Student Number ?

ASCHAM SCHOOL

2014

YEAR 12

TRIAL

EXAMINATION

# Mathematics Extension 2

## General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 100

- Attempt Sections A and B.
- Section A is worth 10 marks.
- Answer Section A on the multiple choice answer sheet.
- Detach the multiple choice answer sheet from the back of the examination paper.
- Section B contains 6 questions worth 15 marks each.
- Answer each question in a new booklet.
- Label all sections clearly with your name/number and teacher.

STUDENT NUMBER/NAME: ........................

## STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

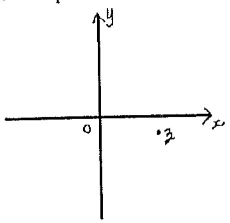
$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

## SECTION A – 10 MULTIPLE CHOICE QUESTIONS 10 MARKS

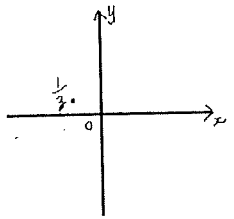
## ANSWER ON THE ANSWER SHEET

1 The complex number z is sketched below.

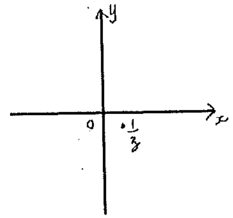


Which of the following sketches could describe $\frac{1}{z}$ ?

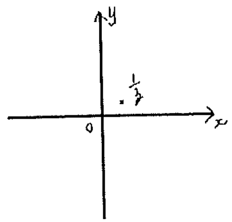
A



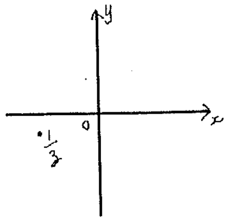
B



C



D





2 If $z$ is complex, a solution to $z^5 = -1$ is:

A $z = cis\frac{\pi}{5}$

B $z = cis\frac{2\pi}{5}$

C $z = cis\frac{8\pi}{5}$

D $z = cis\frac{-2\pi}{5}$

3 If a particle of mass $m$ is projected downwards under gravity and undergoes a resistive force of magnitude $\boxed{kv^3}$ then the acceleration is given by :

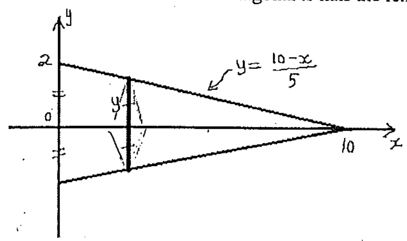
A $\ddot{x} = g - kv^3$

B $\ddot{x} = -g - kv^3$

C $\ddot{x} = g - \frac{kv^3}{m}$

D $\ddot{x} = -g - \frac{kv^3}{m}$

4 The volume of a kookaburra's beak is modelled by taking cross sections which are rhombuses, the longer diagonal of which is the length parallel to the $y$-axis between the lines shown. The smaller diagonal is half the length of the longer diagonal.




The volume would be given by:

A $\int_0^{10} \frac{100 - 20x + x^2}{5} dx$

B $\int_0^{10} \frac{100 - 20x + x^2}{10} dx$

C $\int_0^{10} 4 - \frac{4x}{5} + \frac{x^2}{25} dx$

D $\int_0^{10} 1 - \frac{x}{5} + \frac{x^2}{100} dx$

5 Let $f(x)=x^3+x$ be an increasing function. Let $h(x)$ be the inverse function of $f(x)$. The point $(1,2)$ lies on $y=f(x)$. The value of $h'(2)=$

A $\frac{1}{4}$

B $\frac{1}{13}$

C 4

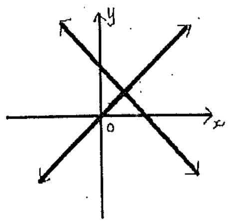
D 13

6 Consider the conic $\frac{x^2}{\lambda-9}+\frac{y^2}{\lambda-4}=1$, where $\lambda$ is a constant. If it is always an ellipse then:

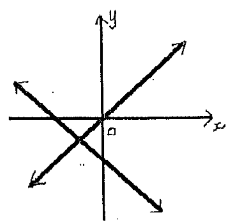
A $\lambda<4$
B $4<\lambda<9$
C $\lambda>9$
D $\lambda\le 4$ or $\lambda\ge 9$.

7 The graph of $|x-1|=|y-1|$ could be:

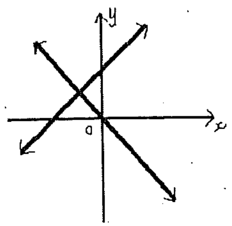
A



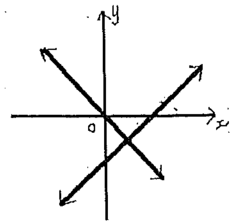

B




C




D






8 The solution to the inequality $\frac{1}{x}+\frac{1}{1-x}>0$ is:

A $x<0$
B $x>1$
C $0<x<1$
D $x<0$ or $x>1$.

9 A possible factor of $15x^7+10x^5-2x^3+14$ would be:

A $2x-7$
B $3x-7$
C $5x+3$
D $3x-5$.

10 The value of $\lim_{x\to\infty} e^{-\frac{1}{x^2}}=$

A $\infty$
B 1
C $-1$
D 0.

## SECTION 2 – 6 QUESTIONS EACH WORTH 15 MARKS

**Question 11 – Begin a new writing booklet**

a Find $\int \cot x \operatorname{cosec}^2 x \, dx$. 2

b Evaluate $\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^4}} \, dx$. 3

c Use the substitution $u = \sqrt{e^x + 1}$ to find $\int \frac{e^{2x}\, dx}{\sqrt{e^x+1}}$. 3

d

(i) Find constants $A$ and $B$ such that $\frac{\cos x}{1-\sin^2 x} = \frac{A\cos x}{1+\sin x} + \frac{B\cos x}{1-\sin x}$. 2

(ii) Hence or otherwise find $\int \sec x \, dx$. 2

e Use the $t$ – results to find $\int_0^{\frac{\pi}{3}} \frac{d\theta}{1+\cos\theta}$. 3



**Question 12 – Begin a new writing booklet**

a Find a square root of $15 + 8i$. 2

b Let $\alpha = 1 - i$ and $\beta = \sqrt{3} + i$.

i Find $\alpha\beta$ in Cartesian form. 1

ii Express $\beta = \sqrt{3} + i$ in mod-arg form. 2

iii If $\alpha = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$, find $\alpha\beta$ in mod-arg form. 1

iv Hence find the exact value of $\cos\left(-\frac{\pi}{12}\right)$. 1

c Sketch the locus of $z$ if $\arg(z+2) = \arg(z-2i)$. 2

d Consider the quadrilateral $ABCD$ representing the complex numbers α,β,γ and δ.

