ASCHAM SCHOOL

YEAR 12
MATHEMATICS EXTENSION 1
TRIAL EXAMINATION 2013

GENERAL INSTRUCTIONS

5 minutes reading time.
Working time **2 hours**.
Use black or blue pen.
A table of standard integrals is provided on the back page.
Approved calculators and templates may be used.

Total Marks - 70

Section 1 – MULTIPLE CHOICE (1 mark each)

- Attempt Questions 1-10.
- Allow 15 minutes.
- Answers on the separate sheet provided.
- Write your name/**BOS number**, teacher's name.

Section 2– Question 11 – 14 (15 marks each)

- Allow 1 hour 45 minutes.
- Start each question in a new booklet.
- If you use a second booklet for a question, place it inside the first.
- Write your name/**BOS number**, teacher's initials and question number on each booklet.

STUDENT NUMBER/NAME: ....

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Section 1 Multiple choice (1 mark each)

(Mark the correct answer on the sheet provided)

1. $\frac{d}{dx}(\cot x)$ is

(A) $\sqrt{1+\tan^2 x}$

(B) $2\sec^2 x \tan x$

(C) $-\cot x \tan x$

(D) $-\text{cosec}^2 x$

2. Solve the inequality $\frac{3}{3x+2} \geq 1$

(A) $x \neq \frac{-2}{3}$ or $x = \frac{1}{3}$

(B) $\frac{-2}{3} \leq x \leq \frac{1}{3}$

(C) $\frac{-2}{3} < x \leq \frac{1}{3}$

(D) $x < \frac{-2}{3}$ or $\frac{1}{3} \leq x$

3. The point M divides the interval PQ externally in the ratio 7: 2. Find M if the coordinates of P and Q are (-3,2) and (7,-3) respectively.

(A) $M = (11, -5)$

(B) $M = (-11, -5)$

(C) $M = (11, 5)$

(D) $M = (-11, 5)$

4. The acceleration of a particle moving in a straight line is given by $\ddot{x} = -9x + 18$, where its displacement from a fixed point O is x m. The motion is simple harmonic. What is the centre of the motion and the period?

(A) centre $= -2$ and period $= \frac{2\pi}{3}$

(B) centre $= 18$ and period $= \frac{2\pi}{9}$

(C) centre $= 2$ and period $= \frac{2\pi}{3}$

(D) centre $= -18$ and period $= 2\pi$

5. What is the domain and range of $f(x) = 3\sin^{-1}(1-2x)$?

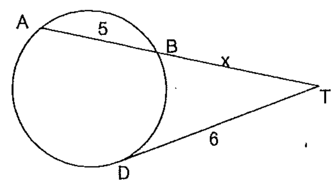
(A) Domain: $-1 \leq x \leq 1$. Range: $\frac{-\pi}{6} \leq y \leq \frac{-\pi}{6}$

(B) Domain: $0 \leq x \leq 1$. Range: $0 \leq y \leq \pi$

(C) Domain: $-1 \leq x \leq 1$. Range: $\frac{-\pi}{2} \leq y \leq \frac{-\pi}{2}$

(D) Domain: $0 \leq x \leq 1$. Range: $\frac{-3\pi}{2} \leq y \leq \frac{3\pi}{2}$

6. The line DT is a tangent to the circle at D and AT is a secant meeting the circle at A and B. Given that DT = 6, AB = 5 and BT = x, find the value of x.



(A) $x = 4$

(B) $x = 5$

(C) $x = 6$

(D) $x = 9$

7. If $f(x)=2\ln(x-3)$ what is the inverse function $f^{-1}(x)$?

(A) $f^{-1}(x)=e^{\frac{x}{2}+3}$

(B) $f^{-1}(x)=e^{\frac{x}{2}}+3$

(C) $f^{-1}(x)=e^{x}+3$

(D) $f^{-1}(x)=e^{x}-3$

8. Which of the following is the correct expression for $\int\frac{dx}{4+3x^2}$?

(A) $\frac{\ln(4+3x^2)}{3}$

(B) $\tan^{-1}\frac{\sqrt{3}x}{2}+c$

(C) $\frac{1}{2\sqrt{3}}\tan^{-1}\frac{2x}{\sqrt{3}}+c$

(D) $\frac{1}{2\sqrt{3}}\tan^{-1}\frac{\sqrt{3}x}{2}+c$

9. Given that $f(x)=3\sin^{-1}x+3\cos^{-1}x$, by finding the derivative of $f(x)$ determine the unique value of $f(x)$?

(A) $f(x)=0$

(B) $f(x)=6$

(C) $f(x)=\frac{3\pi}{2}$

(D) $f(x)=6\pi$

10. The integral of $\sin^2 3x$ is ?

(A) $\frac{\sin^3 3x}{3}+C$

(B) $\frac{x}{2}+\frac{\sin 6x}{6}+C$

(C) $\frac{x}{2}-\frac{\sin 6x}{6}+C$

(D) $\frac{x}{2}-\frac{\sin 6x}{12}+C$

End of Multiple choice. Question 11 begins on the next page

Question 11 **Begin and label a new booklet.** **15 marks**

(a) Evaluate $\lim_{x\to 0}\frac{\sin\frac{x}{2}}{3x}$ 1

(b) Find the gradient of the tangent to the curve $y=\tan^{-1}2x$ at the point $x=1.5$ 2

