{28}{x^2}\right)^2=33$ $x^4-33x^2-28^2=0$ $(x^2-7^2)(x^2+4^2)=0$ $x=\pm7$ and $x=\pm4$	2 Marks: Correct answer. 1 Mark: Finds two equations for x and y or shows some understanding.
11(b) (i)	$P(4,i)$ and $Q(1,4i)$ represent $z_1=4+i$ and $z_2=1+4i$. Point R is constructed by completing the parallelogram. $z_1+z=5+5i$ 	2 Marks: Correct answer. 1 Mark: Constructs an Argand diagram containing z_1 and z_2.
11(b) (ii)	$OPQR$ is a rhombus. Parallelogram with $OP=OQ=\sqrt{17}$	1 Mark: Correct answer.
11(b) (iii)	$PQ=\sqrt{(4-1)^2+(1-4)^2}=3\sqrt{2}$ $OR=\sqrt{(5-0)^2+(5-0)^2}=5\sqrt{2}$ $A=\frac{1}{2}xy$ $=\frac{1}{2}\times3\sqrt{2}\times5\sqrt{2}$ $=15$ square units	1 Mark: Correct answer.

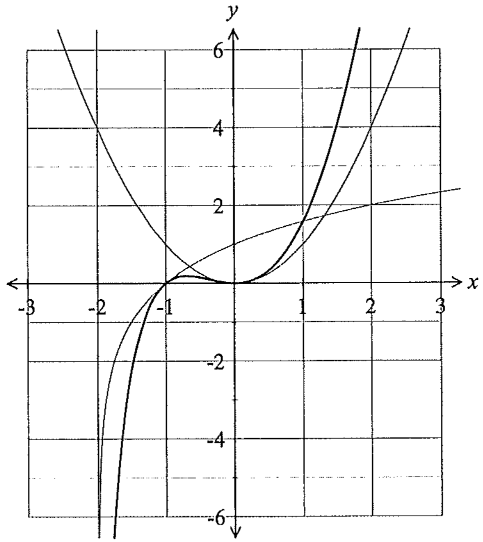
11(c) (i)	$\frac{x}{(x^2+2)(x-3)}=\frac{ax+b}{x^2+2}+\frac{c}{x-3}$ $x=(ax+b)(x-3)+c(x^2+2)$ Let $x=3$ then $3=11c$ or $c=\frac{3}{11}$ Let $x=0$ then $0=-3b+2c$ or $b=\frac{2}{11}$ Let $x=1$ then $1=-2(a+b)+3c$ $1=-2(a+\frac{2}{11})+3\times\frac{3}{11}$ or $a=-\frac{3}{11}$ $\therefore a=-\frac{3}{11}$, $b=\frac{2}{11}$ and $c=\frac{3}{11}$	2 Marks: Correct answer. 1 Mark: Makes some progress in finding a, b or c.
11(c) (ii)	$\int\frac{x}{(x^2+2)(x-3)}dx=\int\frac{-\frac{3}{11}x+\frac{2}{11}}{x^2+2}+\frac{\frac{3}{11}}{x-3}dx$ $=-\frac{3}{22}\int\frac{2x}{x^2+2}dx+\frac{2}{11}\int\frac{1}{x^2+2}dx+\frac{3}{11}\int\frac{1}{x-3}dx$ $=-\frac{3}{22}\ln\lvert x^2+2\rvert+\frac{\sqrt{2}}{11}\tan^{-1}\frac{x}{\sqrt{2}}+\frac{3}{11}\ln\lvert x-3\rvert+C$	2 Marks: Correct answer. 1 Mark: Correctly finds one of the integrals.
11(d)	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ $\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$ or $\frac{dy}{dx}=\frac{b^2x}{a^2y}$ Gradient of the normal at $P(x_1,y_1)$ is $m=-\frac{a^2y_1}{b^2x_1}$ Equation of the normal $y-y_1=m(x-x_1)$ $y-y_1=-\frac{a^2y_1}{b^2x_1}(x-x_1)$ $y-y_1=-\frac{a^2y_1}{b^2x_1}x+\frac{a^2y_1}{b^2}$ $\left(\frac{a^2y_1}{b^2x_1}x+y=y_1+\frac{a^2y_1}{b^2}\right)\times\frac{b^2}{y_1}$ $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Finds the gradient of the tangent to the hyperbola.

