ACE Examinations

Student Name: ____________________

2013

YEAR 12

YEARLY EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time - 5 minutes
- Working time - 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11-16

Total marks - 100

Section I

10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II

90 marks

- Attempt Questions 11-16
- Allow about 2 hour 45 minutes for this section

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Section I

10 marks
Attempt Questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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1 Which of the following is an expression for $\int \frac{1}{\sqrt{x^2-8x+17}}\,dx$?

(A) $\ln\left(x-4-\sqrt{x^2-8x+17}\right)+c$

(B) $\ln\left(x+4-\sqrt{x^2-8x+17}\right)+c$

(C) $\ln\left(x-4+\sqrt{x^2-8x+17}\right)+c$

(D) $\ln\left(x+4+\sqrt{x^2-8x+17}\right)+c$

2 What is the value of $\int -\sec x\,dx$? Use the substitution $t=\tan\frac{x}{2}$.

(A) $\ln\left|\frac{t-1}{t+1}\right|+c$

(B) $\ln\left|\frac{1-t}{t+1}\right|+c$

(C) $\ln|(1-t)(t+1)|+c$

(D) $\ln|(t-1)(t+1)|+c$

3 What is $-1-\sqrt{3}i$ expressed in modulus-argument form?

(A) $\sqrt{2}[\cos(-\frac{\pi}{3})+i\sin(-\frac{\pi}{3})]$

(B) $2[\cos(-\frac{\pi}{3})+i\sin(-\frac{\pi}{3})]$

(C) $\sqrt{2}[\cos(-\frac{2\pi}{3})+i\sin(-\frac{2\pi}{3})]$

(D) $2[\cos(-\frac{2\pi}{3})+i\sin(-\frac{2\pi}{3})]$

4 Let $z=4-i$. What is the value of $\overline{iz}$?

(A) $-1-4i$

(B) $-1+4i$

(C) $1-4i$

(D) $1+4i$

5 The polynomial equation $x^3+x^2-x-4=0$ has roots α, β and γ. Which of the following polynomial equations have roots α^2, β^2 and γ^2?

(A) $x^3-3x^2-8x-16=0$

(B) $x^3-3x^2+9x-16=0$

(C) $x^3-x^2-8x-1=0$

(D) $x^3-x^2+9x-1=0$

6 The points $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ lie on the same branch of the hyperbola $xy=c^2$ $(p \neq q)$. The tangents at P and Q meet at the point T. What is the equation of the normal to the hyperbola at P?

(A) $p^2x-py+c-cp^4=0$

(B) $p^3x-py+c-cp^4=0$

(C) $x+p^2y-2c=0$

(D) $x+p^2y-2cp=0$

7 A conical pendulum consists of a body P of mass m kg and a string of length l metres. Point A is fixed and the body P rotates in a horizontal circle of radius r and centre O at a constant angular velocity of ω radians per second. OA is vertical and has a length of h metres. The angle OAP is θ radians. The body, P, is subject to a gravitational force of mg newtons. The tension in the string is T newtons. Which of the following gives the correct resolution of forces on P in the horizontal and vertical directions?

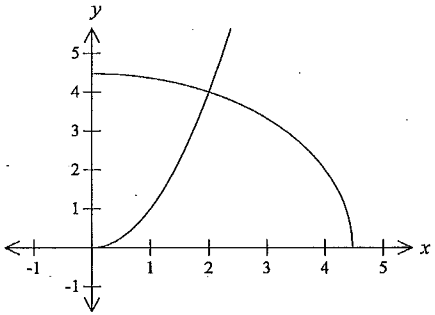
(A) $T\sin\theta-mg=0$ and $T\cos\theta=mr\omega^2$

(B) $T\sin\theta+mg=0$ and $T\cos\theta=mr\omega^2$

(C) $T\cos\theta-mg=0$ and $T\sin\theta=mr\omega^2$

(D) $T\cos\theta+mg=0$ and $T\sin\theta=mr\omega^2$

8 What is the volume of the solid formed when the region bounded by the curves $y = x^2$, $y = \sqrt{20 - x^2}$ and the y-axis is rotated about the y-axis? Use the method of slicing.



What is the correct expression for volume of this solid using the method of cylindrical shells?

(A) $V = \int_0^2 2\pi\left(\sqrt{20 - x^2} - x^2\right)dx$

(B) $V = \int_0^2 2\pi x\left(\sqrt{20 - x^2} - x^2\right)dx$

(C) $V = \int_0^2 2\pi\left(x^2 - \sqrt{20 - x^2}\right)dx$

(D) $V = \int_0^2 2\pi x\left(x^2 - \sqrt{20 - x^2}\right)dx$

9 What is the derivative of $\cos^{-1} x - \sqrt{1 - x^2}$?

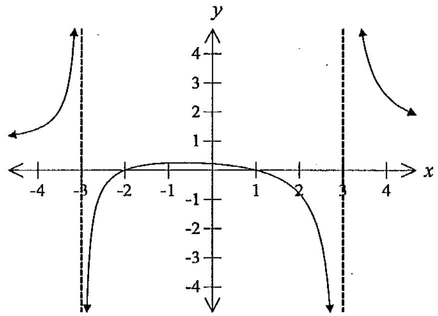
(A) $\dfrac{-\sqrt{1-x}}{\sqrt{1+x}}$

(B) $\dfrac{-\sqrt{1-x}}{1+x}$

(C) $\dfrac{x-1}{\sqrt{1+x}}$

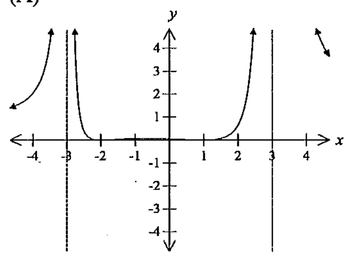
(D) $\dfrac{x-1}{x+1}$

10 The diagram below shows the graph of the function $y = f(x)$.

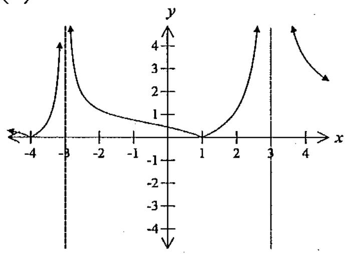


Which of the following is the graph of $y = |f(x)|$?