(c) Find $\frac{d}{dx}(\cos(\tan x))$ 1

(d) Using the substitution $u=x-1$ evaluate $\int_2^5 x\sqrt{x-1}\,dx$. 3

(e) Find the remainder when $P(x)=x^3-5x^2+8x-4$ is divided by $x+3$ 1

(f) If α,β,γ are zeros of $P(x)=x^3+3x^2-2x+1$ find $\frac{2}{\alpha}+\frac{2}{\beta}+\frac{2}{\gamma}$ 2

(g) Write down the equation of the normal to the parametric equations below at $t=-2$, giving your answer in Cartesian form. $x=t+2$ and $y=\frac{t^2}{4}+1$ 3

(h) Find the acute angle between the lines correct to the nearest minute: $x-3y=2$ and $3x-y=4$ 2

Question 12 begins on the next page

Question 12 Begin and label a new booklet. 15 marks

(a) Calculate the term independent of x in the expansion of $\left(x+\frac{1}{x^2}\right)^6$ **2**

(b) (i) Show that $f(x)=2\sin x-10x+5$ has a root between 0.6 and 0.7 **1**

(ii) Use one step of Newton's method and starting with $x=0.6$, find a better approximation. Answer to 2 decimal places. **2**

(c) Prove by mathematical induction, that $3^{3n}+2^{n+2}$ is divisible by 5 for all positive integers n. **3**

(d) By finding any intercepts and asymptotes, draw the graph of $y=\frac{x}{x^2-9}$ showing significant features. *(do not use calculus)* **3**

(e) (i) Express $2\cos x+2\sqrt{3}\sin x$ in the form $R\cos(x-\alpha)$, where $R>0$ and $0<\alpha<\frac{\pi}{2}$. **2**

(ii) Hence solve the equation $2\cos x+2\sqrt{3}\sin x=2$ for $0\le x\le 2\pi$ **1**

(iii) Write down the least value of $\frac{1}{2\cos x+2\sqrt{3}\sin x}$ for $0<x<\frac{\pi}{2}$ **1**

Question 13 begins on the next page

Question 13 Begin and label a new booklet. 15 marks.

(a) Consider the function $f(x)=\cos^{-1}2x$

(i) Sketch the function for $-0.5\le x\le 0.5$ **1**

(ii) Find the volume formed by revolving $y=f(x)$ about the y-axis from $0\le y\le \pi$ **3**

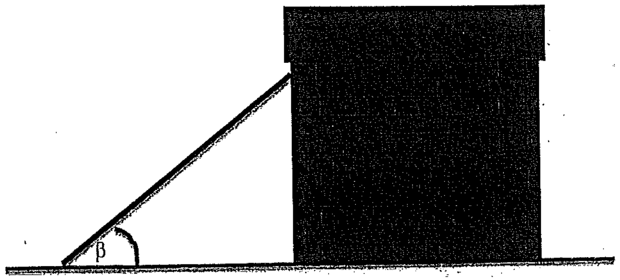
(b) A particle moves in a straight line so that its velocity as a function of x is given by the equation $v^2=x^2(16-x^2)$

(i) Show the acceleration $\ddot{x}$ is equal to $\ddot{x}=2x(8-x^2)$ **2**

(ii) Initially the particle is 4 cm to the left of the origin. Find the direction in which the particle travels immediately after leaving this position giving reasons for your answer. **1**

(iii) Find the maximum speed of the particle. **1**

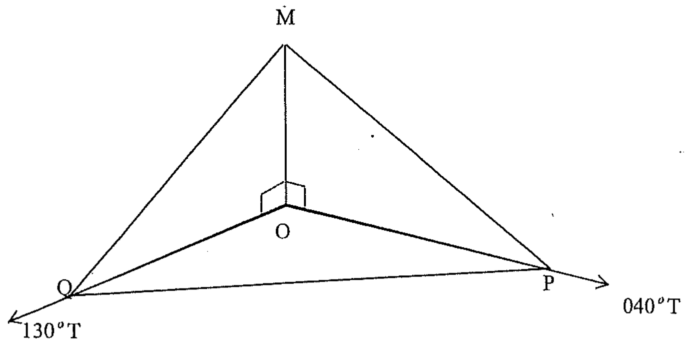
(c) A 25 metre ladder rests with one end against a vertical wall and the other end on horizontal ground which is level with the base of the wall. The end in contact with the ground slips away from the wall at a constant rate of 20 cm/s. Find the rate at which the angle β, between the ladder and the ground is decreasing when the end of the ladder is 10 m from the wall. **3**



Question 13 continues on the next page

Question 13

(d)



A woman walks along the road from P to Q and sees a monument OM in the distance. P and Q have bearings 040°T and 130°T respectively.
The angle of elevation from P to the top of the monument M is 35° and the angle of elevation from Q to the top of the monument M is 50°

The monument OM is of height h metres and its base O lies in the same horizontal plane as POQ.

