ASQUITH BOYS HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

1996

MATHEMATICS

3 Unit (Additional)
and
3/4 Unit (Common)

Time allowed - TWO hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL question are of equal value.
- ALL necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on the last page. These may be removed for your convenience.
- Board-approved calculators may be used.
- Each question should be started on a new page.

Question 1 (Start a new page) Marks

a. Find the exact value of $\int_0^4 \frac{dx}{x^2+16}$ 3

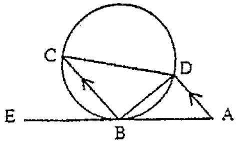
b. Find $\int (1-\cos x)^2 dx$ 3

c. Solve the inequality $\frac{3}{x-2} \geq 1 \quad x \neq 2$ 3

d. Find the first derivative of $y = \log_e\left(\frac{1}{\sqrt{\cos x}}\right)$ 3

Question 2 (Start a new page) Marks

a. 4



AB is a tangent at B and $AD /\!/ BC$. Prove that $\Delta BCD \,|||\, \Delta DBA$.

b. Find $\int \frac{2x}{(x-1)^2} dx$ using the substitution $u = x-1$ 4

c. Prove by the method of Mathematical Induction that 4

$$\sum_{r=1}^{n} 5^{r-1} = \frac{5^n - 1}{4}$$

Question 3 (Start a new page) Marks

a. If $^{12}P_r = 120.{}^{12}C_r$ find r 3

b. The velocity of a particle moving in a straight line is given by 5

$v^2 = 8x - 2x^2$ $m/\sec$

- i. Show that the particle is moving in simple harmonic motion.
- ii. Find the centre of the motion.
- iii. Determine the two end points between which the particle is oscillating.
- iv. Find the maximum speed of the particle.

c. A formula for the rate of change in population of a colony of bacteria, is given by $P = 3200 + 400e^{kt}$ 4

If the population doubles after 20 hours, how long would it take to triple the original population.

Question 4 (Start a new page) Marks

a. At what points on the curve $y = \cos^{-1} x$, is the gradient equal to $-\frac{2}{\sqrt{3}}$ 3

b. Find the middle term in the expansion of $\left(x^3 - \frac{1}{3x}\right)^8$ 4

(c) A capsule is in the shape of a cylinder with hemispherical ends. The radius of the cylindrical section is $r$cm, and the volume of the capsule is 16cm^3. 5

- (i) If the height of the cylinder is 4cm show that $r^3 + 3r^2 = \frac{12}{\pi}$
- ii. Show that one solution of the equation $r^3 + 3r^2 = \frac{12}{\pi}$ lies between 0 and 1.
- iii. The equation $r^3 + 3r^2 = \frac{12}{\pi}$ has a root close to 0·9. Use one application of Newton's method to give a better approximation.

Question 5 (Start a new page) Marks

a. Solve the equation $3x^3 - 17x^2 - 8x + 12 = 0$ given that the product of two of the roots is 4 3

(b) The probability that a vaccine succeeds is $\frac{29}{30}$. An experiment is conducted m times with white mice. 4

- i. What is the probability that the experiment will fail at least once?
- ii. Show that if the probability that the experiment will fail at least once in m trials, is greater than than $\frac{9}{10}$ then $m > \frac{1}{\log_{10} 30 - \log_{10} 29}$

c. For a particular vessel, the rate of increase of the volume with respect to its depth, is given by $\frac{dV}{dh} = \frac{\pi(h+6)^2}{12}$ $0 \le h \le 10$ 5

where $V\text{cm}^3$ is the volume and h is the depth of the water.

- (i) If the container is initially empty, show that the volume as a function of the depth is $V = \frac{\pi h}{36}(h^2 + 18h + 108)$
- ii. Find the volume when the depth is 6cm.
- iii. If water is being poured into the vessel at a constant rate of $8\text{cm}^3/s$ find an expression for the rate of increase in the depth of the water.
- iv. At what rate is the depth increasing when the water level is 6 cm, and how long will it take to the nearest second to reach this level.

Question 6 (Start a new page) Marks

a. The letters of the word **REPETITION** are arranged at random in a row. 3

- i. how many different arrangements are possible?
- ii. what is the probability that one particular arrangement will have vowels and consonants alternating?