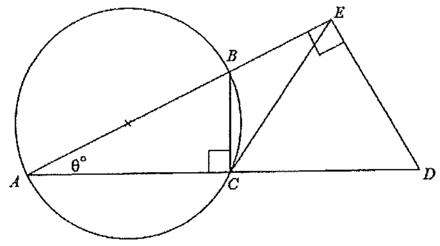
11(e)	Let $z=\cos\theta+i\sin\theta$ with $0\le\theta\le\frac{\pi}{2}$ Now $z+1=1+\cos\theta+i\sin\theta$ $=2\cos^2\frac{\theta}{2}+i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ $=2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$ Therefore $\arg(z+1)=\frac{\theta}{2}$ $\left(\cos\frac{\theta}{2}>0\right)$ $2\arg(z+1)=\theta=\arg z$	2 Marks: Correct answer. 1 Mark: Makes some progress towards the solution.
12(a) (i)	The equation $x^3+x^2-2x-3=0$ has roots α,β,γ. $x=2\alpha,2b,2\gamma$ $\alpha=\frac{x}{2}$ satisfies $x^3+x^2-2x-3=0$ $\left(\frac{x}{2}\right)^3+\left(\frac{x}{2}\right)^2-2\left(\frac{x}{2}\right)-3=0$ $\frac{x^3}{8}+\frac{x^2}{4}-x-3=0$ $x^3+2x^2-8x-24=0$	2 Marks: Correct answer. 1 Mark: Makes significant progress towards the solution.
12(a) (ii)	The equation $x^3+x^2-2x-3=0$ has roots α,β,γ. $x=\alpha-2,b-2,\gamma-2$ $\alpha=x+2$ satisfies $x^3+x^2-2x-3=0$ $(x+2)^3+(x+2)^2-2(x+2)-3=0$ $(x+2)(x^2+4x+4)+(x^2+4x+4)-2x-4-3=0$ $x^3+7x^2+14x+5=0$	2 Marks: Correct answer. 1 Mark: Makes significant progress towards the solution.
12(b) (i)	$16x^2+25y^2=400$ $32x+50y\frac{dy}{dx}=0$ $\frac{dy}{dx}=-\frac{16x}{25y}$ At $P(5\cos\alpha,4\sin\alpha)$ $\frac{dy}{dx}=-\frac{16\times5\cos\alpha}{25\times4\sin\alpha}$ $=-\frac{4\cos\alpha}{5\sin\alpha}$	2 Marks: Correct answer. 1 Mark: Finds $\frac{dy}{dx}$
12(b) (ii)	From (b)(i) gradient of tangent at Q is $-\frac{4\cos\beta}{5\sin\beta}$ Tangents are at right angles ($m_1m_2=-1$) $-\frac{4\cos\alpha}{5\sin\alpha}\times-\frac{4\cos\beta}{5\sin\beta}=-1$ $\frac{16\cot\alpha\cot\beta}{25}=-1$ or $25\tan\alpha\tan\beta=-16$	2 Marks: Correct answer. 1 Mark: Recognises the gradient of the tangent at Q and attempts to use $m_1m_2=-1$

12(c) (i)	Net force $=-mk-mv^2$ $m\ddot{x}=-mk-mv^2$ $\ddot{x}=-k-v^2$ $\frac{1}{2}\frac{dv^2}{dx}=-k-v^2$ $\frac{dv^2}{k+v^2}=-2dx$ $\ln(k+v^2)=-2x+C$ Initial conditions $v=u$, $x=0$ results in $C=\ln(k+u^2)$ Therefore $\ln(k+v^2)=-2x+\ln(k+u^2)$ $x=-\frac{1}{2}\ln\left(\frac{k+v^2}{k+u^2}\right)$	2 Marks: Correct answer. 1 Mark: Resolves forces
12(c) (ii)	$\ddot{x}=-k-v^2$ $\frac{dv}{dt}=-k-v^2$ $\frac{dv}{k+v^2}=-dt$ $\frac{1}{\sqrt{k}}\tan^{-1}\frac{v}{\sqrt{k}}=-t+C$ Initial conditions $t=0$, $v=u$ results in $C=\frac{1}{\sqrt{k}}\tan^{-1}\frac{u}{\sqrt{k}}$ $t=\frac{1}{\sqrt{k}}\tan^{-1}\frac{u}{\sqrt{k}}-\frac{1}{\sqrt{k}}\tan^{-1}\frac{v}{\sqrt{k}}$ Particle is at rest when $v=0$ $\therefore t=\frac{1}{\sqrt{k}}\tan^{-1}\left(\frac{u}{\sqrt{k}}\right)$	3 Marks: Correct answer. 2 Marks: Makes significant progress. 1 Mark: Recognises $\frac{dv}{dt}=-k-v^2$ or has some understanding of the problem.
12(d)	If b and c are real numbers then $z=3+2i$ is a root (conjugate root theorem). $[z-(3-2i)][z-(3+2i)]=z^2-(3-2i+3+2i)z+9+4$ $=z^2-6z+13=0$ Therefore $b=-6$ and $c=13$	2 Marks: Correct answer. 1 Mark: Recognises the conjugate root theorem.
13(a)	Area of the slice is a circle radius is x and height y $A=\pi(R^2-r^2)=\pi(\sqrt{x}^2-x^2)$ $=\pi(x-x^2)$ $\delta V=\delta A.\delta y$	3 Marks: Correct answer.