Copy and complete the diagram on your page showing *all* relevant information.

(i) Show $OP = h\tan 55^\circ$ **1**

(ii) Find the size of angle POQ **1**

(iii) Show that $PQ = h\sqrt{\tan^2 55^\circ + \tan^2 40^\circ}$ **1**

(iv) If $PQ = 400$ metres find h (to the nearest m). **1**

Question 14 begins on the next page

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Question 14 **Begin and label a new booklet.** **15 marks**

(a) The following equations of motion may be assumed in this question:

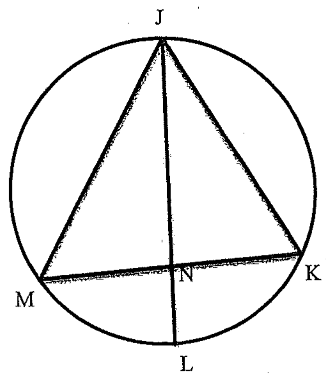
	Horizontal	*Vertical*
Acceleration	$\ddot{x} = 0$	$\ddot{y} = -10$
Velocity	$\dot{x} = V\cos\theta$	$\dot{y} = -10t + V\sin\theta$
Position	$x = Vt\cos\theta$	$y = -5t^2 + Vt\sin\theta$

A particle is projected at a speed of V m/s at an angle of projection θ to the horizontal and $1\frac{1}{2}$ seconds later passes through the point P whose horizontal distance from the point of projection is 36 m and whose vertical height from the point of projection is $3\frac{3}{4}$ m.

(i) Find the speed V and show that the angle of inclination $\theta = \tan^{-1}\left(\frac{5}{12}\right)$. **3**

(ii) Find the range of the projectile. **2**

(b)



In the diagram JKLM lie on the circle. The chord JL bisects ∠KJM and cuts the chord MK at N.

Copy and complete the diagram on your page showing ALL relevant information.

(i) Prove $\Delta JML \,|||\, \Delta JNK$ **3**

(ii) Show that JM × JK = JN × JL **1**

Question 14 continues on the next page

Question 14

(c) A water tank is being filled at a variable rate and the change in volume of water in the tank is given by $\frac{dV}{dt} = k(F - V)$, where F is some constant.

(i) Show that $V = F - Fe^{-kt}$ is a solution to the differential equation. **1**

(ii) What is $\lim_{t\to\infty} V$ and sketch the graph. **1**

(iii) If one quarter of the tank is filled in 10 minutes, find k. **2**

(iv) Following on from part (iii) what fraction of the tank is filled in the next 10 minutes? **2**

The end of the paper

Standard Integrals

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{if } n < 0$

$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$

$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$

$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$

$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$

$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$

NOTE: $\ln x = \log_e x, \quad x > 0$

Solutions to Ascham Extension Trial 2013 BP KS

1 D	✓
2 C	✓
3 A	✓
4 C	✓
5 D	✓
6 A	✓
7 B	✓
8 D	✓
9 C	✓
10 D	✓

(10)

Working:

1. $\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$ or $\frac{d}{dx}(\tan)^{-1} = -1(\tan x)^{-2} \times \sec^2 x$

$= -\frac{\sec^2 x}{\tan^2 x}$

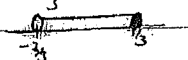
$= -\frac{\frac{1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}} = -\frac{1}{\sin^2 x} = -\text{cosec}$

2. 2 Methods:

C.V method: $x \neq -\frac{2}{3}$

solve $3 = 3x+2$

$\frac{1}{3} = x$



Test $x=0$; $\frac{3}{2} \geq 1$ ✓

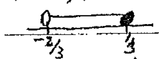
or multiply by denominator squared (Note $x = -\frac{2}{3}$)

$(3x+2)3 \geq (3x+2)^2$

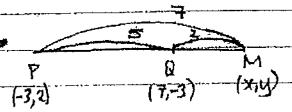
$0 \geq (3x+2)^2 - 3(3x+2)$

$0 \geq (3x+2)(3x+2-3)$

$0 \geq (3x+2)(3x-1)$



3.