Given that $\beta = \frac{1}{2} i\alpha$, $\alpha = -\gamma$, $|\beta| = |\delta|$ and $\arg\left(\frac{\beta}{\delta}\right) = \pi$,

(i) sketch the information on an Argand diagram, 1

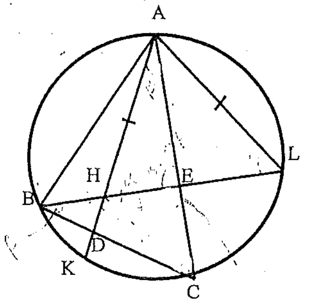
(ii) determine which type of quadrilateral is $ABCD$, giving reasons. 2

e i Sketch $y = e^{2x} + 1$. 1

ii Hence sketch $y = \frac{e^{2x}+1}{x}$. 2

**Question 13 – Begin a new writing booklet**

a Given that $ALCK$ is a cyclic quadrilateral and $H$ is a point on $AK$ such that $AH = AL$. $LH$ produced meets the circle again at $B$ and meets $AC$ at $E$. $BC$ meets $AK$ at $D$.



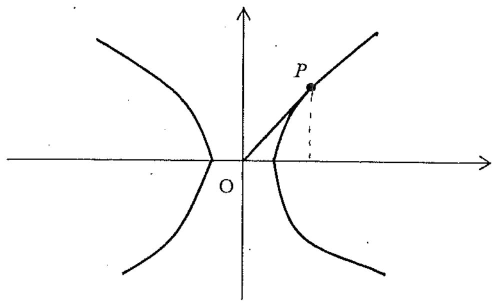

i Prove that $\angle AHL = \angle ACB$. **2**

ii Hence state why that $HECD$ is a cyclic quadrilateral. **1**

iii Given arc $KC$ = arc $CL$, prove that $HC$ is a diameter of $HECD$. **3**

b i Show $\sqrt{\left(e^x + e^{-x}\right)^2 - 4} = e^x - e^{-x}$ (Assume $x > 0$.) **2**

Consider the rectangular hyperbola $x^2 - y^2 = 1$ and the line segment $OP$.




ii Show that the point $P\left(\frac{e^x + e^{-x}}{2}, \frac{e^x - e^{-x}}{2}\right)$ lies on the hyperbola. **1**

iii Using integration by parts (and then the table of standard integrals), show that **3**

$$\int \sqrt{x^2 - 1}\, dx = \frac{x\sqrt{x^2 - 1}}{2} - \frac{\ln\left(x + \sqrt{x^2 - 1}\right)}{2} + C$$

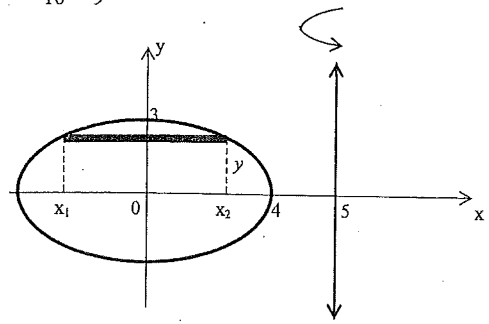
iv Show that the area bounded by the hyperbola, $OP$ and the $x$-axis is $\frac{x}{2}$ units$^2$. **3**





**Question 14 – Begin a new writing booklet**

a The ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is rotated about the line $x = 5$.




(i) Show that the area of one annular type slice taken perpendicular to the axis of rotation is given by $A \approx \frac{80\pi}{3}\sqrt{9 - y^2}$. **2**

(ii) Hence, by summing slices or otherwise, show that the volume of the resulting solid is $120\pi^2$ cubic units. **1**

b A particle of mass $M$ is projected vertically upwards from $O$ with initial speed $I$ m/s. The particle is subjected to a constant gravitational force $g$ m/s$^2$ downwards and a resistance of $Mkv^2$, $k > 0$, where $v$ is the speed at time $t$. Let $x$ m be the displacement above $O$ at time $t$ seconds.

i Show that the greatest height reached, $H$ metres is given by $H = \frac{1}{2k}\ln\left(\frac{g + kI^2}{g}\right)$. **3**

ii The particle then begins to fall. Write down an equation for the acceleration $\ddot{x}$ on its downward journey and find the maximum speed the particle reaches on the downward journey, giving reasons. **2**

iii The particle returns to its point of projection with speed $V$ m/s. Derive the equation for the distance travelled downwards and hence show that $\left(g + kI^2\right)\left(g - kV^2\right) = g^2$. **3**

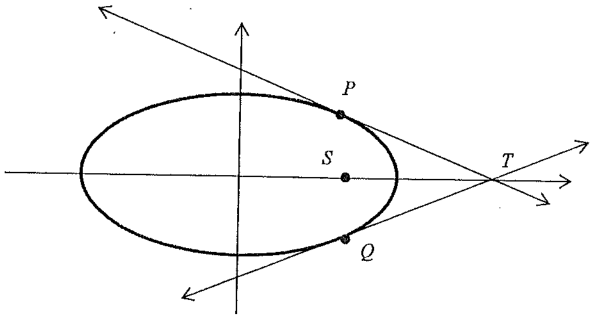
c We know that $A^2 > 0, B^2 > 0, (A \pm B)^2 > 0$ *for* $A, B \neq 0$. Prove for $x, y \neq 0$:

i $4x^2 + 6xy + 4y^2 > 0$, **2**

ii $3x^2 + 5xy + 3y^2 > 0$. **2**

**Question 15 – Begin a new writing booklet**

a Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Tangents to the ellipse are drawn at $P(x_1, y_1)$ and $Q(x_2, y_2)$.




i Derive the equation of the tangent at $P$ and state a similar result for the tangent at $Q$. **2**

ii State the equation of the chord of contact from an external point $T(x_0, y_0)$. **1**

The tangents meet at the point $T\left(\frac{a}{e}, 0\right)$.