(Question 6 continued on page 4)

Question 6 Continued

Marks

b. i. Write the general expansion of $(1+x)^n$ 3

ii. Hence or otherwise prove that

$$^nC_0+\tfrac{1}{2}\,^nC_1,\ +\tfrac{1}{3}\,^nC_2+......+\frac{1}{n+1}\ ^nC_n=\frac{2^{n+1}-1}{n+1}$$

c. The curve $y=\sin^{-1}x$ intersects the curve $y=\cos^{-1}x$ at P, and the latter intersects the x axis at Q 6

i. Draw a neat sketch of this information.

ii. Verify that P has co-ordinates $\left(\frac{1}{\sqrt{2}}\ .\ \frac{\pi}{4}\right)$

iii. Prove $\frac{d}{dx}\left(x\sin^{-1}x+\sqrt{1-x^2}\right)=\sin^{-1}x$

iv. If O is the origin, find the area enclosed by the arcs OP and PQ and the x axis using the results in (iii) and the fact that

$$\frac{d}{dx}\left(x\cos^{-1}x-\sqrt{1-x^2}\right)=\cos^{-1}x$$

Question 7 (Start a new page)

Marks

A projectile fired with velocity V and at an angel $45°$ to the horizontal, just clears the tops of two vertical posts of height $8a^2$, and the posts are $12a^2$ apart. There is no air resistance, and the acceleration due to gravity is g.

a. If the projectile is at the point (x, y) at time t, derive expressions for x and y in terms of t. 3

b. Hence show that the equation of the path of the projectile is $y=x-\frac{gx^2}{V^2}$ 2

c. Using the information in (b) show that the range of the projectile is $\frac{V^2}{g}$ 2

d. If the first post is b units from the origin, show 2

i. $\frac{V^2}{g}=2b+12a^2$

ii. $8a^2=b-\frac{gb^2}{V^2}$

e. Hence or otherwise prove that. $V=6a\sqrt{g}$ 3

Asquith Boys High Maths 3U Trial $\frac{77}{84}$ Dr Ong.

V. Good effort!

(a) $\int_0^4 \frac{dx}{x^2+16}$

$= \left[\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)\right]_0^4$ ✓ ✓

$= \frac{1}{4}\tan^{-1} 1$ ✓

$= \frac{\pi}{16}$ ✓

(b). $\int (1-\cos x)^2 . dx$

$= \int 1 - 2\cos x + \cos^2 x$

$= \int 1 - 2\cos x + \frac{\cos 2x + 1}{2}$

$= \frac{1}{2}\int 3 - 4\cos x + \cos 2x . dx$ ✓

$= \frac{1}{2}\left[3x - \frac{4\sin x}{1} + \frac{\sin 2x}{2}\right] + c.$ ✓

$\cos 2x = 2\cos^2 x - 1$

$\frac{\cos 2x + 1}{2} = \cos^2 x$

$\boxed{12}$

c. $\frac{3}{(x-2)} \geq 1$

$3(x-2) \geq (x-2)^2$ ✓

$3x - 6 \geq x^2 - 4x + 4$

$x^2 - 7x + 10 \leq 0.$

$(x-2)(x-5) \leq 0.$ ✓

$2 < x \leq 5$ ✓

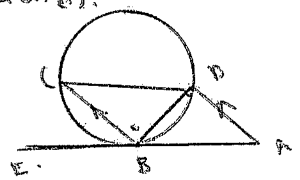
(d). $y = \ln\left(\frac{1}{\sqrt{\cos x}}\right)$

$= \ln(1) - \frac{1}{2}\ln(\cos x)$ ✓

$y' = \frac{\sin x}{2\cos x}$

$= \frac{1}{2}\tan x.$ ✓ ✓

Q2. (a).



In Δ's BCD & DBA

$B\hat{D}A = C\hat{B}D$ (alternate ∠'s). ✓

$D\hat{B}A = B\hat{C}D$ (alternate segment) ✓

∴ $\Delta BCD ||| \Delta DBA$ ✓

(b). $\int \frac{2x}{(x-1)^2}$

Let $u = x-1 \implies u+1 = x.$

$du = dx.$

$I = \int \frac{2(u+1)}{u^2} . du$ ✓

$= \int \frac{2u}{u^2} + \frac{1}{u^2}\, du$

$= \ln|u^2| + -\frac{1}{u} \Longrightarrow 2\ln|x-1| - \frac{1}{x-1} + c.$ ✓

$\sum_{r=1}^{n} 5^{r-1} = \frac{5^n - 1}{4}.$

Let n=1

LHS = 5^0 RHS = $\frac{5^1 - 1}{4}$

= 1 = 1. ✓

$\boxed{11}$

Assume true for n=k.