	$V=\lim_{\delta y\to 0}\sum_{y=0}^{1}\pi(x-x^2)\delta x$ $=\int_0^1 \pi(x-x^2)dx$ $=\pi\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1=\pi\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{\pi}{6}$ cubic units	2 Marks: Correct integral for the volume of the solid. 1Mark: Correct expression for the volume of solid.
13(b) (i)	$P\left(cp,\frac{c}{p}\right)$, $S(c\sqrt{2},c\sqrt{2})$ and $PT:TS=1:2$ Coordinates of T $x=\frac{mx_2+nx_1}{m+n}$ $\quad y=\frac{my_2+ny_1}{m+n}$ $=\frac{1\times c\sqrt{2}+2\times cp}{1+2}$ $\quad =\frac{1\times c\sqrt{2}+2\times\frac{c}{p}}{1+2}$ $=\frac{c(\sqrt{2}+2p)}{3}$ $\quad =\frac{c\left(\sqrt{2}+\frac{2}{p}\right)}{3}$	2 Marks: Correct answer. 1 Mark: Finds one of the coordinates or makes some progress towards the solution.
13(b) (i)	To find the locus of T eliminate p from the above equations. $x=\frac{c(\sqrt{2}+2p)}{3}$ $\quad y=\frac{c\left(\sqrt{2}+\frac{2}{p}\right)}{3}$ $3x=c(\sqrt{2}+2p)$ $\quad \frac{3y}{c}-\sqrt{2}=\frac{2}{p}$ $\frac{3x}{c}-\sqrt{2}=2p$ $\left(\frac{3x}{c}-\sqrt{2}\right)\times\left(\frac{3y}{c}-\sqrt{2}\right)=2p\times\frac{2}{p}=4$ $(3x-\sqrt{2}c)(3y-\sqrt{2}c)=4c^2$	2 Marks: Correct answer. 1 Mark: Uses the coordinates of T and attempts to eliminate p.
13(c)	$u=e^x$ or $du=e^x dx$ or $dx=\frac{1}{u}du$ $\int\frac{e^x+e^{2x}}{1+e^{2x}}dx=\int\frac{u+u^2}{1+u^2}\times\frac{1}{u}du$ $=\int\frac{1+u}{1+u^2}du$ $=\int\frac{1}{1+u^2}du+\int\frac{u}{1+u^2}du$ $=\tan^{-1}u+\frac{1}{2}\ln(u^2+1)+C$ $=\tan^{-1}e^x+\frac{1}{2}\ln(e^{2x}+1)+C$	4 Marks: Correct answer. 3 Marks: Separates and integrates one part correctly. 2 Marks: Correctly expresses the integral in terms of u 1 Mark: Correctly finds dx in terms of du

13(d) (i)	$f(x)=\frac{1-x}{x}=\frac{1}{x}-1$ (asymptote at $x=0$ and $y=-1$) $y=f(\lvert x\rvert)=\frac{1-\lvert x\rvert}{\lvert x\rvert}$ (asymptote at $x=0$ and $y=-1$)  	2 Marks: Correct answer. 1 Mark: Draws $y=f(x)$ or shows some understanding
13(d) (ii)	$x\to 0^+$ then $y\to e^{\infty}=\infty$ $\quad x\to\infty$ then $y\to e^{-1}=\frac{1}{e}$ $x\to 0^-$ then $y\to e^{-\infty}=0$ $\quad x\to-\infty$ then $y\to e^{-1}=\frac{1}{e}$ Asymptote at $x=0$ and $y=\frac{1}{e}$. Also $x\neq 0$  	2 Marks: Correct answer. 1 Mark: Determines the asymptotes or shows some understanding.