iii Prove that the chord $PQ$ passes through the focus $S$. **1**

b It is given that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

i Hence solve the equation $16x^5 - 20x^3 + 5x - 1 = 0$. **2**

ii Show that $\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=-\frac{1}{2}$ and $\cos^2\frac{2\pi}{5}\cos^2\frac{4\pi}{5}=\frac{1}{16}$. **2**

iii Hence deduce that the exact values are $\cos\frac{2\pi}{5}=\frac{-1+\sqrt{5}}{4}$ and $\cos\frac{4\pi}{5}=\frac{-1-\sqrt{5}}{4}$. **2**

*Question 15 continues on the next page.*



**Question 15 continued**

c

i Prove the identity $\dfrac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}+\cos(n+1)x=\dfrac{\sin\left(n+\frac{3}{2}\right)x}{2\sin\frac{x}{2}}$. **2**

[Hint: $\cos(n+1)x=\cos\left(\left(n+\frac{1}{2}\right)x+\frac{1}{2}x\right)$]

ii Hence prove by Mathematical Induction (if $\sin\frac{x}{2}\neq 0$) that for $n = 1, 2, 3,\ldots$ **3**

$$\frac{1}{2}+\cos x+\cos 2x+\cos 3x+\ldots+\cos nx=\frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}.$$

[Hint: $\cos x=\cos\left(\frac{1}{2}x+\frac{1}{2}x\right)$]

**Question 16 – Begin a new writing booklet**

a Solve $\cos 5x = \sin 9x$. **2**

b The black-winged petrel from Lord Howe Island produces chicks of mass 30 grams. The approximate growth pattern of the chick then follows the equation $M = 4t + 30 + 8\sin\frac{\pi t}{2}$, where $M$ is the mass of the chick in grams after $t$ days, $0 \le t \le 70$. After 70 days the parents stop feeding the chick and it must then fend for itself.

i Between which two lines does this function lie? **2**

ii What is the approximate maximum mass the chick can reach in this range? Give reasons. **2**

iii Sketch the function and determine approximately how often the chicks get food. **2**

c Consider the function $G_n = \int_0^\infty e^{-t} t^{n-1}\, dt$, where $n = 1, 2, 3, \ldots$

i State an expression for $G_{n+1}$. **1**

ii Use integration by parts to show that $G_{n+1} = nG_n$. **2**

iii Show $G_1 = 1$. **2**

iv Show $G_n = (n-1)!$ for all $n = 1, 2, 3, \ldots$ **2**

The end! ☺

Student Number ......MC Solutions...... originals

## ASCHAM SCHOOL

### YEAR 12 Trial Mathematics Extension 2 Exam

MULTIPLE-CHOICE ANSWER SHEET

detach

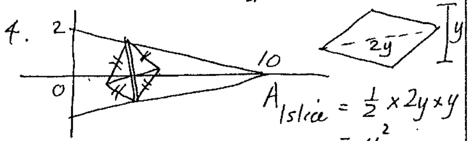
| | | | | |
|---|---|---|---|---|
| 1. | A ○ | B ○ | C ● | D ○ |
| 2. | A ● | B ○ | C ○ | D ○ |
| 3. | A ○ | B ○ | C ● | D ○ |
| 4. | A ○ | B ○ | C ● | D ○ |
| 5. | A ● | B ○ | C ○ | D ○ |
| 6. | A ○ | B ○ | C ● | D ○ denom. has to be neg. |
| 7. | A ● | B ○ | C ○ | D ○ |
| 8. | A ○ | B ○ | C ● | D ○ |
| 9. | A ○ | B ● | C ○ | D ○ |
| 10. | A ○ | B ● | C ○ | D ○ |

Solutions to
Title: Ascham 2014 Y12 Trial Math Ext 2 Exam

MC: 1. $z = r\,\text{cis}\,\theta \therefore \frac{1}{z} = \frac{1}{r}\text{cis}(-\theta)$ reflect angle $\therefore$ (C)

2. $z^5 = -1 \therefore z_1 = \text{cis}\frac{\pi}{5}$ then equally spaced, $\frac{2\pi}{5}$. $= \text{cis}\,\pi \therefore$ (A)

3. $\downarrow g$ $R = ma = -kv^3$ $\therefore \ddot{x} = g - \frac{kv^3}{m} \therefore$ (C)

4. 

$A_{\text{slice}} = \frac{1}{2} \times 2y \times y = y^2$

$V = \int_0^{10} y^2\,dx$ $\quad y^2 = \left(\frac{10-x}{5}\right)^2 = \frac{100 - 20x + x^2}{25}$

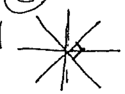
$\therefore$ (C)

5. $f(x) = x^3 + x$ $\therefore$ Inverse of $y = x^3 + x$ is $x = y^3 + y$ $\therefore y = h(x)$ is inverse

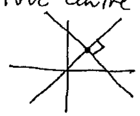
$\therefore$ let $y = x^3 + x$, $\frac{dy}{dx} = 3x^2 + 1$ $\therefore \frac{dx}{dy} = \frac{1}{3x^2+1}$

so $h'(2)$ means when $x = 2$ in $x = y^3 + y \therefore y = 1$. So in original D & R swap so $x = 1$ $\therefore \frac{dx}{dy} = \frac{1}{3(1)^2+1} = \frac{1}{4} \therefore$ (A)

6. $\frac{x^2}{\lambda - 9} + \frac{y^2}{\lambda - 4} = 1$ ellipse then $\lambda - 9 > 0$ and $\lambda - 4 > 0$ $\therefore \lambda > 9$ and $\lambda > 4$ $\therefore \lambda > 9 \therefore$ (C)

7. $|x-1| = |y-1|$ $\quad |x| = |y|$ 

Move centre to (1,1).