solve for x

$7 = \frac{5x + 2(-3)}{7}$

$49 = 5x - 6$

$55 = 5x$

$x = 11$

solve for y

$-3 = \frac{5y + 2(2)}{7}$

$-21 = 5y + 4$

$-25 = 5y$

$y = -5$

4. $\ddot{x} = -9(x-2)$

$\ddot{x} = -n^2(x - x_0)$

$\therefore n = 3$, $x = 2$, period $= \frac{2\pi}{n}$

centre $= 2$, period $= \frac{2\pi}{3}$

5. $f(x) = 3\sin^{-1}(1-2x)$

D: $-1 \leq 1-2x \leq 1$

$-2 \leq -2x \leq 0$

$0 \leq x \leq 1$

R: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

R: $-\frac{3\pi}{2} \leq 3y \leq \frac{3\pi}{2}$

6. $\frac{6}{x} = \frac{x+5}{6}$

$36 = x(x+5)$

$x^2 + 5x - 36 = 0$

$(x-4)(x+9) = 0$

$x = +4$ or $x = -9$

7. $y = 2\ln(x-3)$

interchange x & y

$x = 2\ln(y-3)$

make 'y' the subject

$\frac{x}{2} = \ln(y-3)$

$e^{x/2} = y - 3$

$y = e^{x/2} + 3$

8. $\int \frac{dx}{4+3x^2} = \int \frac{dx}{3(\frac{4}{3} + x^2)}$

$= \frac{1}{3}\int \frac{dx}{(\frac{2}{\sqrt{3}})^2 + x^2}$

$= \frac{1}{3} \cdot \frac{1}{\frac{2}{\sqrt{3}}} \tan^{-1}\frac{x}{\frac{2}{\sqrt{3}}} +$

$= \frac{1}{2\sqrt{3}} \tan^{-1}\frac{\sqrt{3}x}{2} + c$

9. $f(x) = 3\sin^{-1}x + 3\cos^{-1}x$

$f'(x) = \frac{3}{\sqrt{1-x^2}} + \frac{-3}{\sqrt{1-x^2}} = 0$

$\therefore f(x) =$ constant let $x=1$; $x=0$

$f(1) = 3\sin^{-1}1 + 3\cos^{-1}1$

$= \frac{3\pi}{2}$

$f(0) = 3\sin^{-1}0 + 3\cos^{-1}0$

$= \frac{3\pi}{2}$

10. $\cos 2\theta = 1 - 2\sin^2\theta$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\sin^2 3x = \frac{1 - \cos 6x}{2}$

$\frac{1}{2}\int 1 - \cos 6x \, dx = \frac{1}{2}\left(x - \frac{\sin 6x}{6}\right) + c$

$= \frac{x}{2} - \frac{\sin 6x}{12} + c$

Question 11 15 marks CAO (Correct Answer Only)

(a) $\lim_{x \to 0} \frac{\sin \frac{x}{2}}{3x} = \frac{1}{3}\lim_{x\to 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}}\left(\frac{1}{2}\right)$

$= \frac{1}{6}\lim_{x\to 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}}$

$= \frac{1}{6}$ ✓(1) CAO

(b) $y = \tan^{-1}(2x)$

$\frac{dy}{dx} = \frac{1}{1+(2x)^2} \times 2$ ✓1

at $x = 1.5$ (-1) if ×2 missed

$\frac{dy}{dx} = \frac{2}{1+(3)^2}$

$= \frac{2}{10}$

$\frac{dy}{dx} = \frac{1}{5}$ (or 0.2) ✓1 (2)

(c) $\frac{d}{dx}\cos(\tan x) = -\sin(\tan x) \cdot \frac{d}{dx}(\tan x)$

$= -\sin(\tan x)\sec^2 x$ ✓(1)

(d) let $u = x - 1$

limits when $x = 2$, $u = 1$

$x = 5$ $u = 4$

$\frac{du}{dx} = 1$

$\int_2^5 x\sqrt{x-1}\,dx = \int_1^4 (u+1)\sqrt{u}\,du$ ✓1 li… s…

$= \int_1^2 u^{3/2} + u^{1/2}\,du$

$= \left[\frac{2u^{5/2}}{5} + \frac{2u^{3/2}}{3}\right]_1^4$ ✓1 co… inte…

$= \left(\frac{2}{5}\times 4^{5/2} + \frac{2}{3}\times 4^{3/2}\right) - \left(\frac{2}{5} +\right.$

$= \left[\frac{2}{5}\times 32 + \frac{2}{3}\times 8\right] - \left[\frac{16}{15}\right.$

$= \left[\frac{1}{15}(6\times 32 + 10\times 8)\right] - \frac{16}{15}$

$=$

$= \frac{272 - 16}{15}$

$=$

$= \frac{256}{15}$

$= 17\frac{1}{15}$ ✓1 correct answer

(e) $P(x) = x^3 - 5x^2 + 8x - 4$

$P(-3) = (-3)^3 - 5(-3)^2 + 8(-3) - 4$

$= -27 - 45 - 24 - 4$

$= -100$ ✓ 1

(f) $P(x) = x^3 + 3x^2 - 2x + 1$

$\left.\begin{aligned} \alpha+\beta+\gamma &= -3 \\ \alpha\beta+\beta\gamma+\alpha\gamma &= -2 \\ \alpha\beta\gamma &= -1 \end{aligned}\right\}$ ✓1

Now, $\frac{2}{\alpha} + \frac{2}{\beta} + \frac{2}{\gamma} = 2\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)$