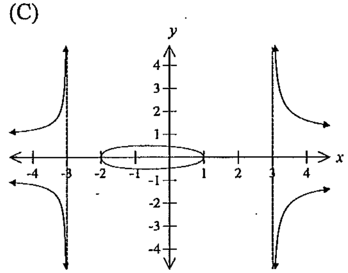
(A)



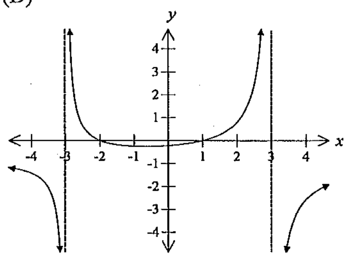
(B)



(C)



(D)



Section II

90 marks
Attempt Questions 11 □ 16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet.

All necessary working should be shown in every question.

Question 11 (15 marks) **Marks**

(a) Find $\int \sqrt{x}\ln x\,dx$. **3**

(b) (i) Find real numbers a, b, c and d such that **2**

$$\frac{x^3-8x^2+9x}{(1+x^2)(9+x^2)}=\frac{ax+b}{1+x^2}+\frac{cx+d}{9+x^2}$$

(ii) Hence evaluate in simplest form **2**

$$\int_0^{\sqrt{3}}\frac{x^3-8x^2+9x}{(1+x^2)(9+x^2)}dx$$

(c) Use the substitution $u=\sqrt{x-1}$ to evaluate $\int_2^3\frac{1+x}{\sqrt{x-1}}dx$. **4**

(d) The polynomial $P(z)$ has the equation $P(z)=z^4-2z^3+az^2+bz+10$, where a and b are real. Given that $2+i$ is a zero of $P(z)$, write $P(z)$ as a product of two real quadratic factors and find all the roots of $P(z)$. **2**

(e) The polynomial $P(x)=x^4+ax^2+bx+28$ has a double root at $x=2$. What are the values of a and b? **2**

Question 12 (15 marks) **Marks**

(a) Let $z=1+i\sqrt{3}$

(i) What is the exact value of $|z|$ and $\arg z$? **2**

(ii) Find the exact value of z^5 in the form $a+ib$, where a and b are real numbers. **2**

(b) Let the point R represent the complex number z on an Argand diagram. Describe and sketch the locus of R specified by $|z|=|z-3|$? **2**

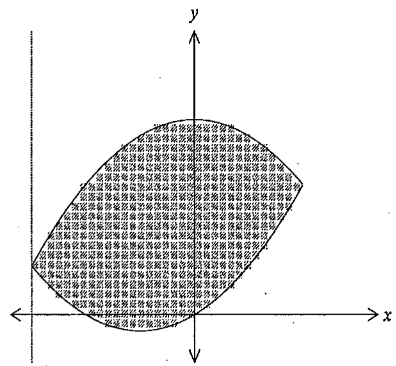
(c) Given $z=r(\cos\theta+i\sin\theta)$ where $z\neq 0$.

(i) Show that $z\bar{z}$ is real **2**

(ii) Use De Moivre's theorem to show that $z^n+\bar{z}^n$ is real for all integers $n\geq 1$ **2**

(iii) Show that $\frac{z}{\bar{z}}+\frac{\bar{z}}{z}$ is real **1**

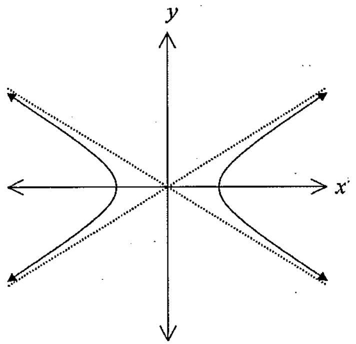
(d) A solid is formed by rotating the shaded region bounded by the curve $y=x^2+x$ and $y=3-x^2$ about the line $x=-1.5$.



Find the volume of this solid using the method of cylindrical shells. **4**

Question 13 (15 marks) **Marks**

(a) The point $P(x_0, y_0)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a > b > 0$.



(i) What are the equations of the asymptotes of the hyperbola? **1**

(ii) Show that the acute angle θ between the two asymptotes satisfies **2**

$$\tan\theta = \frac{2ab}{a^2 - b^2}$$

(iii) If C and D are the feet of the perpendicular drawn for $P(x_0, y_0)$ to the asymptotes show that $CP \times DP = \frac{a^2b^2}{a^2 + b^2}$ **3**

(iv) What is the area of ΔPCD? **2**

(b) A speed boat of mass m is travelling at maximum power. At maximum power, its engine delivers a force F on the speed boat. The water exerts a resistive force proportional to the square of the speed boat's speed v.

(i) Show that $\ddot{x} = \frac{1}{m}\left(F - kv^2\right)$ where k is a positive constant. **2**

(ii) The speed boat increases its speed from v_1 to v_2. Show that the distance travelled during this period is $x = \frac{m}{2k}\ln\left(\frac{F - kv_1^{\,2}}{F - kv_2^{\,2}}\right)$. **3**

(c) The equation $12x^3 - 6x^2 - 2x + 5$ has roots α, β and γ. **2**
What is the value of α if $\alpha = \beta + \gamma$?

Question 14 (15 marks) **Marks**

(a) Consider the function $f(x) = x + \frac{1}{x-1}$.