$\therefore$ (A)

8. $\frac{1}{x} + \frac{1}{1-x} > 0,\ x \neq 0,\ x \neq 1$

$\frac{1 - \cancel{x} + \cancel{x}}{x(1-x)} > 0$

$\frac{1}{x(1-x)} > 0$

Test $-\ \ +\ \ -$ at $0$, $1$ $\therefore 0 < x < 1$ $\therefore$ (C)

MC contd:

9. Factor $ax - b$, "a" factor of $15x^7$, b factor of 14 $\therefore 3x - 7$ $\therefore$ (B)

10. $\lim_{x\to\infty} e^{-\frac{1}{x^2}}$ $\quad \frac{1}{x^2} \to 0$ as $x \to \infty$ $\therefore e^{-0} = e^0 = 1 \therefore$ (B)

Q11 a) $\int \cot x \,\text{cosec}^2 x\,dx$ (2)

$= -\int \cot x(-\text{cosec}^2 x)\,dx$

$= -\frac{\cot^2 x}{2} + C$

b) $\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1 - x^4}}\,dx = \frac{1}{2}\int_0^{\frac{1}{\sqrt{2}}} \frac{2x}{\sqrt{1 - (x^2)^2}}\,dx$ (3)

$= \frac{1}{2}\left[\sin^{-1}(x^2)\right]_0^{\frac{1}{\sqrt{2}}}$

$= \frac{1}{2}\left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)^2 - \sin^{-1}(0^2)\right]$

$= \frac{1}{2}\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right]$

$= \frac{1}{2}\left[\frac{\pi}{6} - 0\right]$

$= \frac{\pi}{12}$

c) $\int \frac{e^{2x}\,dx}{\sqrt{e^x + 1}} = \int \frac{e^x \cdot e^x\,dx}{\sqrt{e^x + 1}}$ (3)

let $u = \sqrt{e^x + 1}$, $du = \frac{1}{2}(e^x + 1)^{-\frac{1}{2}} \times e^x\,dx$, $u^2 = e^x + 1$, $e^x = u^2 - 1$, $2du = \frac{e^x\,dx}{\sqrt{e^x + 1}}$

$= \int (u^2 - 1) \cdot 2\,du$

$= \frac{2u^3}{3} - 2u + C$

$= \frac{2\sqrt{(e^x+1)}^3}{3} - 2\sqrt{e^x + 1} + C$

d) i) Find A & B: (2)

$\frac{\cos x}{1 - \sin^2 x} = \frac{A\cos x}{1 + \sin x} + \frac{B\cos x}{1 - \sin x}$

$\frac{\cos x}{1 - \sin^2 x} = \frac{A(1 - \sin x)\cos x + B(1 + \sin x)\cos x}{(1 + \sin x)(1 - \sin x)}$

$\therefore 1\cos x \equiv \cos x\,(A - A\sin x + B + B\sin x)$

$\therefore A + B = 1,\quad B - A = 0 \quad \therefore A = B = \frac{1}{2}$

Q11 cont'd:

d i) $\therefore \frac{\cos x}{1-\sin^2 x} = \frac{1}{2}\left(\frac{\cos x}{1+\sin x} + \frac{\cos x}{1-\sin x}\right)$

$\therefore$ ii) $\int \sec x\,dx = \int \frac{\cos x}{\cos^2 x}dx$

$= \int \frac{\cos x}{1-\sin^2 x}dx$ (2)

$= \frac{1}{2}\left[\ln(1+\sin x) - \ln(1-\sin x)\right] + C$

$= \ln\sqrt{\frac{1+\sin x}{1-\sin x}} + C$

e) let $t = \tan\frac{\theta}{2}$ $\quad dt = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta$

$\therefore 2dt = (t^2+1)d\theta$

$\therefore d\theta = \frac{2dt}{t^2+1}$

$\theta = \frac{\pi}{3}$, $t = \frac{1}{\sqrt{3}}$

$\theta = 0$, $t = 0$

$\therefore \int_0^{\frac{\pi}{3}} \frac{d\theta}{1+\cos\theta}$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{\frac{2dt}{t^2+1}}{1+\frac{1-t^2}{1+t^2}}$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{2\,dt}{\cancel{t^2+1}} \div \left(\frac{1+\cancel{t^2}+1-\cancel{t^2}}{\cancel{1+t^2}}\right)$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{\cancel{2}\,dt}{\cancel{2}}$ (3)

$= \int_0^{\frac{1}{\sqrt{3}}} 1\,dt$

$= \left[t\right]_0^{\frac{1}{\sqrt{3}}}$

$= \frac{1}{\sqrt{3}}$.

Q12

a) If $\sqrt{15+8i} = a+ib$ where $a, b \in \mathbb{R}$

then $15+8i = (a+ib)^2 = a^2-b^2+2abi$

Equating real & imaginary parts:

$a^2-b^2 = 15$, $8 = 2ab \Rightarrow 4 = ab$ (2)

By inspection, $a = 4, b = 1$ or $a = -4, b = -1$

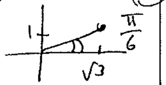
$\therefore$ A root is $4+i$ (or $-4-i$).

Q12 cont'd

b) $\alpha = 1-i$, $\beta = \sqrt{3}+i$

i) $\alpha\beta = (1-i)(\sqrt{3}+i) = \sqrt{3}+1+i(1-\sqrt{3})$ (1)

ii) $\beta = \sqrt{3}+i = 2\,\text{cis}\frac{\pi}{6}$ $\quad |\beta| = \sqrt{\sqrt{3}^2+1^2} = 2$ (2)




iii) $\alpha = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

$\alpha\beta = 2\,\text{cis}\frac{\pi}{6} \times \sqrt{2}\,\text{cis}\left(\frac{-\pi}{4}\right)$ (1)

$= 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{6}-\frac{\pi}{4}\right)$

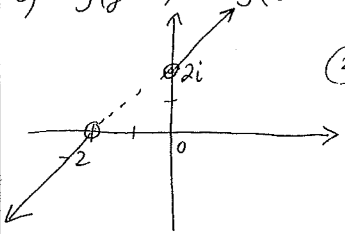
$= 2\sqrt{2}\,\text{cis}\left(\frac{-\pi}{12}\right) = 2\sqrt{2}\left(\cos\left(\frac{-\pi}{12}\right) + i\sin\left(\frac{-\pi}{12}\right)\right)$

iv) $= \sqrt{3}+1+i(1-\sqrt{3})$ Equating reals:

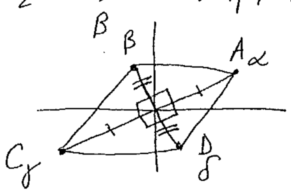
$2\sqrt{2}\cos\left(\frac{-\pi}{12}\right) = \sqrt{3}+1$ (1)

$\therefore \cos\left(\frac{-\pi}{12}\right) = \frac{\sqrt{3}+1}{2\sqrt{2}}$.

