ABBOTSLEIGH

AUGUST 2003
YEAR 12
ASSESSMENT 4
TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics
Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 120
- Attempt Questions 1-8.
- All questions are of equal value.

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Attempt Questions 1-8
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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

QUESTION 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) By completing the square, find $\int \frac{dx}{x^2-4x+8}$ **2**

(b) Use the substitution $x=\sin\theta$ to evaluate

$$\int_0^{\frac{\sqrt{3}}{2}} \frac{x^2\,dx}{\sqrt{1-x^2}}$$ **3**

(c) Use integration by parts to find $\int_1^e \frac{\ln x}{\sqrt{x}}\,dx$ **3**

(d) (i) Find real numbers a, b and c such that

$$\frac{x+7}{(1+x^2)(1+x)} = \frac{ax+b}{1+x^2} + \frac{c}{1+x}$$ **2**

(ii) Find $\int \frac{x+7}{(1+x^2)(1+x)}\,dx$ **2**

(e) Use the substitution $t=\tan\frac{x}{2}$ to find $\int \frac{\tan x}{1+\cos x}\,dx$ **3**

QUESTION 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Express $z=1+\sqrt{3}\,i$ in modulus-argument form. **2**

(ii) Show that $z^7-64z=0$ **3**

(b) Let $z=x+iy$, where x and y are real numbers.

(i) Solve $z\bar{z}+2z=\frac{1}{4}+i$ **4**

(ii) Draw a neat sketch of the locus of $\operatorname{Re}(z)=|z-2|$ **3**

(c) The points A,B,C,D on an Argand diagram represent the complex numbers a,b,c,d respectively. **3**

If $a+c=b+d$ and $a-c=i(b-d)$ find what type of quadrilateral is defined by $ABCD$. Clearly justify your answer.

QUESTION 3 (15 marks) Use a SEPARATE writing booklet.

(a) Given the function $f(x)=x\sqrt{4-x^2}$.

(i) State its natural domain and show that it is an odd function. **2**

(ii) Show that on the curve $y=f(x)$, stationary points occur at $x=\pm\sqrt{2}$. Find the coordinates of the stationary points and determine their nature. **3**

(iii) Draw a neat sketch of the curve $y=f(x)$, indicating the above features, and given that there is a point of inflexion at the origin. **2**

(iv) On seperate diagrams, sketch the curves

1. $y^2=x^2(4-x^2)$ **2**
2. $y=\frac{1}{f(x)}$ **2**

(b) Given that the sum of two of the roots of the equation $x^4-x^3-x^2-x-2=0$ is zero, find all four roots. **4**

Marks

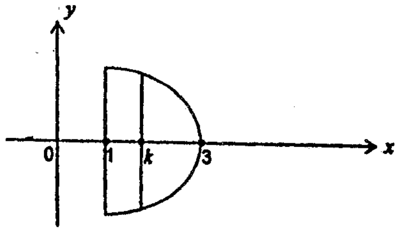
QUESTION 4 (15 marks) Use a SEPARATE writing booklet.

(a) The ellipse E has equation $\frac{x^2}{8}+\frac{y^2}{4}=1$

(i) Write down its eccentricity, the coordinates of its foci, S and S', and the equation of each directrix. Sketch the ellipse E. **4**

(ii) If $P(x_1,y_1)$ is an arbitrary point on E, prove that the sum of the distances SP and $S'P$ is independent of the position of P. **2**

(b)



The base of a particular solid is the region enclosed by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the line $x=1$. Each cross-section of the solid perpendicular to the x-axis is an equilateral triangle.

(i) Show that the area of the triangle at $x=k$ is $\frac{\sqrt{3}}{9}(36-4k^2)$ **2**

(ii) Find the volume of the solid. **3**

(iii) Consider a second solid which is obtained by rotating the region enclosed by the ellipse and the line $x=1$ about the y-axis. Find the volume of the solid formed. **4**

Marks

QUESTION 5 (15 marks) Use a SEPARATE writing booklet.

(a) Factorise x^2+4x+3 and hence, or otherwise, show that the coefficient of x^4 in the expansion of $\left(x^2+4x+3\right)^6$ is 61 695. **4**

(b) (i) Prove that the equation of the tangent to the hyperbola $x^2-y^2=c^2$ at the point $P\left(x_1,y_1\right)$ is $xx_1-yy_1=c^2$. **2**

(ii) This tangent meets the lines $y=x$ and $y=-x$ at Q and R respectively and O is the origin. Prove that the area of triangle OQR is constant. **4**

(c) A particle moves in a straight line and its position x at any time t is given by $x=\sqrt{3}\cos 3t-\sin 3t$

(i) Show that the motion is simple harmonic. **2**

(ii) Determine the period and amplitude of the motion. **3**

Marks

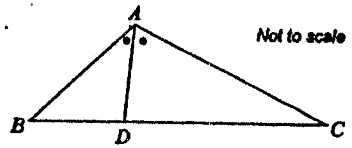
QUESTION 6 (15 marks) Use a SEPARATE writing booklet.