(i) Find the nature of the stationary points on the curve $y = f(x)$. **2**

(ii) Sketch the graph of $y = f(x)$. **2**

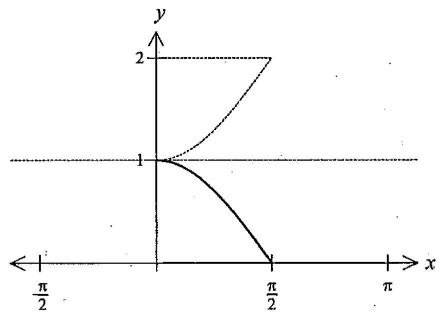
(iii) Show that the line $y = -3x$ is a tangent to the curve and find the coordinates of the point of contact. **2**

(b) (i) Let $I_n = \int_0^1 \sqrt{x}(1-x)^n\,dx$ for $n = 0, 1, 2, 3, \ldots$ **3**

Show that $I_n = \frac{2n}{2n+3} I_{n-1}$.

(ii) Hence or otherwise, find the value of $I_n = \int_0^1 \sqrt{x}(1-x)^3\,dx$. **2**

(c) The area between the curve $y = \cos x$, the x-axis, $x = 0$ and $x = \frac{\pi}{2}$, is rotated about the line $y = 1$ to form a solid. **4**



What is the volume of this solid using the method of slicing?

Question 15 (15 marks) **Marks**

(a) Given the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$. The normal at $A(4\cos\theta, 3\sin\theta)$ meets the x-axis at P and the y-axis at Q.

(i) What is the equation of the normal at A? **2**

(ii) Find the locus of the midpoint of PQ **3**

(b) A class of 25 students is to be divided into four groups consisting of 3, 4, 5 and 6 students.

(i) How many ways can this are done? Leave your answer in unsimplified form. **2**

(ii) Assume that the four groups have been chosen. How many ways can the 25 students be arranged around a circular table if the students in each group are to be seated together? Leave your answer in unsimplified form. **2**

(c) Use the principle of mathematical induction to prove that **3**

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{n}{(n+1)!}=1-\frac{1}{(n+1)!} \text{ for } n\geq 1$$

(d) Prove that ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$ **3**

HSC Mathematics Extension 2

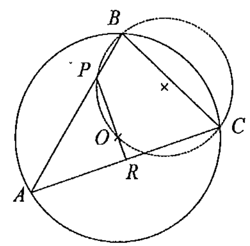
Question 16 (15 marks) **Marks**

(a) A railway line is taken round a circular bend of radius 800 m. The distance between the rails is 1.5 m. At what height above the inner rail should the outer rail be raised in order to eliminate lateral force for an engine travelling at a speed of 50 km/h round the bend? **3**
Assume acceleration due to gravity is 9.8 ms^{-1}

(b) A particle moves with constant angular velocity ω in a horizontal circle of radius r on the inside of a fixed smooth hemispherical bowl of internal radius $2r$. Show that $\omega^2=\frac{g}{r\sqrt{3}}$. **3**

(c) Show that $\frac{a^2+b^2}{a-b}\geq 2\sqrt{2}$ for $a>b$ and $ab>1$. **3**

(d) Triangle ABC is a triangle inscribed in a circle with centre O. A second circle through B, C and O cuts AB at P. PO is produced to AC at R.



(i) Prove that $\angle AOR = \angle ABC$. **3**

(ii) Prove that $AR = CR$. **3**

End of paper

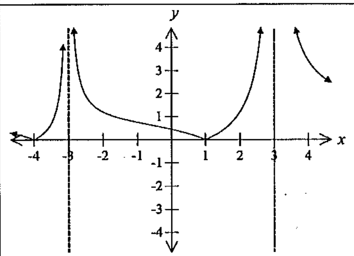
ACE Examination 2013

HSC Mathematics Extension 2 Yearly Examination

Worked solutions and marking guidelines

Section I		
	Solution	**Criteria**
1	$\int \frac{dx}{\sqrt{x^2-8x+17}} = \int \frac{dx}{\sqrt{x^2-8x+16+1}} = \frac{dx}{\sqrt{(x-4)^2+1}}$ $= \ln\left(x-4+\sqrt{(x-4)^2+1}\right)+c$ $= \ln\left(x-4+\sqrt{x^2-8x+17}\right)+c$	1 Mark: C
2	$t = \tan\frac{x}{2}$ $\quad \cos x = \frac{1-t^2}{1+t^2}$ $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$ $\quad \sec x = \frac{1+t^2}{1-t^2}$ $= \frac{1}{2}(1+t^2)$ $dx = \frac{2dt}{1+t^2}$ $\int -\sec x dx = \int \frac{1+t^2}{t^2-1}\times\frac{2dt}{1+t^2}$ $= \int \frac{2dt}{t^2-1}$ $= \int \frac{a}{t-1}+\frac{b}{t+1}dt$ $= \int \frac{1}{t-1}-\frac{1}{t+1}dt$ $= \ln\|t-1\|-\ln\|t+1\|+c$ $= \ln\left\|\frac{t-1}{t+1}\right\|+c$	1 Mark: A
3	$\tan\theta = \frac{-1}{-\sqrt{3}}$ $\quad r^2 = x^2+y^2$ $= (-\sqrt{3})^2+(-1)^2$ $\theta = -\frac{2\pi}{3}$ $\quad r = 2$ $2[\cos(-\frac{2\pi}{3})+i\sin(-\frac{2\pi}{3})]$	1 Mark: D