$5^0 + 5^1 + 5^2 + \ldots 5^{k-1} = \frac{5^k - 1}{4}$

Prove for n=k+1

LHS = $\frac{5^k - 1}{4} + 5^k$ RHS = $\frac{5^{k+1} - 1}{4}.$

$= \frac{5^k - 1 + 4.5^k}{4.}$

$= \frac{5^k[1+4] - 1}{4.}$

$= \frac{5^{k+1} - 1}{4.}$ ✓

= RHS.

3

If true for n=k, and n=k+1, then true for n ≥ 1 integers by the principles of Mathematical Induction.

Q3.

(a). ${}^{12}P_r = 120\ {}^{12}C_r.$

$\frac{12!}{(12-r)!} = 120 . \frac{12!}{(12-r)!\, r!}$

$1 = \frac{120}{r!}$ ✓

$r! = 120.$

$5! = 120.$ ✓

$\therefore r = 5.$

(b). $v^2 = 8x - 2x^2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}(4x - x^2)$

$= 4 - 2x$ ✓

$= -2(x-2) \longrightarrow = -n^2 x$ ✓

$\therefore X = x - 2$ ∴ SHM.

$\ddot{X} = \ddot{x}$ ✓

(ii). Centre = 2. ✓

(iii). $0 = 8x - 2x^2$

$= 2x(4 - x)$

$0 \le x \le 4$. ✓

(iv). ~~at~~ $\ddot{x} = 0$.

ie $x = 2$.

$\therefore v^2 = 16 - 8$

$= 8$

$v = 2\sqrt{2}$. ✓

12

(c). $P = 3200 + 400e^{kt}$.

$t = 0$. ~~[illegible]~~

$P_0 = 3200 + 400$

$= 3600$. ✓

$t = 20$ $P = 7200$

$7200 = 3200 + 400e^{20k}$

$4000 = 400e^{20k}$

$10 = e^{20k}$ ✓

$\frac{\ln(10)}{20} = k$.

For $P = 10800$.

$10800 = 3200 + 400e^{\frac{\ln(10)}{20}t}$.

$19 = e^{\frac{\ln(10)t}{20}}$ ✓

$\frac{20\ln(19)}{\ln(10)} = t$.

$t = 25$h, 35 mins. ✓

4. (b). $y = \cos^{-1}x$

$y' = \frac{-1}{\sqrt{1-x^2}} = \frac{-2}{\sqrt{3}}$

$\sqrt{3} = 2\sqrt{1-x^2}$

$3 = 4(1-x^2)$

$3 = 4 - 4x^2$ ✓

$4x^2 = 1$

$x^2 = \frac{1}{4}$.

$x = \pm 1/2$. ✓

$y = \frac{\pi}{3}, \frac{2\pi}{3}$

$\therefore$ Points are $\left(\frac{1}{2}, \frac{\pi}{3}\right)$ and $\left(-\frac{1}{2}, \frac{2\pi}{3}\right)$

(b) $\left(x^3 - \frac{1}{3x}\right)^8$

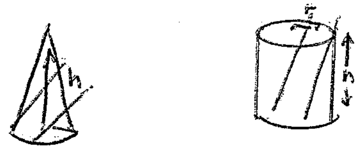
$T_5 = T_{4+1} = \binom{n}{r}(a)^{n-r}(b)^r$ ✓

$= \binom{8}{4}(x^3)^4((3x)^{-1})^4$. ✓

$= 70 \times x^{12} \times \frac{x^{-4}}{81}$

$= \frac{70}{81}x^8$. ✓

(c) (i).



~~$V = \pi r^2 h = 16$~~

~~$4\pi r^2 = 16$~~

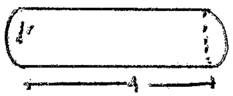
~~$\pi r^2 = 16$~~ $\frac{4}{\pi}$

V = 16cm.

~~$V = \pi r^2 h$~~

~~$16 = \pi r^2$~~ 4.