(a) If α,β,γ are the roots of $2x^3-4x^2-3x-1=0$, find the value of $(\alpha-1)(\beta-1)(\gamma-1)$. **3**

(b) Solve for x,y,z over the complex numbers: **4**

$x+y+z=1$
$xy+yz+zx=9$
$xyz=9$

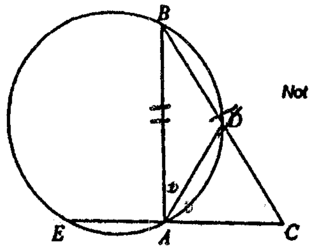
(c) (i) In the triangle ABC, AD bisects angle BAC.



Not to scale

Prove that $\dfrac{BD}{DC}=\dfrac{BA}{AC}$ **4**

(ii)



Not to scale

In the diagram $AB=BC$ and AD bisects angle BAC.

Prove that $BD=CE$. **4**

QUESTION 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The minute hand OP and the hour hand OQ of a clock are 4cm and 3cm long respectively. Let $PQ =$ the distance between the tips of the hands of the clock.

(i) Show that $\frac{dPQ}{d\theta} = \frac{12\sin\theta}{\sqrt{25-24\cos\theta}}$ where θ is the acute angle between the hands of the clock. **2**

(ii) Hence show that the rate of increase (in cm per hour) of the length of PQ at 9 o'clock is $\frac{22\pi}{5}$ cm/h. **3**

(b) (i) If $f(x)$, $g(x)$ and $h(x)$ are distinct non-negative continuous functions of x in the interval $a \le x \le b$ and $f(x) < g(x) < h(x)$, explain why

$$\int_a^b f(x)\,dx < \int_a^b g(x)\,dx < \int_a^b h(x)\,dx$$ **2**

(ii) By considering the interval $0 < x < 1$ as an inequality, use algebra to show that

$$\frac{1}{2}x(1-x)^3 < \frac{x(1-x)^3}{1+x} < x(1-x)^3$$ **3**

(iii) Deduce that $\frac{1}{2}\int_0^1 x(1-x)^3\,dx < \int_0^1 \frac{x(1-x)^3}{1+x}\,dx < \int_0^1 x(1-x)^3\,dx$ **1**

(iv) Given that $\int_0^1 \frac{x(1-x)^3}{1+x}\,dx = \frac{67}{12} - 8\ln 2$, deduce that $\frac{83}{120} < \ln 2 < \frac{667}{960}$ **4**

QUESTION 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A projectile is fired from the origin O with velocity V and angle of elevation α, where α is acute.

(i) By letting $g =$ acceleration due to gravity and $k = \frac{V^2}{2g}$, derive the Cartesian equation of the parabolic path of the projectile. Show that as a quadratic equation in $\tan\alpha$, its Cartesian equation is

$$x^2\tan^2\alpha - 4kx\tan\alpha + \left(4ky + x^2\right) = 0$$ **4**

(ii) Show that the projectile can pass through the point (X, Y) in the first quadrant by firing at two different initial angles α_1 and α_2 if

$$X^2 < 4k^2 - 4kY$$ **2**

(iii) Let $\tan\alpha_1$ and $\tan\alpha_2$ be the two real roots of the quadratic equation in part (i). Show that $\tan\alpha_1 \tan\alpha_2 > 1$, and hence explain why it is impossible for <u>both</u> α_1 and α_2 to be less than 45°. **3**

(b) It is given that $A > 0, B > 0$ and n is a positive integer.

(i) Divide $A^{n+1} - A^nB + B^{n+1} - B^nA$ by $A - B$ **2**

(ii) Deduce that $A^{n+1} + B^{n+1} \ge A^nB + B^nA$ **1**

(iii) Show by induction, that $\left(\frac{A+B}{2}\right)^n \le \frac{A^n + B^n}{2}$ **3**

End of paper

Abbots Leigh 4u. 2003 $\overline{}$ 1 0. D. Ong

1(a). $\int \frac{dx}{x^2 - 4x + 8}$.

$\int \frac{dx}{x^2 - 4x + 4 + 4}$. ✓

$= \int \frac{dx}{(x-2)^2 + 4}$.

$= \frac{1}{2} \tan^{-1} \left(\frac{x-2}{2}\right) + c$. ✓

(b). $\int_0^{\sqrt{3}/2} \frac{x^2 \cdot dx}{\sqrt{1-x^2}}$

Let $x = \sin\theta$ At $x = \sqrt{3}/2$, $\theta = \pi/3$

$\frac{dx}{d\theta} = \cos\theta$ $x = 0$, $\theta = 0$. ✓

$dx = \cos\theta \cdot d\theta$

$I = \int_0^{\pi/3} \frac{\sin^2\theta \cdot \cancel{\cos\theta} \cdot d\theta}{\cancel{\cos\theta}}$ $1 - 2\sin^2\theta = \cos 2x$

$= \frac{1}{2} \int_0^{\pi/3} 1 - \cos 2\theta \cdot d\theta$ ✓

$= \frac{1}{2} [\theta - \sin 2\theta]_0^{\pi/3}$

$= \frac{1}{2} [\pi/3 - \sqrt{3}/2]$

$= \pi/6 - \sqrt{3}/4$. ✓

15

). $\int_1^e \ln x \cdot x^{-1/2} \cdot dx$ $v = 2x^{1/2}$

$u = \ln x$ $v' = x^{-1/2}$ ✓

$u = 1/x$

$I = [\ln x \cdot 2\sqrt{x}]_1^e - 2\int_1^e x^{-1/2} \, dx$ ✓

$= 2\sqrt{e} - [4\sqrt{x}]_1^e$

$= 2\sqrt{e} - 4\sqrt{e} + 4$

$= 4 - 2\sqrt{e}$. ✓

1(i) $\frac{x+7}{(1+x^2)(1+x)} = \frac{ax+b}{1+x^2} + \frac{c}{1+x}$

$x + 7 = (ax+b)(1+x) + c(1+x^2)$

Let $x = -1$

$6 = 2c$ ✓

$c = 3$

Compare x^2

$a + c = 0$.