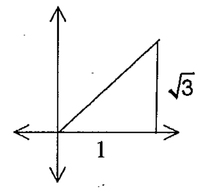
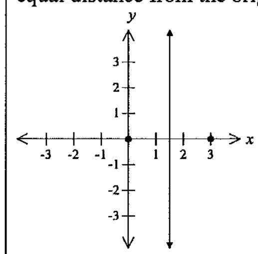
4	$\overline{iz} = \overline{i(4-i)}$ $= \overline{4i+1}$ $= 1-4i$	1 Mark: C
5	If α, β and γ are zeros of $p(x)=x^3+x^2-x-4$ then $p(\alpha)=\alpha^3+\alpha^2-\alpha-4=0$ $p(\beta)=\beta^3+\beta^2-\beta-4=0$ $p(\gamma)=\gamma^3+\gamma^2-\gamma-4=0$ $p(\alpha^2)=(\sqrt{\alpha})^3+(\sqrt{\alpha})^2-(\sqrt{\alpha})-4=0$ $p(\beta^2)=(\sqrt{\beta})^3+(\sqrt{\beta})^2-(\sqrt{\beta})-4=0$ $p(\gamma^2)=(\sqrt{\gamma})^3+(\sqrt{\gamma})^2-(\sqrt{\gamma})-4=0$ Polynomial equation is $(\sqrt{x})^3+(\sqrt{x})^2-(\sqrt{x})-4=0$ $\sqrt{x}(x-1)=4-x$ $x(x^2-2x+1)=16-8x+x^2$ $x^3-3x^2+9x-16=0$	1 Mark: B
6	To find the gradient of the tangent. $xy=c^2$ $x\frac{dy}{dx}+y=0$ $\frac{dy}{dx}=-\frac{y}{x}$ At P $(cp, \frac{c}{p})$ $\frac{dy}{dx}=-\frac{\frac{c}{p}}{cp}=-\frac{1}{p^2}$ Gradient of the normal is p^2 $(m_1m_2=-1)$ Equation of the normal at P $(cp, \frac{c}{p})$ $y-\frac{c}{p}=p^2(x-cp)$ $py-c=p^3x-cp^4$ $p^3x-py+c-cp^4=0$	1 Mark: B

7	Body moving in a horizontal circle. $T\cos\theta - mg = 0$ and $T\sin\theta = mr\omega^2$	1 Mark: C
8	$\sqrt{20-x^2} = x^2$ $20 - x^2 = x^4$ $x^4 + x^2 - 20 = 0$ $(x^2+5)(x^2-4) = 0$ $x = \pm 2$ Cylindrical shells radius is x and height y $V = \lim_{\delta x \to 0} \sum_{x=0}^{2} 2\pi xy\delta x$ $= \int_0^2 2\pi x\left(\sqrt{20-x^2} - x^2\right)dx$	1 Mark: B
9	$y = \cos^{-1}x - \sqrt{1-x^2}$ $\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} - \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times -2x$ $= \frac{-1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}$ $= \frac{-(1-x)}{\sqrt{1-x^2}}$ $= \frac{-(1-x)}{\sqrt{(1+x)(1-x)}}$ $= \frac{-\sqrt{1-x}}{\sqrt{1+x}}$ Result defined for $-1 \le x \le 1$	1 Mark: A
10		1 Mark: B

Section II

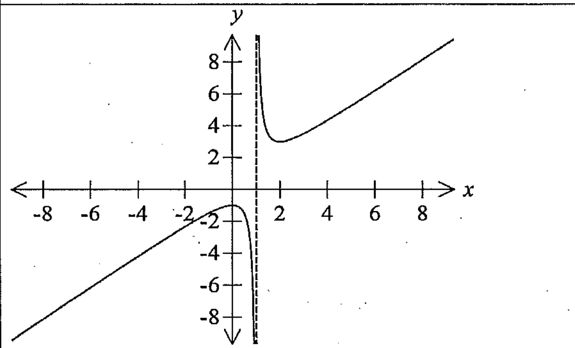
	Solution	Criteria
11(a)	$\int \sqrt{x}\ln x\,dx = \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\int x^{\frac{3}{2}} \times \frac{1}{x}dx$ $= \frac{2}{3}x\sqrt{x}\ln x - \frac{2}{3} \times \frac{2}{3}x^{\frac{3}{2}} + c$ $= \frac{2x\sqrt{x}}{9}(3\ln x - 2) + c$	3 Marks: Correct answer. 2 Marks: Makes significant progress. 1 Mark: Sets up the integration by parts.
11(b) (i)	$\frac{x^3 - 8x^2 + 9x}{(1+x^2)(9+x^2)} = \frac{ax+b}{1+x^2} + \frac{cx+d}{9+x^2}$ $x^3 - 8x^2 + 9x = (ax+b)(9+x^2) + (cx+d)(1+x^2)$ $= ax^3 + bx^2 + 9ax + 9b + cx^3 + dx^2 + cx + d$ $a + c = 1$ (coefficients x^3) $9a + c = 9$ (coefficients x) Therefore $a = 1$ and $c = 0$ Also $b + d = -8$ (coefficients x^2) When $x = 0$ then $9b + d = 0$ Hence $b = 1$ and $d = -9$ $\quad \therefore a = 1,\ b = 1,\ c = 0$ and $d = -9$	2 Marks: Correct answer. 1 Mark: Makes some progress in finding a, b, c or d.
11(b) (ii)	$\int_0^{\sqrt{3}} \frac{x^3 - 8x^2 + 9x}{(1+x^2)(9+x^2)}dx = \int_0^{\sqrt{3}} \frac{x+1}{1+x^2} + \frac{-9}{9+x^2}dx$ $= \int_0^{\sqrt{3}} \frac{x}{1+x^2} + \frac{1}{1+x^2} - \frac{9}{9+x^2}dx$ $= \left[\frac{1}{2}\ln(1+x^2) + \tan^{-1}x - 3\tan^{-1}\frac{x}{3}\right]_0^{\sqrt{3}}$ $= \frac{1}{2}(\ln 4 - \ln 1) + (\tan^{-1}\sqrt{3} - \tan^{-1}0) - 3(\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1}0)$ $= \ln 2 + \frac{\pi}{3} - 3 \times \frac{\pi}{6} = \ln 2 - \frac{\pi}{6}$	2 Marks: Correct answer. 1 Mark: Correctly finds one of the integrals.