$= 2\left(\frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}\right)$

$= 2\left(\frac{-2}{-1}\right)$

$= 4$ ✓

2

(g) $x = t + 2$ $\quad y = \frac{t^2}{4} + 1$

$\frac{dx}{dt} = 1$ $\quad \frac{dy}{dt} = \frac{2t}{4}$

$\frac{dy}{dt} = \frac{t}{2}$

at $t = -2$ $\frac{dy}{dx} = -\frac{2}{2}$ ✓1

$\therefore \frac{dy}{dx} = -1$

$\therefore$ grad of Normal is 1, $t = -2$, $x = 0$, $y = 2$ ✓1

Equation of normal is $y - y_1 = 1(x - x_1)$

$y - 2 = 1(x - 0)$

$y = x + 2$ ✓1

3

Alternative Solution

$y = \frac{(x-2)^2}{4} + 1$

$\frac{dy}{dx} = \frac{2(x-2)^1}{4}$

when $t = -2$, $x = 0$

$\frac{dy}{dx} = -1$

(h) $\left.\begin{aligned} x - 3y = 2;\ m_1 &= \tfrac{1}{3} \\ 3x - y = 4;\ m_2 &= +3 \end{aligned}\right\}$ ½

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{\frac{1}{3} - 3}{1 + \frac{1}{3}\cdot 3}\right|$ ✓½

$= \frac{8/3}{2}$

$= 4/3$ ✓

$\theta = \tan^{-1}(4/3)$

$\theta = 53.13\ldots$ ✓½

$\theta = 53^\circ 8'$ ✓ (nearest minute)

2

Question 12 15 marks

(a) $\left(x + \frac{1}{x^2}\right)^6 = \sum_{r=0}^{6} {}^6C_r\,(x)^{6-r}\left(\frac{1}{x^2}\right)^r$

general term: ${}^6C_r\ x^{6-r}\,(x^{-2})^r$

${}^6C_r\ x^{6-3r}$ ✓1 working

$\therefore$ term independent of x; $6 - 3r = 0$

$r = 2$

term is ${}^6C_2 = \frac{6 \times 5}{2 \times 1}$

$= 15$ ✓1

Alternative Solution using Pascal's △

1 6 15 20 15 6

$1x^6 + 6x^5\frac{1}{x^2} + 15x^4\frac{1}{x^4} + 20x^3\frac{1}{x^6} +$

Term independent of x

is 15

2

(b) (i) $f(x) = 2\sin x - 10x + 5$

$f(0.6) = 2\sin 0.6 - 10(0.6) + 5$

$= 0.129$ (3dp)

$f(0.7) = 2\sin 0.7 - 10(0.7) + 5$

$= -0.712$ (3dp) ½

as $f(x)$ is continuous and $f(0.6) > 0$ and $f(0.7) < 0$ then there lies a root between $x = 0.6$ and $x = 0.7$ ½ explanation

(ii) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$x_1 = 0.6$

$f(x) = 2\sin x - 10x + 5$

$f'(x) = 2\cos x - 10$ ✓½

$f(0.6) = 0.129$ (3dp)

$f'(0.6) = 2\cos 0.6 - 10$

$= -8.349$ (3dp) ½

$x_2 = 0.6 - \frac{0.129}{-8.349}$ ✓½

$x_2 = 0.6 + 0.015$

$x_2 = 0.615$

$x_2 = 0.62$ (2dp) ✓½

2

(c) Prove $3^{3n}+2^{n+2}$ is divisible by 5, by Mathematical Induction

for $n=1$; $3^3+2^3 = 27+8$
$= 35$ (which is divisible by 5)
$= 7\times 5$

$\therefore$ true for $n=1$ ✓½

Assume true for $n=k$

$3^{3k}+2^{k+2} = 5p$ (for some integer p) ie. $3^{3k} = 5p - 2^{k+2}$

Now for $n=k+1$

$$\left.\begin{aligned} 3^{3(k+1)}+2^{k+1+2} &= 3^{3k+3}+2^{k+3} \\ &= 3^{3k}3^3+2^k2^3 \\ &= (5p-2^{k+2})3^3+2^k2^3 \quad \text{(from above)} \\ &= 3^3(5p) - 2^k2^2 3^3 + 2^k 2^3 \\ &= 27(5p) - 2^k(4\times 27 - 8) \\ &= 5(27p) - 2^k(108-8) \\ &= 5(27p) - 5(2^k)(20) \\ &= 5(27p - 20\times 2^k) \end{aligned}\right\} \text{2 for working}$$

this expression is divisible by 5. ✓½ (3)

$\therefore$ By the principle of Mathematical Induction, this statement holds for all.