$a = -3$ ✓

Compare Constants.

$7 = b + c$.

$\therefore b = 4$.

$$I = \int \frac{-3x+4}{1+x^2} + \frac{3}{1+x} \, . dx$$

$$= -\frac{3}{2}\int \frac{2x}{1+x^2} dx + \int \frac{4\,dx}{1+x^2} + 3\int \frac{dx}{1+x} \quad ✓$$

$$= -3/2 \ln |1+x^2| + 4\tan^{-1}(x) + 3\ln|1+x| \quad ✓ + c.$$

(c). $\int \frac{\tan x}{1+\cos x} . dx$

Let $t = \tan x/2$.

$$\therefore \tan x = \frac{2t}{1-t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2 x/2$$

$$= \frac{1}{2}(1 + \tan^2 x/2) \quad ✓$$

$$\frac{dt}{dx} = \frac{1}{2}(1+t^2)$$

$$\frac{2dt}{1+t^2} = dx$$

$$I = \int \frac{\frac{2t}{1-t^2}}{1+\frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$=$

$$= \int \frac{\frac{\cancel{2}t}{1-t^2}}{\frac{\cancel{2}}{\cancel{1+t^2}}} \cdot \frac{2dt}{\cancel{1+t^2}} \quad ✓$$

$$= \int \frac{2t}{1-t^2} . dt.$$

$$= -\int \frac{-2t}{1-t^2} . dt$$

$$= -\ln|1-t^2| + c \quad ✓$$

$$= -\ln|1-\tan^2 x/2| + c.$$

22.

(i) $z = 1 + \sqrt{3}i$ ✓

$= 2 \text{cis } \pi/3$ ✓

(ii) $z^7 - 64z = 0.$

$= 2^7 \text{cis } \frac{7\pi}{3} - 128 \text{ cis } \pi/3$ ✓

$= 2^7 \text{cis } \pi/3 - 128 \text{ cis } \pi/3$

$= 128 \text{cis } \pi/3 - 128 \text{cis } \pi/3$ ✓

$= 0$ ✓

$=$ RHS

b)(i) $z = x + iy$

$z\bar{z} + 2z = \frac{1}{4} + i$

$(x+iy)(x-iy) + 2(x+iy)$

$x^2 + y^2 + 2x + 2iy = \frac{1}{4} + i$

Compare $Re(z)$ & $Im(z)$

$x^2 + y^2 + 2x = \frac{1}{4}$. $\quad 2y = 1$

$4x^2 + 4y^2 + 8x - 1 = 0$. $\quad y = 1/2$. ✓

$4x^2 \; 1 + 8x - 1 = 0$. ✓

$4x^2 + 8x = 0$.

$4x(x+2) = 0$.

$x = 0, \quad x = -2$. ✓

$\therefore z = \underline{\underline{1/2\, i, \; -2 + i \cdot 1/2}}$

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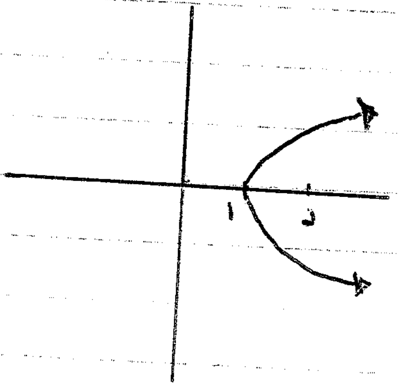
(ii). $Re(z) = |z-2|$



✓

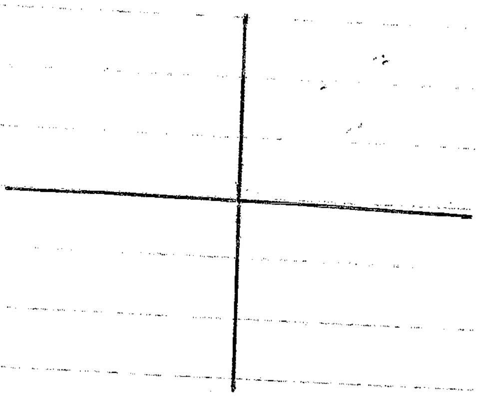
$x = \sqrt{(x-2)^2 + (y^2)}$ ✓

$x^2 = (x^2 - 4x + 4 + y^2$

$y^2 = 4x - 4$.

$y^2 = 4(x-1)$ ✓

(iii).



Using a vector diagram, show that ABCD is a rhombus.

3.(a)(i) $f(x) = x\sqrt{4-x^2}$

$D: -2 \le x \le 2.$ ✓

$f(-x) = -x\sqrt{4-(-x)^2}$

$= -x\sqrt{4-x^2}$

$= -f(x)$ ✓

$\therefore$ Odd.