11(c)	$u=(x-1)^{\frac{1}{2}}$ $u^2=x-1$ $du=\frac{1}{2}(x-1)^{-\frac{1}{2}}dx$ $1+x=u^2+2$ $2du=\frac{dx}{\sqrt{x-1}}$ When $x=2$ then $u=1$ and when $x=3$ then $u=\sqrt{2}$ $\int_2^3\frac{1+x}{\sqrt{x-1}}dx=\int_1^{\sqrt{2}}(u^2+1)2du$ $=2\left[\frac{u^3}{3}+2u\right]_1^{\sqrt{2}}$ $=2\left[(\frac{2\sqrt{2}}{3}+2\sqrt{2})-(\frac{1}{3}+2)\right]$ $=\frac{2}{3}(8\sqrt{2}-7)$	4 Marks: Correct answer. 3 Marks: Correctly determines the primitive function. 2 Marks: Correctly expresses the integral in terms of u 1 Mark: Correctly finds du in terms of dx and determines the new limits.
11(d)	Coefficients are all real, hence $2-i$ is also a root (Conjugate root theorem). $\therefore(z-2+i)(z-2-i)=z^2-4z+5$ is a factor. $z^4-2z^3+az^2+bz+10=(z^2-4z+5)(z^2+2z+2)$ Therefore the other roots are $z=\frac{-2\pm\sqrt{4-4\times1\times2}}{2}$ $=-1\pm i$	2 Marks: Correct answer. 1 Mark: Recognises that $2-i$ is a root.
11(e)	$P(x)=x^4+ax^2+bx+28$ $P'(x)=4x^3+2ax+b$ Root at $x=2$ $P(2)=2^4+a\times2^2+b\times2+28=0$ $44+4a+2b=0$ (1) $P'(2)=4\times2^3+2a\times2+b=0$ $32+4a+b=0$ (2) Eqn (1) – (2) $12+b=0$ $b=-12$ Substitute $b=-12$ into Eqn (2) $32+4a-12=0$ $a=-5$ Therefore $a=-5$ and $b=-12$	2 Marks: Correct answer. 1 Mark: Shows some understanding of the problem.

12(a) (i)	$\|z\|=\sqrt{1^2+(\sqrt{3})^2}$ $=\sqrt{4}=2$ $\arg z=\frac{\pi}{3}$ 	2 Marks: Correct answer. 1 Mark: Determines $\|z\|$ or arg z.
12(a) (ii)	$z^5=2^5(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3})$ $=2^5\left[\cos(-\frac{\pi}{3})+i\sin(-\frac{\pi}{3})\right]$ $=2^5(\frac{1}{2}-i\frac{\sqrt{3}}{2})$ $=16-i16\sqrt{3}$ $\therefore a=16$ and $b=-16\sqrt{3}$	2 Marks: Correct answer. 1 Mark: Makes some progress towards the solution.
12(b)	The locus of $\|z\|=\|z-3\|$ consists of the set of points that are an equal distance from the origin as they are from the point (3, 0).  It is the perpendicular bisector of (0,0) and (3,0) on the Argand diagram.	2 Marks: Correct answer. 1 Mark: Shows some understanding.
12(c) (i)	$z\bar{z}=r(\cos\theta+i\sin\theta)\times r(\cos\theta-i\sin\theta)$ $=r^2(\cos^2\theta+\sin^2\theta)$ $=r^2$ (real)	2 Marks: Correct answer. 1 Mark: Correct expression for the conjugate
12(c) (ii)	$z^n=[r(\cos\theta+i\sin\theta)]^n$ and $\bar{z}^n=[r(\cos\theta-i\sin\theta)]^n$ $=r^n(\cos n\theta+\sin n\theta)$ $=r^n(\cos n\theta-\sin n\theta)$ $z^n+\bar{z}^n=r^n(\cos n\theta+\sin n\theta)+r^n(\cos n\theta-\sin n\theta)$ $=2r^n\cos n\theta$ (real with $n\geq1$)	2 Marks: Correct answer. 1 Mark: Uses De Moivre's theorem
12(c) (iii)	$\frac{z}{\bar{z}}+\frac{\bar{z}}{z}=\frac{z^2+\bar{z}}{z\bar{z}}$ $=\frac{2r^2\cos2\theta}{r^2}=2\cos2\theta$ (real)	1 Mark: Correct answer.

12(d)	Point of intersection of the two curves. $3-x^2=x+x^2$ $2x^2+x-3=0$ $(2x+3)(x-1)=0$ $\therefore x=1$ or $x=-1.5$ Cylindrical shell has a radius $x+1.5$. Height is $(3-x^2)-(x+x^2)=3-x-2x^2$ $V=\lim_{\delta x\to 0}\sum_{x=-1.5}^{1}2\pi rh\delta x$ $=2\pi\int_{-1.5}^{1}(x+1.5)(3-x-2x^2)dx$ $=2\pi\int_{-1.5}^{1}(3x-x^2-2x^3+4.5-1.5x-3x^2)dx$ $=2\pi\int_{-1.5}^{1}(4.5+1.5x-4x^2-2x^3)dx$ $=2\pi\left[\frac{9}{2}x+\frac{3x^2}{4}-\frac{4x^3}{3}-\frac{x^4}{2}\right]_{-1.5}^{1}$ $=2\pi\left[(\frac{9}{2}+\frac{3}{4}-\frac{4}{3}-\frac{1}{2})-(\frac{9}{2}\times1.5+\frac{3\times1.5^2}{4}-\frac{4\times1.5^3}{3}-\frac{1.5^4}{2})\right]$ $=\frac{355\pi}{48}$	4 Marks: Correct answer. 3 Marks: Correct integral for the volume of the solid. 2 Marks: Correct expression for δV. 1 Mark: Determines the radius or height of the cylindrical shell.
13(a) (i)	Asymptotes are $y=\pm\frac{b}{a}x$	1 Mark: Correct answer.
13(a) (ii)	$\tan\theta=\left\lvert\frac{m_1-m_2}{1+m_1m_2}\right\rvert$ $=\left\lvert\frac{\frac{b}{a}-(-\frac{b}{a})}{1+\frac{b}{a}\times(-\frac{b}{a})}\right\rvert$ $=\left\lvert\frac{2\frac{b}{a}}{1-\frac{b^2}{a^2}}\right\rvert=\left\lvert\frac{2\frac{b}{a}}{\frac{a^2-b^2}{a^2}}\right\rvert=\frac{2ab}{a^2-b^2}$	2 Marks: Correct answer. 1 Mark: Makes some progress towards the solution.
13(a) (iii)	Perpendicular distance from (x_0,y_0) to $bx-ay=0$ ($y=\frac{b}{a}x$) $CP=\left\lvert\frac{bx_0-ay_0}{\sqrt{b^2+(-a)^2}}\right\rvert$ Perpendicular distance from (x_0,y_0) to $bx+ay=0$ $y=-\frac{b}{a}x$ $DP=\left\lvert\frac{bx_0+ay_0}{\sqrt{b^2+(a)^2}}\right\rvert$	3 Marks: Correct answer.