14(a)	Resolving the forces vertically and horizontally at P $T_1\cos\theta - T_2\cos\theta - mg = 0$ (1) $T_1\sin\theta + T_2\sin\theta = mr\omega^2$ (2) Resolving the forces vertically and horizontally at R $T_2\cos\theta - mg = 0$ (3) $T_2\sin\theta - N = 0$ (4) Eqn (1) – (3) $T_1\cos\theta - 2T_2\cos\theta = 0$ $T_1 = 2T_2$ Sub $T_1 = 2T_2$ into eqn (2) $2T_2\sin\theta + T_2\sin\theta = mr\omega^2$ $3T_2\sin\theta = mr\omega^2$ (5) Divid eqn (5) by eqn (3) $\frac{3T_2\sin\theta}{T_2\cos\theta} = \frac{mr\omega^2}{mg}$ $3\tan\theta = \frac{r\omega^2}{g}$ $\theta = \tan^{-1}\left(\frac{r\omega^2}{3g}\right)$	4 Marks: Correct answer. 3 Marks: Makes significant progress towards the solution. 2 Marks: Resolves forces in the vertical and horizontal directions at both P and R. 1 Mark: Resolves forces in the vertical and horizontal directions at P or R.
14(b)		2 Marks: Correct answer. 1 Mark: Finds asymptote or x-intercepts or shows some understanding.

14(c)	$I_n = \int x(\ln x)^n dx$ $= (\ln x)^n \frac{x^2}{2} - \int \frac{x^2}{2} \times \frac{n}{x}(\ln x)^{n-1} dx$ $= (\ln x)^n \frac{x^2}{2} - \frac{n}{2}\int x(\ln x)^{n-1} dx$ $= (\ln x)^n \frac{x^2}{2} - \frac{n}{2} I_{n-1}$ for $n \geq 1$ When $n = 0$ $I_0 = \int x(\ln x)^0 dx = \frac{x^2}{2} + C$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Sets up the integration and shows some understanding.
14(d)	 Let $\angle BAC = \theta$ $\angle ACB = 90^\circ$ (angle in a semi-circle) $\angle BCD = 90^\circ$ (adjacent angles on a straight line) $\angle BCE = \angle BAC$ (angle between a tangent and a chord equals the angle in the alternate segment) $\therefore \angle BCE = \theta$ $\therefore \angle ECD = 90 - \theta$ (angle sum of $\angle BCD$) $\therefore \angle EDC = 90 - \theta$ ((angle sum of ΔAED) $\therefore \angle ECD = \angle EDC = 90 - \theta$ ΔECD is isosceles (base angles of the triangle are equal)	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Draws the diagram and applies a relevant circle theorem.
14(e)	Let $f(x) = x - \ln(1+x)$ and $f'(x) = 1 - \frac{1}{1+x}$ Minimum occurs if $f'(x) = 0$ $1 - \frac{1}{1+x} = 0$ $\frac{1+x}{1+x} - \frac{1}{1+x} = 0$ $\therefore 1 + x - 1 = 0$ or $x = 0$ $(x \neq -1)$ Test $f''(x) = \frac{1}{(1+x)^2}$, $f''(0) = 1 > 0$ Minima Therefore the least value of $f(x)$ is at $x = 0$ $f(0) = 0 - \ln(1+0) = 0$ hence $f(x) \geq 0$ $f(x) = x - \ln(1+x) \geq 0$ $\therefore x \geq \ln(1+x)$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 2 Marks: 1 Mark: Sets up the function and correctly uses calculus.