(ii). $f'(x) = x\sqrt{4-x^2}$

$u = x \qquad v = \sqrt{(4-x^2)^{1/2}}$

$u' = 1 \qquad v' = \frac{1}{2} \times -2x\ (4-x^2)^{-1/2}$

$= -\frac{x}{\sqrt{4-x^2}}$

$f'(x) = -\frac{x^2}{\sqrt{4-x^2}} + \sqrt{4-x^2} = 0.$ ✓

$-x^2 + 4 - x^2 = 0.$

$2x^2 = 4.$

$x^2 = 2.$

$x = \pm\sqrt{2}.$

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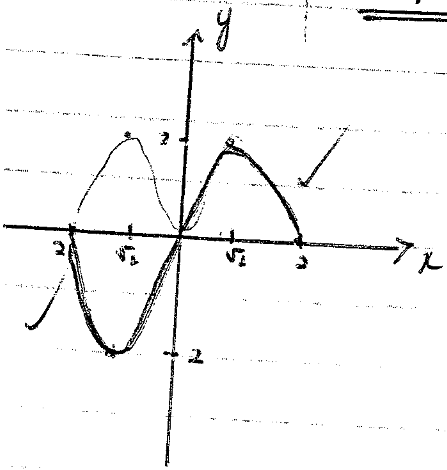
$f(\sqrt{2}) = 2, \quad f(-\sqrt{2}) = -2.$

Test	$-\frac{3}{2}$	$-\sqrt{2}$	0	$\sqrt{2}$	$\frac{3}{2}$
	-0.377	0	2	0	$-.377$
	+	0	+	0	−
	\	—	/	—	\

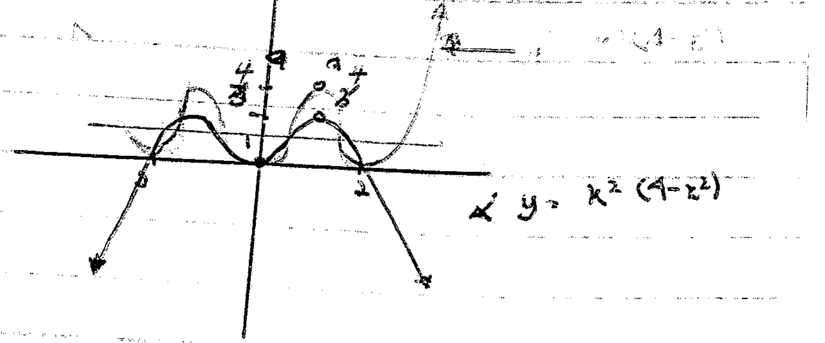
$\therefore (\sqrt{2}, -2)$ Min TP ✓

$(-\sqrt{2}, +2)$ Max TP.



(iv). $y^2 = x^2(4-x^2)$

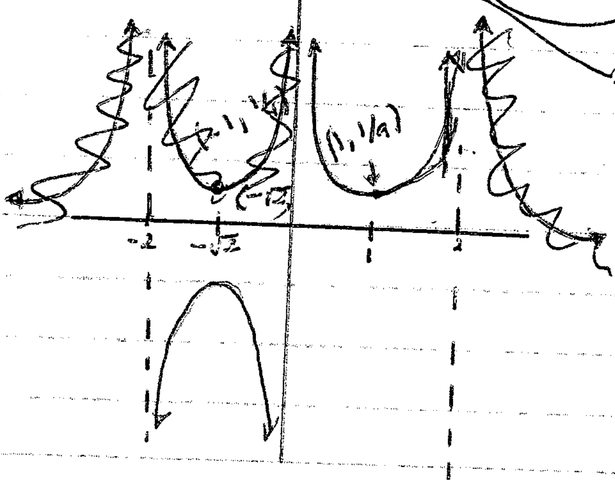
$= 4x^2 - x^4.$



(iv)(a) $y = \frac{1}{f(x)}$

$\Rightarrow \frac{1}{y^2} \quad \frac{1}{x^2(4-x^2)}$

$y = \frac{1}{x\sqrt{4-x^2}}$ $\quad D: -2 < x < 2$



(b) $x^4 - x^3 - x^2 - x - 2 = 0.$

$\alpha, \beta, \gamma, \delta$

$\alpha + \beta = 0.$

$\Sigma\alpha = 1 \Rightarrow \alpha + \beta + \gamma + \delta$

$\gamma + \delta = 1$

$\Sigma\alpha\beta = -1 \Rightarrow \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \delta\gamma$

$= \alpha(\gamma+\delta) + \beta(\gamma+\delta) + \alpha\beta + \delta\gamma$

$= (\alpha+\beta)(\gamma+\delta) + \alpha\beta + \delta\gamma$

$= \alpha\beta + \delta\gamma = -1 \quad \Rightarrow \quad -1 - \delta\gamma = \alpha\beta$

$\Sigma\alpha\beta\gamma = 1 \Longrightarrow \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