13(a) (iii)	$CP\times DP=\left\lvert\frac{bx_0-ay_0}{\sqrt{b^2+(-a)^2}}\right\rvert\times\left\lvert\frac{bx_0+ay_0}{\sqrt{b^2+(a)^2}}\right\rvert$ $=\left\lvert\frac{b^2x_0^{\ 2}-a^2y_0^{\ 2}}{a^2+b^2}\right\rvert$ Now $P(x_0,y_0)$ satisfies the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. $\frac{x_0^{\ 2}}{a^2}-\frac{y_0^{\ 2}}{b^2}=1$ $\frac{b^2x_0^{\ 2}-ay_0^{\ 2}}{a^2b^2}=1$ or $b^2x_0^{\ 2}-ay_0^{\ 2}=a^2b^2$ Therefore $CP\times DP=\left\lvert\frac{b^2x_0^{\ 2}-a^2y_0^{\ 2}}{a^2+b^2}\right\rvert=\frac{a^2b^2}{a^2+b^2}$	2 Marks: Makes significant progress 1 Mark: Uses the perpendicular distance formula or makes some progress.
13(a) (iv)	Let the angle between PC and PD be α $\alpha+90^\circ+90^\circ+\theta=360^\circ$ (angles sum of quadrilateral is 360) $\alpha=180^\circ-\theta$ If $\tan\theta=\frac{2ab}{a^2-b^2}$ then $\sin\theta=\frac{2ab}{a^2+b^2}$ (Pythagoras theorem) Then $\sin(180-\theta)=\frac{2ab}{a^2+b^2}$ ($\sin\theta=\sin(180-\theta)$) $A=\frac{1}{2}CP\times DP\times\sin(180^\circ-\theta)$ $=\frac{1}{2}\times\frac{a^2b^2}{a^2+b^2}\times\frac{2ab}{a^2+b^2}$ $=\frac{a^3b^3}{(a^2+b^2)^2}$	2 Marks: Correct answer. 1 Mark: Makes some progress towards the solution.
13(b) (i)	Net force $=F-kv^2$ $m\frac{dv}{dt}=F-kv^2$ or $\ddot{x}=\frac{1}{m}(F-kv^2)$	2 Marks: Correct answer. 1 Mark: Resolving forces
13(b) (ii)	$v\frac{dv}{dx}=\frac{1}{m}(F-kv^2)$ $\frac{dv}{dx}=\frac{(F-kv^2)}{mv}$ $\frac{dx}{dv}=\frac{mv}{(F-kv^2)}$	3 Marks: Correct answer

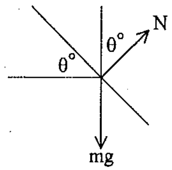
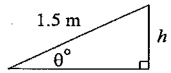
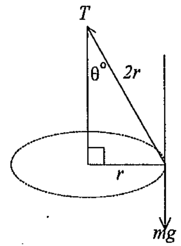
13(b) (ii)	$\int_{x_1}^{x_2} 1dx = \int_{v_1}^{v_2} \frac{mv}{(F-kv^2)}dv$ (Distance travelled is $x_2 - x_1$) $x_2 - x_1 = -\frac{m}{2k}\left[\ln(F-kv^2)\right]_{v_1}^{v_2}$ $= \frac{m}{2k}\left[\ln(F-kv^2)\right]_{v_2}^{v_1}$ $= \frac{m}{2k}\left[\ln(F-kv_1^2) - \ln(F-kv_2^2)\right]$ $x = \frac{m}{2k}\ln\left(\frac{F-kv_1^2}{F-kv_2^2}\right)$	2 Marks: Makes significant progress. 1 Mark: Finds an expression for $\frac{dx}{dv}$
13(c)	Sum of the roots $\alpha+\beta+\gamma = -\frac{b}{a} = -\frac{-6}{12}$ $\alpha+(\beta+\gamma) = \frac{1}{2}$ $\alpha+(\alpha) = \frac{1}{2}$ or $\alpha = \frac{1}{4}$	2 Marks: Correct answer. 1 Mark: Calculates the sum of the roots
14(a) (i)	$f(x) = x + \frac{1}{x-1}$ $f'(x) = 1 - \frac{1}{(x-1)^2}$ Stationary points $f'(x) = 0$ $1 - \frac{1}{(x-1)^2} = 0$ or $\frac{1}{(x-1)^2} = 1$ or $x = 0$ or 2 $f''(x) = \frac{2}{(x-1)^3}$ $f''(0) = -2 < 0$ (0,-1) is a Maxima. $f''(2) = 2 < 0$ (2,3) is a Minima	2 Marks: Correct answer. 1 Mark: Finds stationary points.
14(a) (ii)		2 Marks: Correct answer. 1 Mark: Shows vertical asymptote