15(a) (i)	$a^2+b^2-2ab=(a-b)^2\ge 0$ $a^2+b^2\ge 2ab$	1 Mark: Correct answer.
15(a) (ii)	$a^2+b^2\ge 2ab$ $\quad$ $b^2+c^2\ge 2bc$ $a^2+c^2\ge 2ac$ $\quad$ $b^2+d^2\ge 2bd$ $a^2+d^2\ge 2ad$ $\quad$ $b^2+c^2\ge 2cd$ $3(a^2+b^2+c^2+d^2)\ge 2(ab+ac+ad+bc+bd+cd)$	2 Marks: Correct answer. 1 Mark: Uses the result in part (i).
15(a) (iii)	$a^2+b^2+c^2+d^2=(a+b+c+d)^2-2(ab+ac+ad+bc+bd+cd)$ Multiply the above equation by 3 and use $a+b+c+d=1$ $3(a^2+b^2+c^2+d^2)=3\times 1^2-6(ab+ac+ad+bc+bd+cd)$ Now using the result in part (ii) $2(ab+ac+ad+bc+bd+cd)\le 3-6(ab+ac+ad+bc+bd+cd)$ $8(ab+ac+ad+bc+bd+cd)\le 3$ $(ab+ac+ad+bc+bd+cd)\le\frac{3}{8}$	2 Marks: Correct answer. 1 Mark: Writes the sum of the squares in terms of the products taken two at a time.
15(b) (i)	$P(x)=(x-\alpha)^2Q(x)$ $P'(x)=(x-\alpha)^2Q'(x)+2(x-\alpha)Q(x)$ $\quad=(x-\alpha)\left[(x-\alpha)Q'(x)+2Q(x)\right]$ Therefore $P'(\alpha)=0$ and $x=\alpha$ is a root of $P'(x)$.	2 Marks: Correct answer. 1 Mark: Finds $P'(x)$
15(b) (ii)	$P(x)=x^4+ax^3+bx+21$ $P(1)=1^4+a\times 1^3+b\times 1+21$ $\quad=a+b+22=0$ (1) $P'(x)=4x^3+3ax^2+b$ $P'(1)=4\times 1+3a\times 1^2+b$ $\quad=4+3a+b=0$ (2) Solving equations (1) and (2) simultaneously $a=9$ and $b=-31$	2 Marks: Correct answer. 1 Mark: Shows some understanding of the problem.

15(b) (iii)	Double root at $x=1$ then $(x-1)^2=x^2-2x+1$ is a factor. Quotient: $x^2+11x+21$ $x^2-2x+1\,\overline{)\,x^4+9x^3+0x^2-31x+21}$ $\underline{x^4-2x^3+1x^2}$ $11x^3-1x^2$ $\underline{11x^3-22x^2+11x}$ $21x^2-42x$ $\underline{21x^2-42x+21}$ $P(x)=(x-1)^2(x^2+11x+21)$ $\quad=(x-1)^2(x^2+11x+21)$ $\quad=(x-1)\left[x+\left(\frac{-11\pm\sqrt{37}}{2}\right)\right]$	2 Marks: Correct answer. 1 Mark: Makes significant progress towards the solution.
15(c)	Cylindrical shell has an outer radius of $3-x$, inner radius $3-(x+\delta x)$ and height $\frac{6}{\sqrt{4-x^2}}$. Volume of the shell: $\delta V=\pi h(R^2-r^2)$ $\quad=\pi h(R+r)(R-r)$ $\quad=\pi h\left[2(3-x)-\delta x\right]\delta x$ $V=\lim_{\delta x\to 0}\sum_{x=0}^{1}\pi\left[2(3-x)-\delta x\right]\delta x\frac{6}{\sqrt{4-x^2}}$ $\quad=\lim_{\delta x\to 0}\sum_{x=0}^{1}\pi\frac{6}{\sqrt{4-x^2}}2(3-x)\delta x$ igoring δx^2 $\quad=12\pi\int_0^1\frac{3-x}{\sqrt{4-x^2}}dx$	4 Marks: Correct answer. 3 Marks: Writes V as the correct limiting sum 2 Marks: Correct expression for δV. 1 Mark: Determines the radius or height of the cylindrical shell.