$\Rightarrow \alpha\beta(\gamma+\delta) + \delta\gamma(\alpha+\beta)$

$= \alpha\beta(\gamma+\delta)$

$1 = -1 - \delta\gamma \Longrightarrow -2 = \delta\gamma$

$\Sigma\alpha\beta\gamma\delta = -2 = \alpha\beta\gamma\delta$

$-2 = -2(\alpha\beta)$

$\alpha\beta = 1$

$\alpha = \frac{1}{\beta}$

$\frac{1}{\beta} + \beta = 0.$

$1 + \beta^2 = 0.$

$\beta^2 = -1$

$\beta = -i, i$

$\gamma = \frac{-2}{\delta}$

$\delta + \frac{-2}{\delta} = 1$

$\delta^2 - 2 = \delta$ ✓

$\delta^2 - \delta - 2 = 0.$

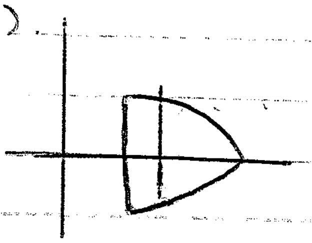
$(\delta+1)(\delta-2) = 0$

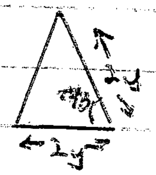
$\delta = -1, 2$

$\therefore$ Roots: $\underline{\underline{-i, i, -1, 2.}}$

1(a) ___

(b).





$$A(x) = \frac{1}{2} \times 2y \times 2y \sin \pi/3$$
$$= 2y^2 \cdot \sqrt{3}/2$$
$$= y^2\sqrt{3}$$ ✓

$$f(x): \frac{x^2}{9} + \frac{y^2}{4} = 1$$
$$\frac{4x^2}{9} + y^2 = 4.$$
$$y^2 = 4 - \frac{4x^2}{9}.$$

$$\therefore A(x) = \sqrt{3}\left(4 - \frac{4x^2}{9}\right)$$
$$= \frac{\sqrt{3}(36 - 4x^2)}{9}$$ ✓

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A+ x = k.

$$\underline{\underline{A(x) = \frac{\sqrt{3}}{9}(36 - 4k^2)}}$$

). $V = A(x) \cdot \delta x$

$$\lim_{\delta x \to 0} \sum_{1}^{3} A(x)\,dx$$ ✓
$$= \int_1^3 \frac{\sqrt{3}}{9}(36 - 4x^2)\cdot dx$$
$$= \sqrt{3}/9 \int 36 - 4x^2\, dx$$
$$= \sqrt{3}/9 \left[36x - \frac{4x^3}{3}\right]_1^3$$ ✓
$$= \frac{\sqrt{3}}{9}\left[(108 - 36) - (36 - 4/3)\right]$$
$$= \frac{\sqrt{3}}{9}\left[\frac{112}{3}\right]$$
$$= \underline{\underline{\frac{112\sqrt{3}}{27}}}$$ ✓

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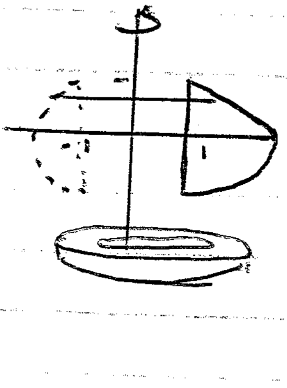

A+ x = 1, $\frac{1}{9} + \frac{y^2}{4} = 1 \Rightarrow \frac{y^2}{4} = 8/9$

$y^2 = 32/9$ ✓

$= \frac{2\sqrt{8}}{3}.$

$A(x) = (x_2^2 - x_1^2).$ ✓

$x_2^2 = 9 - \frac{9y^2}{4} \quad x_1^2 = 1.$

$V = A(x)\,dy$

$$= \pi \int_{-\frac{2\sqrt{8}}{3}}^{+2\sqrt{8}/3} 8 - \frac{9y^2}{4}\cdot dy$$
$$= 2\pi \left[8y - \frac{9y^3}{12}\right]^{\cancel{\frac{2\sqrt{8}}{3}}\ 4\sqrt{2}/3}$$
$$= 2\pi\left[\frac{32\sqrt{2}}{3} - \frac{9 \times 64 \times 2\sqrt{2}}{12 \times 27}\right]$$
$$= \frac{40\sqrt{2}}{9}\pi$$

5(a). x^2+4x+3

$= (x+1)(x+3)$

$= [(x+1)(x+3)]^6$ ✓

$= (x+1)^6\ (x+3)^6$ $\quad \therefore n=6.$

$= \left[\binom{6}{0} + \binom{6}{1}x + \binom{6}{2}x^2 + \ldots\ldots \binom{6}{6}x^6\right]\left[\binom{6}{0}3^6 + \binom{6}{1}3^5x + \binom{6}{2}3^4x^2 + \ldots\binom{6}{6}x^6\right]$ ✓

For x^4 coefficient

$= \binom{6}{0}\binom{6}{4}3^2 + \binom{6}{1}\binom{6}{3}3^3 + \binom{6}{2}\binom{6}{2}3^4 + \binom{6}{3}\binom{6}{1}3^5 + \binom{6}{4}\binom{6}{0}3^6$