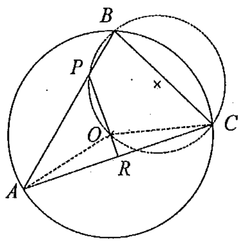
14(a) (iii)	Find the intersection of $y = -3x$ and the curve. $x + \frac{1}{x-1} = -3x$ $4x(x-1)+1 = 0$ $4x^2 - 4x + 1 = 0$ $(2x-1)^2 = 0$ $x = \frac{1}{2}$ Line intersects the curve at one point hence it is a tangent. Point of contact is $(\frac{1}{2}, -\frac{3}{2})$	2 Marks: Correct answer. 1 Mark: Makes some progress towards the solution.
14(b) (i)	$I_n = \int_0^1 \sqrt{x}(1-x)^n dx$ $= \left[\frac{2}{3}x^{\frac{3}{2}}(1-x)^n\right]_0^1 - \int_0^1 \frac{2}{3}x^{\frac{3}{2}}\{-n(1-x)^{n-1}\}dx$ $= 0 - \frac{2n}{3}\int_0^1 \{x^{\frac{1}{2}} \times -x \times (1-x)^{n-1}\}dx$ $= -\frac{2n}{3}\int_0^1 \{x^{\frac{1}{2}} \times (1-x-1) \times (1-x)^{n-1}\}dx$ $= -\frac{2n}{3}\int_0^1 \{x^{\frac{1}{2}} \times (1-x)^n - x^{\frac{1}{2}}(1-x)^{n-1}\}dx$ $= -\frac{2n}{3}(I_n - I_{n-1})$ $3I_n = -2n(I_n - I_{n-1})$ $(2n+3)I_n = 2nI_{n-1}$ $I_n = \frac{2n}{(2n+3)}I_{n-1}$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Sets up the integration and shows some understanding.
14(b) (ii)	$I_3 = \frac{6}{9}I_2$, $I_2 = \frac{4}{7}I_1$, $I_1 = \frac{2}{5}I_0$ Also $I_0 = \int_0^1 \sqrt{x}dx$ $= \frac{2}{3}\left[x^{\frac{2}{3}}\right]_0^1$ $= \frac{2}{3}$ Therefore $I_3 = \frac{6}{9} \times \frac{4}{7} \times \frac{2}{5} \times \frac{2}{3}$ $= \frac{32}{315}$	2 Marks: Correct answer. 1 Mark: Using the result from (b)(i) to obtain the definite integral.

14(c)	Area of the slice is an annulus Outer radius is 1 and inner radius is $1-\cos x$ and height x $A=\pi\left(R^2-r^2\right)$ $=\pi\left(1^2-(1-\cos x)^2\right)$ $=\pi\left(1-1+2\cos x-\cos^2 x\right)$ $=\pi\left(2\cos x-\cos^2 x\right)$ $\delta V=\delta A.\delta y$ $V=\lim_{\delta y\to 0}\sum_{y=0}^{\frac{\pi}{2}}\pi\left(2\cos x-\cos^2 x\right)\delta x$ $=\int_0^{\frac{\pi}{2}}\pi\left(2\cos x-\cos^2 x\right)dx$ $=\int_0^{\frac{\pi}{2}}\pi\left(2\cos x-\frac{1}{2}(1+\cos 2x)\right)dx$ $=\pi\left[2\sin x-\frac{x}{2}-\frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}}$ $=2\pi-\frac{\pi^2}{4}$	4 Marks: Correct answer. 3 Marks: Correct integral for the volume of the solid. 2 Marks: Correct expression for area of the annulus. 1 Mark: Recognises that the area of the slice is an annulus.
15(a) (i)	$\frac{2x}{16}+\frac{2y}{9}\times\frac{dy}{dx}=0$ $\frac{dy}{dx}=-\frac{9x}{16y}$ At $A(4\cos\theta,3\sin\theta)$ $m_1=-\frac{9\times 4\cos\theta}{16\times 3\sin\theta}=-\frac{3\cos\theta}{4\sin\theta}$ Gradient of the normal at $A(4\cos\theta,3\sin\theta)$ $m_2=\frac{4\sin\theta}{3\cos\theta}$ Equation of the normal at $A(4\cos\theta,3\sin\theta)$ $y-3\sin\theta=\frac{4\sin\theta}{3\cos\theta}\times(x-4\cos\theta)$ $3\cos\theta y-9\sin\theta\cos\theta=4\sin\theta x-16\sin\theta\cos\theta$ $4\sin\theta x-3\cos\theta y-7\sin\theta\cos\theta=0$	2 Marks: Correct answer 1 Mark: Calculates the gradient of the normal or shows some understanding.