~~$4 = \pi r^2$~~



V = 16

$V = \frac{4}{3}\pi r^3 + \pi r^2 4 = 16$. ✓

$\frac{\pi r^3}{3} + \pi r^2 = 4$

$\pi r^3 + 3\pi r^2 = 12$ ✓

$r^3 + 3r^2 = \frac{12}{\pi}$

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(ii) $P(r) = r^3 + 3r^2 - \frac{12}{\pi}$

$P(0) = -\frac{12}{\pi}$ ∴ −ve ✓

$P(1) = 4 - \frac{12}{\pi}$

= +ve.

∴ root between 0, 1 ✓

$P'(x) = 3r^2 + 6r$.

(iii). $x_2 = 0.9 - \frac{f(0.9)}{f'(0.9)}$

$= 0.9 - \frac{(-0.6607)}{7.83}$ ✓

$= 0.9844$.

5.(b). $3x^3 - 17x^2 - 8x + 12 = 0$.

$4\gamma = -4$. ✓

$\gamma = -1$

~~$\alpha\beta + \beta\gamma + \alpha\gamma = -\frac{8}{3}$.~~

~~$4 - \beta - \alpha = -8/3$.~~

~~$-\beta - \alpha = -20/3$~~

~~$\alpha + \beta = 20/3$.~~

$$x+1 \overset{\displaystyle 3x^2 - 20x + 12.}{\overline{\big)\,3x^3 - 17x^2 - 8x + 12}}$$ ✓

$3x^3 + 3x^2$

$-20x^2 - 8x$

$-20x^2 - 20x$

$12x + 12$

$\Rightarrow$ ~~$\alpha + \beta + \gamma = \frac{17}{3}$~~

$(x+1)(3x^2-20x+12)=0$

$(x+1)(3x-2)(x-6)=0$ ✓

$\therefore \underline{\underline{x=-1, \ 3/2, \ 6.}}$

(b)(i) $p = 29/30$. $q = 1/30$ $n=m$.

$(29/30 + \frac{1}{30})^m$

$P(\text{At least one}) = P(\text{All}) - P(\text{None})$

$= \underline{1 - \binom{m}{0}\left(\frac{29}{30}\right)^m}$ ✓

(ii). $1-\left(\frac{29}{30}\right)^m > \frac{9}{10}$

$\therefore \frac{1}{10} > \left(\frac{29}{30}\right)^m$

$\log_{10}(10^{-1}) > m \log_{10}\left(\frac{29}{30}\right)$

$-1 > m\left[\log_{10}\frac{29}{30}\right]$

$\therefore \frac{-1}{\log_{10}\left(\frac{29}{30}\right)} < m$

$\therefore \frac{1}{\log_{10}\left(\frac{30}{29}\right)} < m$

$\therefore m > \frac{1}{\log_{10}30 - \log_{10}29}$

(c) (i) $\frac{dV}{dh} = \frac{\pi(h+6)^2}{12}.$

$V = \int \underline{\pi}(h+6)^2 \cdot dh$

$= \frac{\pi}{12}\int (h+6)^2 \cdot dh$ ✓

$= \frac{\pi}{12}\frac{(h+6)^3}{3} + c.$ ✓

$= \frac{\pi(h+6)^3}{36} + c$

At $h=0, V=0.$

$V = \frac{\pi}{36}(0^3+6)^3 + c.$

$\therefore c = -6\pi$ ✓

$\therefore V = \pi/36\,(h+6)^3 - 6\pi$

$= \frac{\pi}{36}(h^3 + 18h^2 + 108 + 108) - 6\pi$

$= \underline{\underline{\frac{\pi h}{36}(h^2+18h+108).}}$ ✓

(ii). $V = \frac{\pi h}{36}(h^2+18h+108)$

At $h=6$,

$V = \frac{\pi}{6}(36+108+108)$ ✓

$= 42\pi$

(iii). $\frac{dV}{dt} = 8.$

$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$ ✓

$= 8 \times \frac{12}{\pi(h+6)^2} \Rightarrow \frac{96}{\pi(h+6)^2}$

⑨

(iv) $\frac{dh}{dt} = \frac{96}{\pi(h+6)^2}$

At $h=6.$

$\frac{dh}{dt} = \frac{96}{\pi 144}$

$= \frac{2}{3\pi}$ ✓

$\underline{\underline{h = \frac{2t}{3\pi}}}$

$\frac{dt}{dh} = \frac{\pi(h+6)^2}{96}$

At $h=6$

$t = \frac{\pi}{96}\int (h+6)^2 \, dh$ ✓

$= \frac{\pi}{96}\frac{[h+6]^3}{3\times 96}$

At $h=6.$ ✓

$\underline{\underline{t = 6\pi.}} = 19s$ (to the nearest sec)