c) $\arg(z+2) = \arg(z-2i)$ (2)

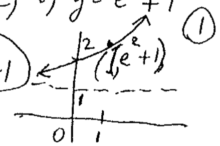



d) $\beta = \frac{1}{2}i\alpha$, $\alpha = -\gamma$, $|\beta| = |\delta|$, $\arg\left(\frac{\beta}{\delta}\right) = \pi$ (1)

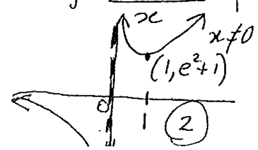



ABCD is a rhombus since diagonals bisect at ⊥ and that's it! (2)

e) i) $y = e^{2x} + 1$ (1)




$y = \frac{e^{2x}+1}{x}$ $\quad x \neq 0$

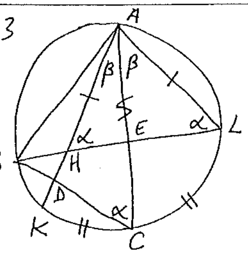



(2)

Solutions to
Title: Ascham 2014 Y12 Trial Math Ext 2 Exam

Q13

a)




Let $\angle AHL = \alpha$

i) $\angle ALH = \alpha$ (base ∠s of isosceles △AHL are equal)

Also $\angle ACB = \alpha$ (∠s standing at circumference on arc AB equal)

(2)

$\therefore \angle AHL = \angle ACB$ (both equal $\angle ALH$)

ii) $\therefore$ HECD is a cyclic quad

(1) ($\angle AHL = \angle ACB$, exterior ∠ of quad HECD = interior opposite ∠)

iii) In △AHE and △ALE,

1. AE common
2. $\angle HAE = \angle LAE$ (equal arcs subtend equal ∠s at circumference)

(3)

3. $AH = AL$ (given)

$\therefore \triangle AHE \equiv \triangle ALE$ (SAS)

$\therefore \angle AEH = \angle AEL$ (matching ∠s in congruent △s)

but $\angle AEH + \angle AEL = 180°$ (straight line)

$\therefore 2x = 180$ $\quad (x = \angle AEH)$

$x = 90$

$\therefore \angle AEH = 90°$ then $\angle HEC = 90°$ (vertically opposite ∠AEH)

$\therefore$ HC is diameter (∠ in semicircle is 90°)

b) i) $\sqrt{(e^x+e^{-x})^2-4} = \sqrt{e^{2x}+2+e^{-2x}-4}$

(2)

$= \sqrt{e^{2x}-2+e^{-2x}}$

$= \sqrt{(e^x-e^{-x})^2}$

$= e^x - e^{-x}$ if $x > 0$

ii) Sub: $\left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2$ (1)

$= \frac{e^{2x}+2+e^{-2x}}{4} - \frac{e^{2x}-2+e^{-2x}}{4}$ ...

Q13 cont'd: $\int\sqrt{x^2-1}\,dx$ $\quad$ let $u = \sqrt{x^2-1}$

$du = \frac{1}{\cancel{2}}(x^2-1)^{-\frac{1}{2}}.\cancel{2}x\,dx$

$dv = 1\,dx$

$v = x$

iii) $= uv - \int v\,du$

$= x\sqrt{x^2-1} - \int\frac{x.x\,dx}{\sqrt{x^2-1}}$

$= x\sqrt{x^2-1} - \int\frac{x^2}{\sqrt{x^2-1}}dx$ (3)

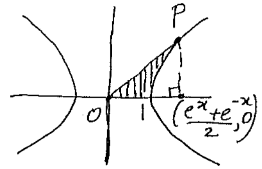
$= x\sqrt{x^2-1} - \int\frac{x^2-1}{\sqrt{x^2-1}} + \frac{1}{\sqrt{x^2-1}}dx$

$= x\sqrt{x^2-1} - \int\sqrt{x^2-1}\,dx + -\int\frac{1}{\sqrt{x^2-1}}dx$

$\therefore 2\int\sqrt{x^2-1}\,dx = x\sqrt{x^2-1} - \ln(x+\sqrt{x^2-1}) + C$

$\therefore \int\sqrt{x^2-1}\,dx = \frac{x\sqrt{x^2-1}}{2} - \frac{\ln(x+\sqrt{x^2-1})}{2} + C$

iv)




$x^2-y^2 = 1$

$y^2 = x^2-1$

$y = \pm\sqrt{x^2-1}$

Area = Triangle $- \int_1^{\frac{e^x+e^{-x}}{2}} \sqrt{x^2-1}\,dx$

$= \frac{1}{2}bh - \int_1^{\frac{e^x+e^{-x}}{2}} \sqrt{x^2-1}\,dx$

$= \frac{1}{2}\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^x-e^{-x}}{2}\right) - \left[\frac{x\sqrt{x^2-1}}{2} - \frac{\ln(x+\sqrt{x^2-1})}{2}\right]_1^{\frac{e^x+e^{-x}}{2}}$

$= \frac{1}{2}\left(\frac{e^{2x}-e^{-2x}}{4}\right) - \left(\left[\frac{e^x+e^{-x}}{4}\sqrt{\left(\frac{e^x+e^{-x}}{2}\right)^2-1} - \frac{\ln\left(\frac{e^x+e^{-x}}{2}+\sqrt{\left(\frac{e^x+e^{-x}}{2}\right)^2-1}\right)}{2}\right] - \left[\frac{1\sqrt{1^2-1}}{2} - \frac{\ln(1+\sqrt{1^2-1})}{2}\right]\right)$ (3)

$= \frac{e^{2x}-e^{-2x}}{8} - \left(\frac{e^x+e^{-x}}{4}\sqrt{\frac{e^{2x}+2+e^{-2x}-4}{4}} - \frac{\ln\left(\frac{e^x+e^{-x}}{2}+\sqrt{\frac{e^{2x}+2+e^{-2x}-4}{4}}\right)}{2}\right) - 0$

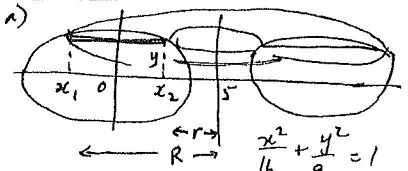
Using (i)

$= \frac{e^{2x}-e^{-2x}}{8} - \left(\frac{e^x+e^{-x}}{4} \times \frac{e^x-e^{-x}}{2} - \frac{\ln\left(\frac{e^x+\cancel{e^{-x}}+e^x-\cancel{e^{-x}}}{2}\right)}{2}\right)$

$= \frac{e^{2x}-e^{-2x}}{8} - \frac{e^{2x}-e^{-2x}}{8} + \frac{\ln\frac{\cancel{2}e^x}{\cancel{2}}}{2}$

$= 0 + \frac{\ln e^x}{2} = \frac{x\ln e}{2} = \frac{x}{2}$ unit$^2$.