15(a) (ii)	To find the x-intercept ($y=0$). $4\sin\theta x-7\sin\theta\cos\theta=0$ $x=\frac{7\cos\theta}{4}$ $\therefore P(\frac{7\cos\theta}{4},0)$ To find the y-intercept ($x=0$). $-3\cos\theta y-7\sin\theta\cos\theta=0$ $y=\frac{-7\sin\theta}{3}$ $\therefore Q(0,\frac{-7\sin\theta}{3})$ Let M be the midpoint of PQ or $(\frac{7\cos\theta}{8},\frac{-7\sin\theta}{6})$ Hence $x=\frac{7\cos\theta}{8}$ or $\cos\theta=\frac{8x}{7}$ and $y=\frac{-7\sin\theta}{6}$ or $\sin\theta=\frac{-6y}{7}$ Since $\cos^2\theta+\sin^2=1$ then $(\frac{8x}{7})^2+(\frac{-6y}{7})^2=1$ $\frac{64x^2}{49}+\frac{36y^2}{49}=1$ Therefore the locus of M is an ellipse.	3 Marks: Correct answer 2 Mark: Finds the midpoint of PQ or makes significant progress. 1 Mark: Finds the coordinates of P and Q.
15(b) (i)	There are 25! ways to arrange the students. But those in the group of 3 can be arranged in 3! ways, those in the group of 4 can be arranged in 4! ways, those in the group of 5 can be arranged in 5! ways and those in the group of 6 can be arranged in 6! ways. Therefore the number of arrangements is $\frac{25!}{3!4!5!6!}$	2 Marks: Correct answer. 1 Mark: Shows some understanding of the problem.
15(b) (ii)	There are 4 groups to be arranged in a circle. Number of arrangements in a circle is $(4-1)!=3!$ There are also numerous arrangements in each group. Therefore the number of arrangements is $3!\times(3!4!5!6!)$	2 Marks: Correct answer. 1 Mark: Calculates the arrangements for 4 objects in a circle.

15(c)	Test the result for $n=1$ $\frac{1}{2!}=1-\frac{1}{2!}=\frac{1}{2}$ The result is true for $n=1$ Assume the result is true for $n=k$ $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{k}{(k+1)!}=1-\frac{1}{(k+1)!}$ To prove the result is true for $n=k+1$ $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{k}{(k+1)!}+\frac{k+1}{(k+2)!}=1-\frac{1}{(k+2)!}$ $\text{LHS}=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{k}{(k+1)!}+\frac{k+1}{(k+2)!}$ $=1-\frac{1}{(k+1)!}+\frac{k+1}{(k+2)!}$ $=1-\frac{k+2-(k+1)}{(k+2)!}$ $=1-\frac{1}{(k+2)!}$ = RHS Therefore the result holds true for $n=k+1$ Thus if the result is true for $n=k$, it is true for $n=k+1$. It has been shown that the result is $n=2$, hence it is correct for $n=3$, then $n=4$ and so on.	3 Marks: Correct answer. 2 Marks: Proves the result true for $n=k+1$ 1 Mark: Tests the result for $n=2$
15(d)	${}^{n+1}C_r={}^nC_r+{}^nC_{r-1}$ $\text{LHS}={}^{n+1}C_r=\frac{(n+1)!}{(n+1-r)!r!}$ $\text{RHS}={}^nC_r+{}^nC_{r-1}$ $=\frac{n!}{(n-r)!r!}+\frac{n!}{(n-r+1)!(r-1)!}$ $=\frac{n!(n+1-r)+n!r}{(n+1-r)!r!}$ $=\frac{n!(n+1)}{(n+1-r)!r!}$ $=\frac{(n+1)!}{(n+1-r)!r!}=$ LHS	3 Marks: Correct answer. 2 Marks: Makes significant progress. 1 Mark: Uses formula for ${}^nC_r$

16(a)	Resolving forces $N\cos\theta-mg=0$ (1) $N\sin\theta=m\frac{v^2}{r}$ (2)  Eqn (2) ÷ (1) $\tan\theta=\frac{v^2}{rg}$ $=\frac{13.\dot{8}^2}{800\times 9.8}=0.0246047...$ $\sin\theta=\theta=\tan\theta$ (small angles)  $\tan\theta=\frac{h}{1.5}$ $h=1.5\times 0.0246027$ $=0.03690712...\text{m}$ ≈ 36.9 mm	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Resolves forces in the vertical and horizontal directions.
16(b)	Body moving in a horizontal circle. $T\cos\theta-mg=0$ and $T\sin\theta=mr\omega^2$ Dividing the two equations $\frac{T\sin\theta}{T\cos\theta}=\frac{mr\omega^2}{mg}$ or $\tan\theta=\frac{r\omega^2}{g}$ Also $\tan\theta=\frac{r}{\sqrt{4r^2-r^2}}$ $=\frac{r}{\sqrt{3}r}=\frac{1}{\sqrt{3}}$ $\therefore\frac{r\omega^2}{g}=\frac{1}{\sqrt{3}}$ $\omega^2=\frac{g}{r\sqrt{3}}$  	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Resolves forces in the vertical and horizontal directions.
16(c)	If $a-b>0$ then $a^2+b^2\geq 2\sqrt{2}(a-b)$ Squaring both sides $(a^2+b^2)^2\geq 8(a^2+b^2-2ab)$ $\geq 8(a^2+b^2-2)$ $(a^2+b^2)^2-8(a^2+b^2)+16\geq 0$ This quadratic inequation is true if $\Delta\geq 0$. $\Delta=(-8)^2-4\times 1\times 64$ $=0$	3 Marks: Correct answer. 2 Marks: Makes significant progress. 1 Mark: Squares both sides or shows some understanding

<table>
<tr><td>16(d)
(i)</td><td>

$\angle COR = \angle ABC$ (Exterior angle of cyclic quadrilateral is equal to the interior opposite angle)
$\angle AOC = 2\angle ABC$ (Angle subtended at the centre is twice the angle at the circumference standing on the same arc)
$\angle AOC = \angle AOR + \angle COR$ (Adjacent angles)
$\therefore \angle AOR = \angle ABC$</td><td>3 Marks: Correct answer.

2 Marks: Makes significant progress towards the solution.

1 Mark: States a relevant circle geometry property.</td></tr>
<tr><td>16(d)
(ii)</td><td>Consider ΔAOR and ΔCOR
$\angle AOR = \angle COR$ (both equal to $\angle ABC$ from above)
$AO = CO$ (radii of circle ABC)
$OR = OR$ (common side)

$\therefore \Delta AOR \equiv \Delta COR$ (SAS)
$AR = CR$ (matching sides in congruent triangles)</td><td>3 Marks: Correct answer.

2 Marks: Makes significant progress

1 Mark: States a relevant property.</td></tr>
</table>