16(a)	$\int\sqrt{x^2+8}dx = x\sqrt{x^2+8} - \int x\frac{1}{2}(x^2+8)^{-\frac{1}{2}}2xdx$ $= x\sqrt{x^2+8} - \int(x^2+8-8)(x^2+8)^{-\frac{1}{2}}dx$ $= x\sqrt{x^2+8} - \int\left[(x^2+8)-8(x^2+8)^{-\frac{1}{2}}\right]dx$ $= x\sqrt{x^2+8} - \int\sqrt{x^2+8}dx + 8\int\frac{1}{\sqrt{x^2+8}}dx$ $2\int\sqrt{x^2+8}dx = x\sqrt{x^2+8} + 8\int\frac{1}{\sqrt{x^2+8}}dx$ $= x\sqrt{x^2+8} + 8\ln\left(x+\sqrt{x^2+8}\right)+c$ $\int\sqrt{x^2+8}dx = \frac{1}{2}x\sqrt{x^2+8} + 4\ln\left(x+\sqrt{x^2+8}\right)+c$	3 Marks: Correct answer. 2 Marks: Makes significant progress. 1 Mark: Sets up the integration by parts.
16(b) (i)	Numbers greater than 45321 begin with a 5. Number of arrangements $= 1\times4\times3\times2\times1$ $= 24$	1 Mark: Correct answer.
16(b) (ii)	All of the arrangements minus the numbers greater than and equal 45321. Number of arrangements $= 5!-4!-1$ $= 95$	1 Mark: Correct answer.
16(c) (i)	$z^n = [\cos\theta + i\sin\theta]^n$ and $\frac{1}{z^n} = [\cos\theta+i\sin\theta]^{-n}$ $= \cos n\theta + i\sin n\theta$ $\quad = \cos n\theta - i\sin n\theta$ $z^n + \frac{1}{z^n} = \cos n\theta + \sin n\theta + \cos n\theta - \sin n\theta$ $= 2\cos n\theta$	2 Marks: Correct answer. 1 Mark: Uses De Moivre's theorem
16(c) (ii)	$\left(z+\frac{1}{z}\right)^5 = z^5 + 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z}\right)^2 + 10z^2\left(\frac{1}{z}\right)^3 + 5z\left(\frac{1}{z}\right)^4 + \left(\frac{1}{z}\right.$ $= \left(z^5+\frac{1}{z^5}\right) + 5\left(z^3+\frac{1}{z^3}\right) + 10\left(z+\frac{1}{z}\right)$ $(2\cos\theta)^5 = 2\cos5\theta + 10\cos3\theta + 20\cos\theta$ $\cos^5\theta = \frac{1}{16}(\cos5\theta + 5\cos3\theta + 10\cos\theta)$	3 Marks: Correct answer. 2 Marks: Makes significant progress. 1 Mark: Writes the binomial expansion.

16(c) (iii)	$\int_0^{\frac{\pi}{2}}\cos^5\theta d\theta = \int_0^{\frac{\pi}{2}}\frac{1}{16}(\cos5\theta + 5\cos3\theta + 10\cos\theta)d\theta$ $= \frac{1}{16}\left[\frac{1}{5}\sin5\theta + \frac{5}{3}\sin3\theta + 10\sin\theta\right]_0^{\frac{\pi}{2}}$ $= \frac{1}{16}\left(\frac{1}{5} - \frac{5}{3} + 10\right)$ $= \frac{8}{15}$	2 Marks: Correct answer. 1 Mark: Finds the primitive function.
16(d)	Step 1: To prove the statement true for $n=1$ and $n=2$ $T_1 = (1+3)2^1 = 8$ $\quad T_2 = (2+3)2^2 = 20$ Result is true for $n=1$ $\quad$ Result is true for $n=2$ Step 2: Assume the result true for $n=k$ $T_k = (n+3)2^k$ To prove the result is true for $n=k+1$ $T_{k+1} = (n+4)2^{k+1}$ $T_{k+1} = 4T_k - 4T_{k-1}$ $= 4(k+3)2^k - 4(k+2)2^{k-1}$ $= 4k2^k + 12\times2^k - 4k2^{k-1} - 8\times2^{k-1}$ $= 4k2^k + 12\times2^k - 2k2^k - 4\times2^k$ $= 2^{k+1}(2k+6-k-2)$ $= (k+4)2^{k+1}$ Result is true for $n=k+1$ if true for $n=k$ Step 3: Result true by principle of mathematical induction.	3 Marks: Correct answer. 2 Marks: Proves the result true for $n=1$ and attempts to use the result of $n=k$ to prove the result for $n=k+1$. 1 Mark: Proves the result true for $n=1$ and $n=2$.