Q14 a)




i) Area of 1 slice $\doteqdot \pi R^2 - \pi r^2$ (2)

$= \pi(R+r)(R-r)$

$= \pi\big((5-x_1)+(5-x_2)\big)\big((5-x_1)-(5-x_2)\big)$

$= \pi\big(10-(x_1+x_2)\big)(x_2-x_1)$

Now make $x$ subject to find $x_1, x_2$:

$\frac{x^2}{16} = 1-\frac{y^2}{9} \Rightarrow x = \pm\sqrt{16\left(\frac{9-y^2}{9}\right)}$

$= \pm\frac{4}{3}\sqrt{9-y^2}$

so $x_1 = -\frac{4}{3}\sqrt{9-y^2}$ $x_2 = +\frac{4}{3}\sqrt{9-y^2}$

$\therefore A \doteqdot \pi\big(10-(0)\big)\left(2\times\frac{4}{3}\sqrt{9-y^2}\right)$

$\doteqdot \pi\times\frac{80}{3}\sqrt{9-y^2}$ #

ii) $V \doteqdot \sum_{y=-3}^{3} \frac{80\pi}{3}\sqrt{9-y^2}\,\Delta y$

$= 2\int_0^3 \frac{80\pi}{3}\sqrt{9-y^2}\,dy$


$= 2\times\frac{\cancel{80}\pi}{\cancel{3}}\times\frac{1}{\cancel{4}}\pi 3^2$

$= 120\pi^2$ unit$^3$ (1)

OR Using Pappus' Theorem:

$V =$ Area of Cross-section $\times$ Average Radius Circumference

$= \pi\times 3\times 4\times 2\pi\times 5$

$= 120\pi^2$ $u^3$

b) $\uparrow x$, $v=I$, $-g$, $-Mkv^2$, 0

i) $F = ma = -Mg - Mkv^2$

$\therefore \ddot{x} = -g-kv^2$

$\cancel{v}\frac{dv}{dx} = \frac{-g-kv^2}{v}$

$\therefore \int_0 2k\,dx\,\frac{dv}{dv} = \int_I^0 -\frac{2kv}{g+kv^2}\,dv$ (3)

$\therefore 2k\,x = -\left[\ln(g+kv^2)\right]_I^0$

Q14 b) i) cont'd $x = \frac{-1}{2k}\left[\ln(g+k0^2)-\ln(g+kI^2)\right]$

$\therefore H = \frac{1}{2k}\ln\left(\frac{g+kI^2}{g}\right)$

ii) 0, $g$, $x$, $-Mkv^2$, $V$

$F = ma = Mg - Mkv^2$

$\therefore \ddot{x} = g-kv^2$ (2)

Max speed should be terminal speed as starting from rest $\therefore \ddot{x} = 0 = g-kv^2$

$\therefore v = \sqrt{\frac{g}{k}}$ since $v>0$ (speed)

iii) $v = V$ when $x = H$ down.

$\therefore \ddot{x} = g-kv^2$

$v\frac{dv}{dx} = g-kv^2$

$\frac{dv}{dx} = \frac{g-kv^2}{v}$ (3)

$\therefore \int_0 -2k\frac{dx}{dv} = \int_0^V \frac{-2kv}{g-kv^2}$

$x = \frac{-1}{2k}\left[\ln(g-kv^2)\right]_0^V$

$= -\frac{1}{2k}\left[\ln(g-kV^2)-\ln(g-k0^2)\right]$

$H = \frac{1}{2k}\ln\left(\frac{g}{g-kV^2}\right)$

Distance up = distance down

$\therefore \frac{1}{\cancel{2k}}\cancel{\ln}\left(\frac{g+kI^2}{g}\right) = \frac{1}{\cancel{2k}}\cancel{\ln}\left(\frac{g}{g-kV^2}\right)$

$(g+kI^2)(g-kV^2) = g^2$ #.

c) i) $4x^2+6xy+4y^2 = 3x^2+3y^2+6xy+x^2+y^2$

$= 3(x^2+2xy+y^2)+x^2+y^2$ (2)

$= 3(x+y)^2+x^2+y^2$

$>0$ if $x,y\neq 0$ since $(x+y)^2>0$ and $x^2,y^2>0$

ii) $3x^2+5xy+3y^2 = 2x^2+4xy+2y^2+x^2+xy+y^2$

$= 2(x+y)^2+\frac{x^2}{2}+\frac{y^2}{2}+\frac{x^2+2xy+y^2}{2}$

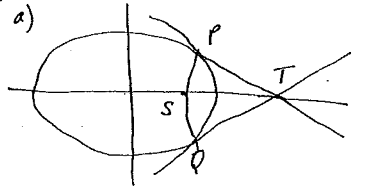
$= 2(x+y)^2+\frac{x^2}{2}+\frac{y^2}{2}+\frac{(x+y)^2}{2}$ (2)

$>0$ since all terms $>0$ for $x,y\neq 0$

QED.