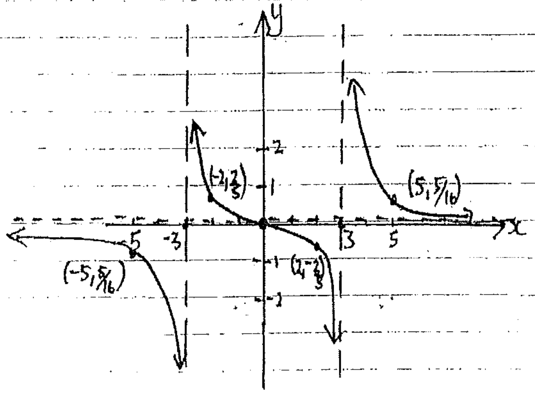
(d) $y = \dfrac{x}{x^2-9}$ root at $x=0$, (when $x=0, y=0$.)
asymptotes at $x=3$ or $x=-3$

$y=f(x)$ is an odd function

$f(a) = \dfrac{a}{a^2-9}$

$f(-a) = \dfrac{-a}{(-a)^2-9}$

$\therefore f(a) = -f(-a)$ odd fn



✓1 asymptotes

✓1 shape (3)

✓1 working (as above)

(e) **(i)** $2\cos x + 2\sqrt{3}\sin x = R\cos(x-\alpha)$
$= R\cos x\cos\alpha + R\sin x\sin\alpha$

equating parts.

$R\cos\alpha = 2$ ...①
$R\sin\alpha = 2\sqrt{3}$ ...②

1 working
½ for R
½ for α
(2)

② ÷ ① ⇒ $\therefore \tan\alpha = \sqrt{3}$
$\alpha = \frac{\pi}{3}$ $(0<\alpha<\frac{\pi}{2})$

①² + ②² ⇒ $4+12 = R^2(\cos^2\alpha+s$
$R^2 = 16$
$R = 4$, $R>0$

$\therefore R\cos(x-\alpha) = 4\cos(x-\frac{\pi}{3})$

(ii) $4\cos(x-\frac{\pi}{3}) = 2$
$\cos(x-\frac{\pi}{3}) = \frac{1}{2}$

(−½ each solution mis
or each error)

$(x-\frac{\pi}{3}) = -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$

$x = 0, \frac{2\pi}{3}, 2\pi$ ✓(1) for all 3 solutions

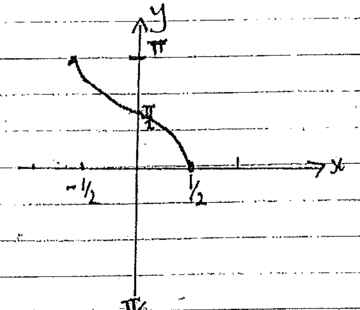
(iii) max value of $4\cos(x-\frac{\pi}{3})$ is 4

$\therefore$ min value of $\dfrac{1}{2\cos x+2\sqrt{3}\sin x} = \dfrac{1}{4\cos(x-\frac{\pi}{3})}$
$= \dfrac{1}{4}$ ✓(1) CAO

Question 13 15 marks

(a) $f(x) = \cos^{-1} 2x$

(i)



✓½ domain + range
✓½ shape } ①

(ii) $V = 2\pi \int_0^{\pi/2} x^2 \, dy$

$y = \cos^{-1} 2x$

$\cos y = 2x$

$\left(\frac{\cos y}{2}\right)^2 = x^2$

$\frac{1}{4}\cos^2 y = x^2$

$= \frac{2\pi}{4}\int_0^{\pi/2} \cos^2 y \, dy$

$\cos 2y = 2\cos^2 y - 1$

$\frac{\cos 2y + 1}{2} = \cos^2 y$

} ✓1 for working

$= \frac{\pi}{2}\int_0^{\pi/2} \frac{\cos 2y + 1}{2} \, dy$

$= \frac{\pi}{4}\left[\frac{\sin 2y}{2} + y\right]_0^{\pi/2}$ } ✓1 correct integration

$= \frac{\pi}{4}\left[\left(\frac{\sin \pi}{2} + \frac{\pi}{2}\right) - \left(\frac{\sin 0}{2} + 0\right)\right]$

$= \frac{\pi^2}{8}$ units^3 ✓1 ③

(b)(i) $v^2 = x^2(16 - x^2)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2}\frac{d}{dx}(16x^2 - x^4)$

$= \frac{1}{2}(32x - 4x^3)$ ✓1

$= \frac{4x}{2}(8 - x^2)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x(8 - x^2)$ ✓1 ②

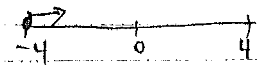
$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x}$

$\therefore \ddot{x} = 2x(8 - x^2)$ (as required)

(bii) particles stops when $v = 0$

$0 = x^2(16 - x^2)$

$\therefore x = -4, 0, 4$ cm



initially $x = -4$, $t = 0$, $v = 0$

particle can only move between $x = -4$ and $x = 4$

particle stops moving at $x = -4$, $x = 0$, and $x = 4$ and changes direction.