$= 3^2[15 + 3\times120 + 9\times225 + 120\times27 + 15\times81]$

$= 9\times6855$ ✓

$= 61695.$

b) (i) $x^2-y^2=c^2$ $\qquad$ (ii) —— *Please ask me!*

$2x - 2y\frac{dy}{dx} = 0.$

$2x = 2y\cdot\frac{dy}{dx}$

$\frac{x}{y} = \frac{dy}{dx}$

At (x_1,y_1)

$m = \frac{x_1}{y_1}$ ✓

$y-y_1 = \frac{x_1}{y_1}(x-x_1)$

$y_1y - y_1y_1 = x_1x - x_1x_1$

$x_1x_1 - y_1y_1 = x_1x - y_1y$

$c^2 = x_1x - y_1y$

But $P(x_1,y_1)$ is on the hyperbola $\therefore x_1^2-y_1^2=c^2$

c). (i) $x = \sqrt{3}\cos 3t - \sin 3t.$

$= 2\cos(3t+\pi/6)$

$\dot{x} = -6\sin(3t+\pi/6)$ ✓

$\ddot{x} = -18\cos(3t+\pi/6)$

$= -9x$ ✓

$-n^2x$ form, therefore SHM

(ii) $n=3$, $T = \frac{2\pi}{n}$ $\qquad$ Amp. 2. ✓

$= \frac{2\pi}{3}$ ✓

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d). $2x^3 - 4x^2 - 3x - 1 = 0.$

$(\alpha-1)(\beta-1)(\gamma-1) = (\alpha\beta - (\alpha+\beta) + 1)(\gamma-1)$ ✓

$= \alpha\beta\gamma - (\alpha\gamma + \beta\gamma) + \gamma - \alpha\beta + \alpha + \beta + 1$

$= \alpha\beta\gamma - (\alpha\beta + \beta\gamma + \alpha\gamma) + \alpha + \beta + \gamma + 1$ ✓

$= 1/2 + 3/2 + 4/2 - 1 = 3$

(b). $x + y + z = 1.$

$xy + yz + zw = 9$

$xyz = 9.$ ✓

$\therefore x^3 - x^2 + 9x - 9 = 0.$

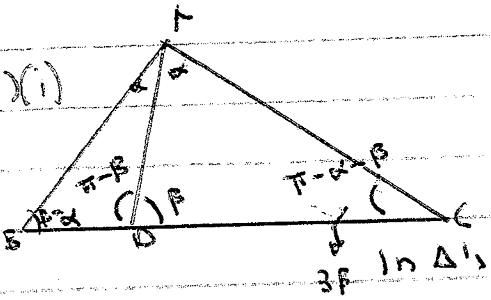
$p(1) = 0.$

$$\begin{array}{r} x^2 \qquad +9 \\ \therefore (x-1)\overline{)\,x^3 - x^2 + 9x - 9} \\ \underline{x^3 - x^2} \qquad\qquad \\ 0 + 9x - 9 \\ \underline{9x - 9} \\ 0. \end{array}$$ ✓

$\therefore (x-1)(x^2+9)$

$\therefore$ Roots: $\underline{\underline{1, 3i, -3i}}$ ✓

(c)(i)



Try again using sine rule in $\triangle ABD$ and $\triangle ADC$...

In Δ's ABD & ADC

$D\hat{A}B = D\hat{A}C$ (given)

Let $\beta = A\hat{D}C$, Let $\alpha = B\hat{A}D$

Then $A\hat{D}B = \pi - \beta$ (supp).

$\therefore A\hat{B}D = \pi - (\pi - \beta) - \alpha$ (Angle Sum)

$= \beta - \alpha$

$A\hat{C}D = \pi - (\alpha + \beta)$

$= \pi - \alpha - \beta$

$\pi - \alpha - \beta + \pi - \beta + 2\alpha = \pi$

$2\pi + \alpha - 2\beta = \pi$

$\alpha - 2\beta = \pi \implies 2\beta - \alpha = -\pi$

$\beta - \alpha = -\pi - \beta$

$\alpha - \beta - \beta = \pi$

$\therefore A\hat{B}D = \beta - \alpha = -(\pi + \beta)$

$-3\beta = \pi - \alpha - \beta$

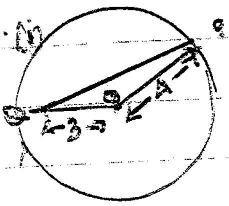
$\therefore A\hat{C}D = -3\beta$

i) From part(i)

$$\frac{BD}{DC} = \frac{BA}{AC}$$

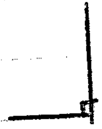
$$BD = \frac{DC \times BA}{AC}$$

7.(a).(i)



$$PQ^2 = OP^2 + OQ^2 \cancel{+}2.OP\,OQ\cos\theta$$
$$= 16 + 9 - 2\times3\times4\cos\theta$$
$$PQ^2 = 25 - 24\cos\theta \quad ✓$$
$$PQ = \sqrt{25-24\cos\theta}$$
$$= (25-24\cos\theta)^{\frac{1}{2}}$$
$$\frac{dPQ}{d\theta} = \frac{1}{2}\times 24\sin\theta\,[25-24\cos\theta]^{-1/2} \quad ✓$$
$$= \frac{12\sin\theta}{\sqrt{25-24\cos\theta}}$$

(ii) At 9 o'clock.