6.(a)(i) REPETITION

2 Es, 2 Is, 2 Ts.

$\frac{10!}{2!2!\times 2} = \underline{\underline{604\,000.}}$ ✓

(ii) 5 Vowels, 5 consonants

$\frac{5! \times 5! \times 2}{604000} \longleftarrow$ i.e. R E P E T I T I N O

$= \underline{\underline{\frac{36}{755}}}$ ✓ ✗

$\frac{\frac{5!}{\cancel{2!}} \times \frac{5!}{\cancel{2!}} \times 2}{\frac{10!}{\cancel{2!}\cancel{2!}\cancel{2!}}} = \frac{[illegible]}{\cancel{10}\cdot 9\cdot\cancel{8}\cdot 7\cdot\cancel{6}} = \frac{1}{7}$

(b)(i) $(1+x)^n$

$= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots \binom{n}{k}x^k + \ldots \binom{n}{n}x^n$

Integrating

$\frac{(1+x)^{n+1}}{n+1} + c_1 = \binom{n}{0}x + \binom{n}{1}\frac{x^2}{2} + \binom{n}{2}\frac{x^3}{3} + \ldots \binom{n}{k}\frac{x^{k+1}}{k+1} + \ldots \binom{n}{n}\frac{x^{n+1}}{n+1} + c_2$

$\frac{(1+x)^{n+1}}{n+1} + k = \binom{n}{0}x + \binom{n}{1}\frac{x^2}{2} + \binom{n}{2}\frac{x^3}{3} +$ ✓ $\ldots \binom{n}{k}\frac{x^{k+1}}{k+1} + \ldots \binom{n}{n}\frac{x^{n+1}}{n+1}$

Let $x=0.$

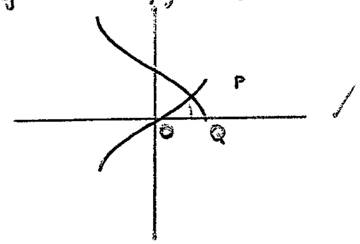
$\frac{1}{n+1} = -k$

$\frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = \binom{n}{0}x + \binom{n}{1}\frac{x^2}{2}$ ✓ $+ \binom{n}{2}\frac{x^3}{3} + \ldots \binom{n}{k}\frac{x^{k+1}}{k+1} + \ldots \binom{n}{n}\frac{x^n}{n}$

Let $x=1$

$\frac{2^{n+1}-1}{1+n} = \binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \ldots \frac{1}{n+1}\binom{n}{n}$

(c). $y = \sin^{-1}x$, $y = \cos^{-1}x$.



~~$\sin^{-1}x = \cos^{-1}x$.~~

$y = \sin^{-1}x$

$x = \sin y \qquad x = \cos y$

$\therefore \sin y = \cos y$

$\tan y = 1$

$y = \pi/4$.

At $y = \pi/4$, $x = \frac{1}{\sqrt{2}}$.

$\therefore \underline{\underline{\left(\frac{1}{\sqrt{2}}, \pi/4\right)}}$.

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(iii). $\frac{d}{dx}\left(x\sin^{-1}x + \sqrt{1-x^2}\right)$

$u = x \qquad v = \sin^{-1}x$

$u' = 1 \qquad v' = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\left[(1-x^2)^{1/2}\right] = \frac{x}{2}x - 2\left[(1-x^2)^{-1/2}\right]$

$= \frac{-x}{\sqrt{1-x^2}}$

$= \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x \quad - \frac{x}{\sqrt{1-x^2}}$

$\underline{\underline{= \sin^{-1}x.}}$

(iv). $A = \int_0^{\frac{1}{\sqrt{2}}} \sin^{-1}x + \int_{\frac{1}{\sqrt{2}}}^{1} \cos^{-1}x$.