$\therefore$ particle is moving to the right (towards $x = 0$) ✓①

also acceleration is positive, just after leaving $x = -4$

Explanation

(biii) Max speed when $\ddot{x} = 0$

$2x(8 - x^2) = 0$

$x = 0$, $x = -2\sqrt{2}$, $x = 2\sqrt{2}$ ✓½

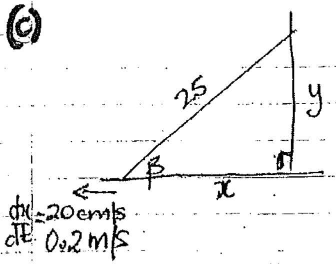
when $x = 0$; $v = 0$

$x = \sqrt{8}$, $v^2 = 8(16 - 8)$

$x = \sqrt{8}$, $v^2 = 8(16 - 8)$ } $v^2 = 64$; $v = \pm 8$ cm/s

$\therefore$ max speed $|v| = 8$ cm/s ✓½ ①

(c)



$y = \sqrt{25 - x^2}$

$\cos\beta = \frac{x}{25}$

$\beta = \cos^{-1}\left(\frac{x}{25}\right)$

$\frac{d\beta}{dx} = \frac{-1}{\sqrt{1-\left(\frac{x}{25}\right)^2}} \times \frac{1}{25}$ ✓1 setting up problem + differentiation

when $x = 10$. $\frac{d\beta}{dx} = \frac{-1}{\sqrt{1-\left(\frac{2}{5}\right)^2}} \times \frac{1}{25}$

$= \frac{-1}{\sqrt{\frac{25-4}{25}}} \times \frac{1}{25}$

$= \frac{-5}{\sqrt{21}} \times \frac{1}{25}$

$= \frac{-1}{5\sqrt{21}}$ ✓½ correct substitution

$\frac{d\beta}{dt} = \frac{d\beta}{dx} \times \frac{dx}{dt}$

$= \frac{-1}{5\sqrt{21}} \times 0.20$ ✓1 chain rule

$= \frac{-1}{25\sqrt{21}}$ rads/sec

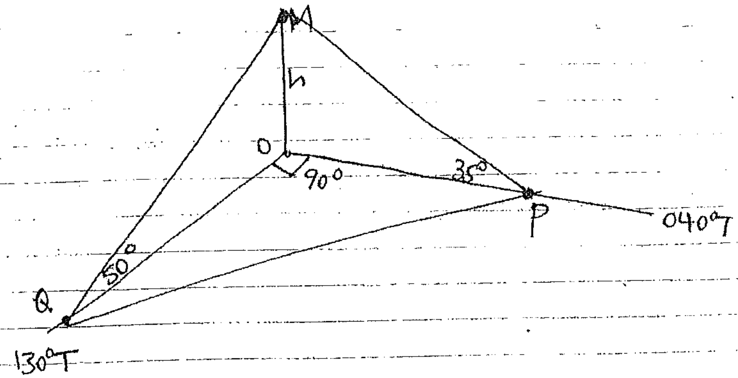
$= \frac{-1}{25\sqrt{21}} \times \frac{180}{\pi}$ degrees/sec

accept either rads/deg.

✓½ CAO

3

(d)



(i) $\tan 35° = \frac{h}{OP}$

$OP = \frac{h}{\tan 35°}$

$= h \cot 35°$ ✓1

$OP = h \tan 55°$ (as required)

(ii) $\angle POQ = 130° - 40°$

$= 90°$

(angle between bearings)

✓1

(iii) similarly $\tan 50° = \frac{h}{OQ}$

$OQ = \frac{h}{\tan 50°}$

$OQ = h \tan 40°$ ✓½

By Pythagoras' theorem $PQ^2 = h^2\tan^2 55° + h^2\tan^2 40°$ ✓½ 1

$PQ = h\sqrt{\tan^2 55° + \tan^2 40°}$ as required.

(iv) $PQ = 400$

$400 = h\sqrt{\tan^2 55° + \tan^2 40°}$

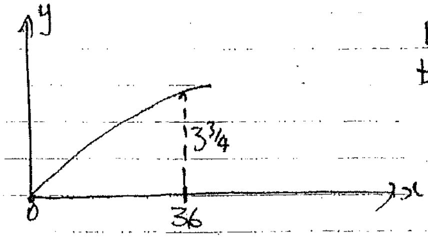
$\frac{400}{\sqrt{\tan^2 55 + \tan^2 40}} = h$

$h = 241.486\ldots$

$h = 241$ m (nearest m) ✓1 CAO

Question 14 15 marks

(a)



$t=0,\ x=0,\ y=0,\ v=V$ m/s
$t=1\frac{1}{2},\ x=36,\ y=3\frac{3}{4}$

(i) $t=1\frac{1}{2},\ x=36$

$36 = V\cdot\frac{3}{2}\cos\theta$ ①

$24 = V\cos\theta$ .. ① ✓½

$t=1\frac{1}{2},\ y=\frac{15}{4}$

$\frac{15}{4} = -5\left(\frac{3}{2}\right)^2 + V\left(\frac{3}{2}\right)\sin\theta$