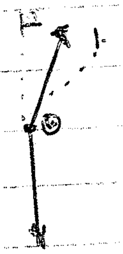


$\theta = \pi/2.$

$\frac{d\theta}{dt} =$

At 12 o'clock $\theta = 0.$

At 12:30 θ



$$\theta = \pi - \left(2\pi - \frac{2\pi}{24}\right) \quad ✓$$
$$= \pi - 2\pi + \frac{\pi}{12}$$
$$= -\frac{11\pi}{12} \qquad = \frac{11\pi}{12}.$$

$\therefore\ \theta$ increases $\frac{11\pi}{12}$ over 30

$\therefore\ \cancel{\frac{22\pi}{12}}\ \frac{22}{12}\pi$ per h

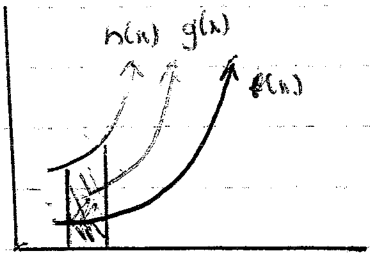
$$\frac{d\theta}{dt} = \frac{11\pi}{6}.$$

$$\frac{dPQ}{dt} = \frac{12\sin\theta}{\sqrt{25-24\cos\theta}} \times \frac{11\pi}{6} \quad ✓$$

At 9:00 $\theta = \pi/2.$

$$\frac{dPQ}{dt} = \frac{12}{5}\times\frac{11\pi}{6} \quad ✓$$
$$= \frac{22\pi}{5}.$$

)(i)



From graph

If $f(x) < g(x) < h(x)$ *Using Area under each curve*

Then $\int_a^b f(x)\,.dx < \int_a^b g(x)\,dx < \int_a^b h(x)\,dx$ ✓

(ii). If $0 < x < 1$

$x < 1$

$1 + x < 2$

$\frac{1}{1+x} > \frac{1}{2}$

$\therefore \frac{f(x)}{1+x} > \frac{f(x)}{2}$

$\frac{f(x)}{2} < \frac{f(x)}{1+x}$

$\therefore \frac{x(1-x)^3}{2} < \frac{x(1-x)^3}{1+x}$ ✓

If $0 < x < 1$

$x > 0$

$0 < x$

Now $1 < 1 + x$

$\frac{1}{1+x} < 1$

$\frac{f(x)}{1+x} < f(x)$ ✓

$\therefore \frac{x(1-x)^3}{1+x} < x(1-x)^3$

$$\therefore \frac{x(1-x)^3}{2} < \frac{x(1-x)^3}{1+x} < x(1-x)^3$$ ✓

(iii) From part(i)

$$\int_a^b f(x)\,.dx < \int_a^b g(x)\,dx < \int_a^b h(x)\,dx$$

$$\therefore \frac{1}{2}\int_0^1 x(1-x^3)\,dx < \int_0^1 \frac{x(1-x)^3}{1+x}\,.dx < \int_0^1 x(1-x)^3\,dx$$

(iv) If $\int_0^1 \frac{x(1-x)^3}{1+x} = \frac{67}{12} - 8\ln 2$

Then For $\int_0^1 x(1+x)^3\,.dx$ → *Use by parts! Try again*

Show that $\int = \frac{1}{20}$

$= \frac{1}{2}\int_0^1 \frac{(x(1-x)^3}{1+x} \times \int_0^1 1+x\,dx$

$= \frac{1}{2}\left[\frac{67}{12} - 8\ln 2\right]\left[x + \frac{x^2}{2}\right]_0^1$

$= \frac{1}{2} \times 3/2\left[\frac{67}{12} - 8\ln 2\right]$

$= 3/4\left[\frac{67}{12} - 8\ln 2\right]$

$= \frac{67}{16} - 6\ln 2$

3 (a). $\ddot{x} = 0$ $\quad\quad$ $\ddot{y} = -g$

$\dot{x} = V\cos\alpha$ $\quad\quad$ $\dot{y} = -gt + V\sin\alpha$

$x = Vt\cos\alpha$ $\quad\quad$ $y = \frac{-gt^2}{2} + Vt\sin\alpha$ ✓

$$\frac{x}{V\cos\alpha} = t$$

14

$$y = \frac{-gx^2}{2V^2\cos^2\alpha} + \frac{xV\sin\alpha}{V\cos\alpha}$$

$$= -\frac{gx^2(1+\tan^2\alpha)}{2V^2} + x\tan\alpha$$

$$= \frac{-g^2x^2}{2V^2} + x\tan\alpha - \frac{gx^2\tan^2\alpha}{2V^2}$$ ✓

$$\text{But } k = \frac{V^2}{2g}$$

$$2gk\ V^2$$

$$y = \frac{-g^2x^2}{4gk} + x\tan\alpha - \frac{gx^2\tan^2\alpha}{4gk}$$ ✓

$$y = \frac{-x^2}{4k} + x\tan\alpha - \frac{x^2\tan^2\alpha}{4k}$$

$$4ky = -x^2 + 4kx\tan\alpha - x^2\tan^2\alpha$$ ✓

$$\underline{x^2\tan^2\alpha - 4kx\tan\alpha + (x^2+4ky)} = 0.$$

(ii). $\therefore \Delta > 0$ For Two α values.