Q15 a)




i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$

$\therefore \frac{dy}{dx} = -\frac{b^2x}{a^2y}$ at $x=x_1$ $m = \frac{-b^2x_1}{a^2y_1}$

$\therefore$ Eq$^n$ of tangent is:

$y-y_1 = m(x-x_1)$

$y-y_1 = \frac{-b^2x_1}{a^2y_1}(x-x_1)$ (2)

$\frac{yy_1}{b^2}-\frac{y_1^2}{b^2} = -\frac{xx_1}{a^2}+\frac{x_1^2}{a^2}$

$\therefore \frac{xx_1}{a^2}+\frac{yy_1}{b^2} = \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2} = 1$

since $(x_1,y_1)$ lies on ellipse

$\therefore$ Eq$^n$ of tangent at P is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

Similarly tangent at Q is

$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1.$

ii) Eq$^n$ of chord of contact is (1)

$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1.$

iii) chord of contact from $T\left(\frac{a}{e},0\right)$ is chord PQ — sub $\left(\frac{a}{e},0\right) \Rightarrow$ (1)

$\frac{x\left(\frac{a}{e}\right)}{a^2}+\frac{y(0)}{b^2}=1$

$\therefore \frac{x}{ae}=1 \Rightarrow x=ae$ ie. PQ is line $x=ae$ $\therefore S(ae,0)$ lies on vertical PQ. QED.

b) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$

i) $16x^5-20x^3+5x-1=0$ if $x=\cos\theta$ then equivalent to $\cos 5\theta = 1$

$\therefore 5\theta = 0+2k\pi$ where $k\in\mathbb{Z}$

$\theta = \frac{2k\pi}{5}$ ie. $\theta = 0, \pm\frac{2\pi}{5}, \pm\frac{4\pi}{5}, \ldots$

etc. so $x = \cos\left(\frac{2k\pi}{5}\right)$ where values are 5 (distinct) roots of equation

$\therefore \cos 0, \cos\frac{2\pi}{5}, \cos\left(\frac{-2\pi}{5}\right), \cos\left(\frac{4\pi}{5}\right), \cos\left(\frac{-4\pi}{5}\right),$

$\cos\frac{6\pi}{5}, \cos\frac{-6\pi}{5}, \cos\frac{8\pi}{5}, \cos\left(-\frac{8\pi}{5}\right)\ldots$

(Distinct) roots are $\cos 0 = 1, \cos\frac{2\pi}{5}, \cos\frac{4\pi}{5},$ (2)

$\cos\frac{6\pi}{5}\left(\cos\left(-\frac{4\pi}{5}\right)\right), \cos\frac{8\pi}{5}\left(\cos\left(-\frac{2\pi}{5}\right)\right)$

but (ii) $\cos\left(-\frac{4\pi}{5}\right) = \cos\left(\frac{4\pi}{5}\right), \cos\frac{2\pi}{5} = \cos\left(-\frac{2\pi}{5}\right)$

$\therefore$ Sum of roots $= \frac{-b}{a} = \frac{0}{16} = 0$

$\therefore \cos 0 + \cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}+\cos\left(\frac{-2\pi}{5}\right)+\cos\left(\frac{-4\pi}{5}\right)=0$

$\therefore 1+2\cos\frac{2\pi}{5}+2\cos\frac{4\pi}{5}=0$

$\therefore \cos\frac{2\pi}{5}+\cos\frac{4\pi}{5} = -\frac{1}{2}.$

and product $= \frac{-f}{a} = \frac{1}{16}$ (2)

$\therefore 1\times\cos\frac{2\pi}{5}\times\cos\frac{4\pi}{5}\times\cos\frac{-2\pi}{5}\times\cos\left(\frac{-4\pi}{5}\right) = +\frac{1}{16}$

$\therefore \cos^2\frac{2\pi}{5}\cos^2\frac{4\pi}{5} = \frac{1}{16}$

iii) let $\cos\frac{2\pi}{5} = a$, $\cos\frac{4\pi}{5} = b$

$\therefore a+b = -\frac{1}{2}$ $a^2b^2 = \frac{1}{16}$ (2)

$ab = \pm\frac{1}{4}$

why?

but $\boxed{\cos\frac{4\pi}{5}<0}$ so $ab = -\frac{1}{4}$ only

$\therefore a = \frac{-1}{4b}$ $\therefore \frac{-1}{4b}+b = -\frac{1}{2}$

$\therefore 1-4b^2 = 2b \Rightarrow 4b^2+2b-1=0$

$\Rightarrow b = \frac{-2\pm\sqrt{20}}{8} = \frac{-1\pm\sqrt{5}}{4}$ but $b<0$ $\therefore b = \frac{-1-\sqrt{5}}{4}$

Q15 cont'd:

b iii) cont'd $\therefore a = \frac{-1}{2} - \left(\frac{-1-\sqrt{5}}{4}\right) = \frac{-1+\sqrt{5}}{4}$

$\therefore \cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$, $\cos\frac{4\pi}{5} = \frac{-1-\sqrt{5}}{4}$

c) i) RTP: $\frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}} + \cos(n+1)x = \frac{\sin(n+\frac{3}{2})x}{2\sin\frac{x}{2}}$

Proof: $\frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}} + \cos(n+1)x$

$= \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}} + \cos(\underbrace{(n+\frac{1}{2})x}_{A} + \underbrace{\frac{1}{2}x}_{B})$ (2)

$= \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}} + \cos(n+\frac{1}{2})x\cos\frac{1}{2}x - \sin(n+\frac{1}{2})x\sin\frac{1}{2}x$

$= \frac{\sin(n+\frac{1}{2})x + 2\sin\frac{x}{2}\cos(n+\frac{1}{2})x\cos\frac{1}{2}x - 2\sin(n+\frac{1}{2})x\sin^2\frac{1}{2}x}{2\sin\frac{x}{2}}$

$= \frac{\sin(n+\frac{1}{2})x + \cos(n+\frac{1}{2})x\sin x - 2\sin(n+\frac{1}{2})x\sin^2\frac{1}{2}x}{2\sin\frac{x}{2}}$

$= \frac{\sin(n+\frac{1}{2})x\left[1-2\sin^2\frac{1}{2}x\right] + \cos(n+\frac{1}{2})x\sin x}{2\sin\frac{x}{2}}$

$= \frac{\sin(n+\frac{1}{2})x\cos x + \cos(n+\frac{1}{2})x\sin x}{2\sin\frac{x}{2}}$

$= \frac{\sin(n+\frac{1}{2}+1)x}{2\sin\frac{x}{2}} = \frac{\sin(n+\frac{3}{2})x}{2\sin\frac{x}{2}}$. QED

ii) RTP: let $P(n)$ be the proposition that $\frac{1}{2} + \cos x + \cos 2x + \cos 3x + \dots + \cos nx = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}}$.

Proof: Let $P(1)$ be $\frac{1}{2} + \cos x = \frac{\sin(\frac{3}{2}x)}{2\sin\frac{x}{2}}$.