$\frac{15}{4} = -\frac{45}{4} + \frac{3V}{2}\sin\theta$ ✓½

$15 = \frac{3V}{2}\sin\theta$

$10 = V\sin\theta$ .. ② ✓½

$①^2 + ②^2 \Rightarrow 24^2+10^2 = V^2\cos^2\theta + V^2\sin^2\theta$

$26^2 = V^2(\cos^2\theta+\sin^2\theta)$

$V = \pm 26$ m/s

speed $= |V| = 26$ m/s ✓1

$② \div ① \Rightarrow \frac{V\sin\theta}{V\cos\theta} = \frac{10}{24}$

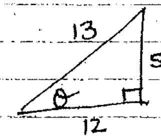
$\tan\theta = \frac{5}{12}$ ✓½ (3)

$\theta = \tan^{-1}\left(\frac{5}{12}\right)$ (as required)

(ii) To find range, $y=0$,

$\theta = \tan^{-1}\left(\frac{5}{12}\right)$

$V = 26$



$\sin\theta = \frac{5}{13}$
$\cos\theta = \frac{12}{13}$

$0 = -5t^2 + 26t\times\frac{5}{13}$

$0 = -5t^2 + 10t$

$0 = 5t(2-t)$ ✓1 working

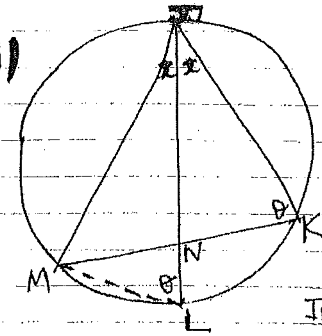
$t=0$ or $t=2$,

when $t=2$, Range $= x = Vt\cos\theta$ (2)

$= 26\times 2\times\frac{12}{13}$ ✓1 CAO

$= 48$

(b) (i)



let $\angle MJN = \angle NJK = x$
(given) JL bisects $\angle KJM$
join LM
} 1 const + working

In $\triangle JML$ and $\triangle JNK$ ✓
$\angle MJL = \angle NJK = x$ ✓ (given)
$\angle JLM = \angle JKN = \theta$ ✓ (equal angles subtended by arc JM)
$\therefore \triangle JML \parallel \triangle JNK$ (equiangular)

(ii) as the $\triangle$'s above are similar, their sides are in proportion.

$\frac{JK}{JL} = \frac{NK}{ML} = \frac{JN}{JM}$ ✓½

$\therefore \frac{JK}{JL} = \frac{JN}{JM}$ ✓½ (1)

$JM\times JK = JN\times JL$ (as required)

(c) $\frac{dV}{dt} = k(F-V)$

(i) To show $V = F - Fe^{-kt}$ is a solution, differentiate equation

$$\frac{d}{dt}V = \frac{d}{dt}\left(F - Fe^{-kt}\right)$$

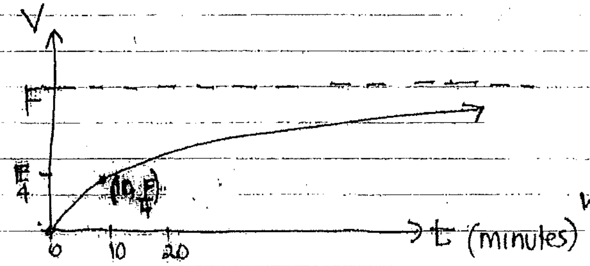
$$\frac{dV}{dt} = 0 - -kFe^{-kt}$$

$$= kFe^{-kt} \quad \text{but } F - V = Fe^{-kt}$$

$$\frac{dV}{dt} = k(F-V) \text{ as required}$$ ✓ 1

(ii) $\lim_{t\to\infty} V = \lim_{t\to\infty}\left(F - Fe^{-kt}\right)$ $\quad e^{-kt} \to 0$ as $t \to 0$

$$= F$$ ✓ 1/2



✓ 1/2 1

(iii) $t = 10$, $V = \frac{F}{4}$,

$$\therefore V = F - Fe^{-kt}$$

$$\frac{F}{4} = F - Fe^{-10k}$$

$$Fe^{-10k} = \frac{3F}{4}$$ ✓1 working

$$e^{-10k} = \frac{3}{4}$$

$$-10k = \ln\frac{3}{4}$$

$$k = \frac{-1}{10}\ln(3/4)$$ ✓1

2

(iv) when $t = 20$

$V = F - Fe^{-kt}$ $\quad k = -\frac{1}{10}\ln\left(\frac{3}{4}\right)$

$$V = F - Fe^{+\frac{1}{10}\ln\left(\frac{3}{4}\right)\times 20}$$

$$= F - Fe^{2\ln\left(\frac{3}{4}\right)}$$ (−1/2 each error)

$$= F - Fe^{\ln\left(\frac{3}{4}\right)^2}$$

$$= F - F\frac{9}{16}$$

$$= \frac{7F}{16}$$ ✓1

$\therefore$ Fraction filled in the next 10 minutes is $\frac{7F}{16} - \frac{F}{4} = \frac{3F}{16}$

$\frac{3}{16}$ of the Tank is filled in the next 10 minutes ✓1

2