$$16k^2x^2 - 4x^2(x^2+4ky) > 0.$$

$$4x^2[4k^2 - (x^2+4ky) > 0.$$ ✓

$$4k^2 - 4ky - x^2 > 0.$$

$$4k^2 - 4ky > x^2$$

$$x^2 < 4k^2 - 4ky$$ ✓

$$\text{At } (X, Y)$$

$$\underline{\underline{X^2 < 4k^2 - 4kY}}$$

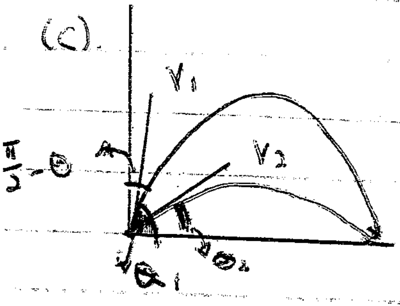
~~$x^2 + 4ky < 4k^2$~~

(iii). ~~$\tan\alpha_1 . \tan\alpha_2$~~ $= \frac{c}{a}$

$$= \cancel{\frac{x^2+4ky}{\frac{x^2}{4k}}} \quad \cancel{< \frac{4k^2}{x}}$$

$=$

(c).



Max Range: $gR = V^2 \sin 2\theta$

$\sin 2\theta = \frac{gR}{V^2}$ ——— ①.

$\sin(\pi - 2\theta) = \frac{gR}{V^2}$ ——— ②

$2\theta = \sin^{-1}\left(\frac{gR}{V^2}\right)$ ——— ①.

$\theta_1 = \frac{1}{2}\sin^{-1}\left(\frac{gR}{V^2}\right)$

$\pi - 2\theta = \sin^{-1}\left(\frac{gR}{V^2}\right)$ ——— ②.

$2\theta = \pi - \sin^{-1}\left(\frac{gR}{V^2}\right)$

$\therefore \theta_2 = \pi/2 - \frac{1}{2}\sin^{-1}\left(\frac{gR}{V^2}\right)$

Quicker to use product of roots from ~~(iii)~~ (i) i.e.

$\tan\alpha_1 \cdot \tan\alpha_2 = \frac{4ky + x^2}{x^2} > 1$

$\therefore$ if $\alpha_1 < \frac{\pi}{4}$ then $\tan\alpha_1 < 1$ $\therefore$ $\theta_1 + \alpha_2 = \pi/2$

$\Rightarrow \tan\alpha_2 > 1$ & vice versa. $\therefore$ Both can't be less than $\pi/4$.

). (i) $\frac{A^{n+1} - A^n B + B^{n+1} - B^n A}{A - B}$

$\frac{A^n[A - B] - B^n[A - B]}{A - B}$ ✓

$\frac{[A^n - B^n][A - B]}{A - B}$

$= A^n - B^n$ ✓

(ii). If $A > 0$, $B > 0$.

$A^n - B^n > 0$.

$(A^n - B^n)(A - B) > 0$.

$A^{n+1} - A^n B + B^{n+1} - B^n A > 0$.

$A^{n+1} + B^{n+1} > A^n B + B^n A$

Note: $\frac{A^{n+1} - A^n B + B^{n+1} - B^n}{(A - B)}$

$= \frac{(A^n - B^n)(A - B)}{A - B}$

if $A - B > 0 \Rightarrow A^n - B^n > 0$

if $A - B < 0 \Rightarrow A^n - B^n > 0$.

$\therefore \frac{A^{n+1} - A^n B + B^{n+1} - B^n A}{A - B} > 0$

$\Rightarrow A^{n+1} + B^{n+1} > A^n B + B^n A$

(iii) $\left(\frac{A+B}{2}\right)^n \leq \frac{A^n + B^n}{2}$

Let $n = 1$

$\frac{A+B}{2} \leq \frac{A+B}{2}$

True.

Assume for $n = k$

$\left(\frac{A+B}{2}\right)^k \leq \frac{A^k + B^k}{2}$

Prove for $n = k+1$

LHS: $\left(\frac{A+B}{2}\right)^{k+1}$

$= \left(\frac{A+B}{2}\right)^k \left(\frac{A+B}{2}\right) \leq \left(\frac{A^k + B^k}{2}\right)\left(\frac{A+B}{2}\right)$ ✓

$\leq \frac{A^{k+1} + A^k B + A B^k + B^{k+1}}{4}$

$$\left(\frac{A+B}{2}\right)^k \left(\frac{A+B}{2}\right) \leq A^{k+1} + \frac{A^kB}{4} + AB^k + B^{k+1} \checkmark$$

$$\leq 2\left(\frac{A^{k+1}}{4} + B^{k+1}\right)$$

(from part(ii))

$$\therefore \left(\frac{A+B}{2}\right)^k \left(\frac{A+B}{2}\right) \leq \frac{A^{k+1}}{2} + B^{k+1} \checkmark$$

$$\therefore \left(\frac{A+B}{2}\right)^{k+1} \leq \frac{A^{k+1}}{2} + B^{k+1}$$

$$\leq A^{k+1} + B^{k+1}$$

if true for $n=k$, then shown ~~too~~ true for $n=k+1$. Therefore by the principles of Mathematical Induction, true for $n \geq 1$ integers.