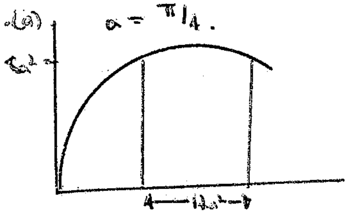
$= \left[x\sin^{-1}x + \sqrt{1-x^2}\right]_0^{\frac{1}{\sqrt{2}}} + \left[x\cos^{-1}x - \sqrt{1-x^2}\right]_{\frac{1}{\sqrt{2}}}^{1}$.

$= \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 + \left[-\left(\frac{\pi}{4\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right.$

$= \frac{2}{\sqrt{2}} - 1$

$\underline{\underline{= \sqrt{2} - 1}}$

7.(a) $\alpha = \pi/4$.



$\ddot{x} = 0$.

$\dot{x} = 0t + c$

At $\dot{x} = -V\cos\alpha + c$

$\therefore \dot{x} = V\cos\alpha$

$x = Vt\cos\alpha$

$\ddot{y} = -g$

$\dot{y} = -gt + V\sin\alpha$

$y = \frac{-gt^2}{2} + Vt\sin\alpha$

$\alpha = \pi/4$.

$x = \frac{Vt}{\sqrt{2}}$, $y = \frac{-gt^2}{2} + \frac{Vt}{\sqrt{2}}$.

(b). $x = \frac{Vt}{\sqrt{2}} \qquad y = \frac{-gt^2}{2} + \frac{Vt}{\sqrt{2}}$.

$\frac{\sqrt{2}x}{V} = t$.

$y = \frac{-2x^2g}{2V^2} + x$

$\underline{\underline{= x - \frac{gx^2}{V^2}}}$

(c). when $y = 0$,

$x - \frac{gx^2}{V^2} = 0$.

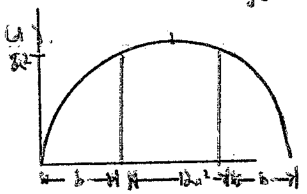
$x\left[1 - \frac{gx}{V^2}\right] = 0$.

$1 = \frac{gx}{V^2}$

$V^2 = gx$

Range: $\underline{\underline{x = \frac{V^2}{g}}}$

12



(i) Range $= 2b + 12a^2$ (Symmetry)

$\underline{\underline{\frac{V^2}{g} = 2b + 12a^2}}$

(ii) Passes through $(b, 8a^2)$

$\underline{\underline{8a^2 = b - \frac{gb^2}{V^2}}}$

~~(e) $8a^2 + \frac{gb^2}{V^2} = b$.~~

~~$\therefore \frac{V^2}{g} = 2\left(8a^2 + \frac{gb^2}{V^2}\right) + 12a^2$~~

~~$V^2 = 16a^2g + \frac{2gb^2}{V^2} + 12a^2g$~~

~~$V^4 - V^2(28a^2g) + \frac{2gb^2}{4} = 0$.~~

~~$V^2 = 28a^2g \pm \sqrt{784a^4g^2 - 8g^2}$~~

PTO.

$V^2 = g(2b + 12a^2)$

$\frac{V^2}{2g} - \frac{12a^2}{2} = b.$

$\frac{V^2}{2g} - 6a^2 = b.$

$\therefore 8a^2 = \frac{V^2}{2g} - 6a^2 - \frac{g(\frac{V^2}{2g} - 6a^2)^2}{V^2}$ ✓

$= \frac{V^2}{2g} - \cancel{6a^2} - \frac{gV^4}{4g^2V^2} + \frac{\cancel{6a^2V^2}}{V^2} + \frac{36a^4}{V^2}g$

$= \frac{V^2}{2g} - \frac{V^2}{4g} + \frac{36a^4}{V^2}g$

$8a^2 = \frac{V^2}{4g} + \frac{36a^4}{V^2}g$

$V^2(32a^2g) = V^4 + 144a^4g^2$ ✓

$V^4 - V^2(32a^2g) + 144a^4g$

$V^2 = \frac{32a^2g \pm \sqrt{1024a^4g^2 + 576a^4g^2}}{2}$

$V^2 = \frac{32a^2g \pm a^2g\sqrt{1600}}{2}$

$= \frac{32a^2g \pm 40a^2g}{2}$

$= \frac{72a^2g}{2}$

$V^2 = 36a^2g$ ✓

$\underline{\underline{v = 6a\sqrt{g}.}}$