LHS $= \frac{1}{2} + \cos x$ [or use (i) result where $n=0$]

$= \frac{1}{2} + \cos(\frac{1}{2}x + \frac{1}{2}x)$

$= \frac{1}{2} + \cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x$

$= \frac{1 + 2\cos^2\frac{1}{2}x - 2\sin^2\frac{1}{2}x}{2} \times \frac{\sin\frac{x}{2}}{\sin\frac{x}{2}}$

$= \frac{\sin\frac{x}{2} + 2\cos^2\frac{1}{2}x\sin\frac{x}{2} - 2\sin^2\frac{1}{2}x\sin\frac{x}{2}}{2\sin\frac{x}{2}}$

$= \frac{\sin\frac{x}{2}(1-2\sin^2\frac{x}{2}) + 2\sin\frac{x}{2}\cos\frac{x}{2}\cos\frac{x}{2}}{2\sin\frac{x}{2}}$

$= \frac{\sin\frac{x}{2}\cos x + \sin x\cos\frac{x}{2}}{2\sin\frac{x}{2}} = \frac{\sin(\frac{x}{2}+x)}{2\sin\frac{x}{2}}$ = RHS

$\therefore P(1)$ true.

Assume $P(k)$ true ie. $k \in N, k \geq 1$

$\frac{1}{2} + \cos x + \cos 2x + \dots + \cos kx = \frac{\sin(k+\frac{1}{2})x}{2\sin\frac{x}{2}}$

RTP: $P(k+1)$ true ie.

$\frac{1}{2} + \cos x + \cos 2x + \dots + \cos kx + \cos(k+1)x = \frac{\sin(k+1+\frac{1}{2})x}{2\sin\frac{x}{2}}$. (3)

Proof: Consider the LHS of $P(k+1)$:

$\frac{1}{2} + \cos x + \cos 2x + \dots + \cos kx + \cos(k+1)x$

$= \frac{\sin(k+\frac{1}{2})x}{2\sin\frac{x}{2}} + \cos(k+1)x$ using $P(k)$

$= \frac{\sin(k+\frac{3}{2})x}{2\sin\frac{x}{2}}$ from (i)

= RHS of $P(k+1)$ $\therefore P(k+1)$ true.

$\therefore P(n)$ is true for all $n \in N, n \geq 1$ by Math Induction.

Q16

a) $\cos 5x = \sin 9x$ (2)

$\therefore \cos 5x = \cos(\frac{\pi}{2} - 9x)$

$\therefore 5x = 2k\pi \pm (\frac{\pi}{2} - 9x)$ $k \in \mathbb{Z}$

$\therefore 5x = 2k\pi + \frac{\pi}{2} - 9x$ OR $5x = 2k\pi - (\frac{\pi}{2} - 9x)$

$\therefore 14x = 2k\pi + \frac{\pi}{2}$ OR $4x = -(2k\pi - \frac{\pi}{2})$

$\therefore x = \frac{2k\pi + \frac{\pi}{2}}{14}$ OR $x = -\frac{(4k\pi - \pi)}{4}$

$\therefore x = \frac{\pi(4k+1)}{28}$ OR $x = \frac{-\pi(4k-1)}{4}$

b) $M = 4t + 30 + 8\sin\frac{\pi t}{2}$, $0 \leq t \leq 70$

i) $-1 \leq \sin\left(\frac{\pi t}{2}\right) \leq 1$

$\therefore -8 \leq 8\sin\frac{\pi t}{2} \leq 8$

$\therefore 4t + 30 - 8 \leq 4t + 30 + 8\sin\frac{\pi t}{2} \leq 4t + 30 + 8$

$\therefore 4t + 22 \leq y \leq 4t + 38$ (2)

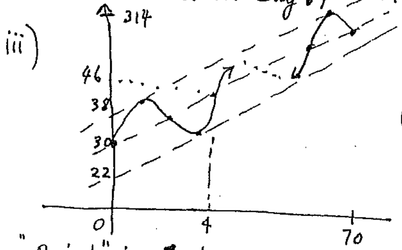
$\therefore$ Lines are $y = 4t + 22$ and $y = 4t + 38$

16 b) ii) If $t = 70$ $y = 4(70) + 30 + 8\sin(35\pi) = 310 + 0 = 310$

But if $t = 69$ $y = 4(69) + 30 + 8\sin\left(\frac{68\pi + \pi}{2}\right) = 306 + 8 \times 1 = 314$

$\therefore$ Max mass is on Day 69 314 g.

iii)




(2)

"Period" is 4 days so chick probably gets fed about every 4 days.

c) $G_n = \int_0^\infty e^{-t} t^{n-1}\,dt$ $n = 1, 2, 3, \dots$

i) $G_{n+1} = \int_0^\infty e^{-t} t^n\,dt$ (1)

ii) $G_n = \int_0^\infty e^{-t} t^{n-1}\,dt$

$= uv - \int v\,du$ where $u = e^{-t}$, $du = -e^{-t}dt$, $dv = t^{n-1}dt$, $v = \frac{1}{n}t^n$

$= \left[e^{-t}\frac{1}{n}t^n\right]_0^\infty - \int_0^\infty \frac{1}{n}t^n \cdot -e^{-t}\,dt$

$= 0 \cdot \frac{1}{n}t^\infty - e^0 \cdot \frac{1}{n}0^n + \frac{1}{n}\int_0^\infty t^n \cdot e^{-t}\,dt$

$= 0 + \frac{1}{n}\int_0^\infty e^{-t} t^n\,dt$

$\therefore G_n = \frac{1}{n}G_{n+1}$ (2)

$\therefore G_{n+1} = nG_n$ QED

iii) $G_1 = \int_0^\infty e^{-t} t^{1-1}\,dt$

$= \int_0^\infty e^{-t} \cdot 1\,dt$

$= \left[-e^{-t}\right]_0^\infty$ (2)

$= -0 - -e^{-0}$

$= e^0 = 1$ QED

16 cont'd

c) iii) RTP: $G_n = (n-1)!$

Now $G_1 = 1$ and $G_{n+1} = nG_n$.

$\therefore G_2 = 1.G_1 = 1 \times 1 = 1$

$G_3 = 2.G_2 = 2.1$

$G_4 = 3.G_3 = 3.2.1 = 3!$

$\therefore G_5 = 4.G_4 = 4.3.2.1 = 4!$

$\vdots$

$G_n = (n-1)G_{n-1}$

$= (n-1).(n-2)(n-3)\dots \times 2 \times 1$

$\therefore G_n = (n-1)!$ $n = 1, 2, 3, \dots$ QED

The End! ☺