ACE Examinations

Student Name: ____________________

2011

YEAR 12

TRIAL HSC EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time - 5 minutes
- Working time - 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks - 120

- Attempt Questions 1-8
- All questions are of equal value

Total marks - 120
Attempt Questions 1 - 8
All questions are of equal value

Answer each question in the appropriate writing booklet.

Question 1 (15 marks) **Marks**

(a) By completing the square, find $\int \frac{2}{x^2+4x+13}dx$ **2**

(b) Use integration by parts to evaluate $\int 3xe^x\,dx$. **2**

(c) (i) Find real numbers a, b and c such that **3**

$$\frac{7x+4}{(x^2+1)(x+2)}=\frac{ax+b}{x^2+1}+\frac{c}{x+2}$$

(ii) Hence find $\int \frac{7x+4}{(x^2+1)(x+2)}dx$ **2**

(d) Evaluate $\int_0^1 xe^{-x^2}dx$ **2**

(e) Use the substitution $t=\tan\frac{\theta}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta+2\sin\theta+3}d\theta$ **4**

Express your answer correct to 3 significant figures.

Question 2 (15 marks) **Marks**

(a) Let $z=1+2i$ and $w=-2+i$. Express the following in the form $a+ib$, where, a and b are real numbers:

(i) zw **1**

(ii) $\frac{5}{iw}$ **1**

(b) If $z_1=4+i$ and $z_2=1+2i$ show geometrically how to construct the vectors representing.

(i) z_1+z_2 **1**

(ii) z_1-z_2 **1**

(c) Sketch the locus of z on the Argand diagram where the inequalities $|z-1|\leq 3$ and $\text{Im}(z)\geq 3$ hold simultaneously. **3**

(d) Let $z=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$. Find z^6. **2**

(e) It is given that $3+i$ is a root of $P(z)=z^3+az^2+bz+10$ where a and b are real numbers.

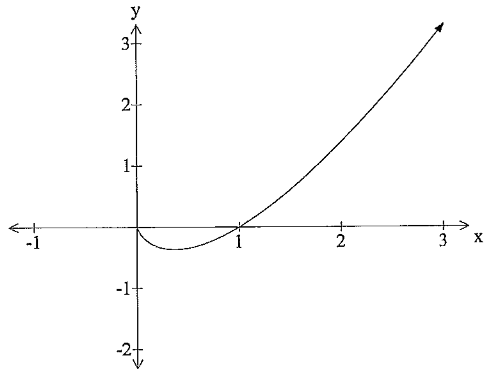
(i) State why $3-i$ is also a root of $P(z)$. **1**

(ii) Factorise $P(z)$ over the real numbers. **2**

(f) Solve the equation $z^2=i\bar{z}$ **3**

Question 3 (15 marks) **Marks**

(a) The diagram shows the graph of the function $y=f(x)$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y=|f(x)|$ **1**

(ii) $y=\frac{1}{f(x)}$ **2**

(iii) $y=(f(x))^2$ **2**

(b) (i) Sketch the graphs of $y=\ln x$ and $y=\frac{1}{x}$ for $x>0$. **2**

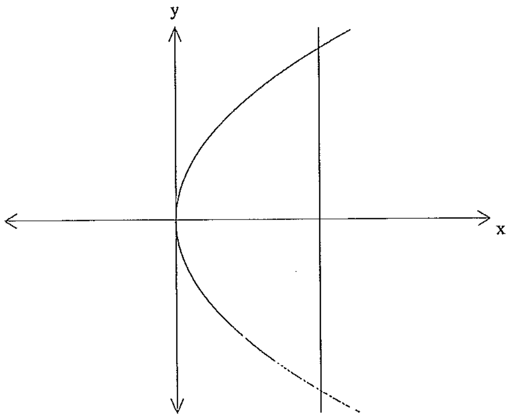
(ii) Hence sketch the graph of $y=\ln x+\frac{1}{x}$ **1**

(c) The polynomial equation $x^3-3x^2-x+2=0$ has roots α, β and γ.

(i) Find the polynomial equation with roots $2\alpha+\beta+\gamma$, $\alpha+2\beta+\gamma$ and $\alpha+\beta+2\gamma$. **2**

(ii) Find the polynomial equation with roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$. **2**

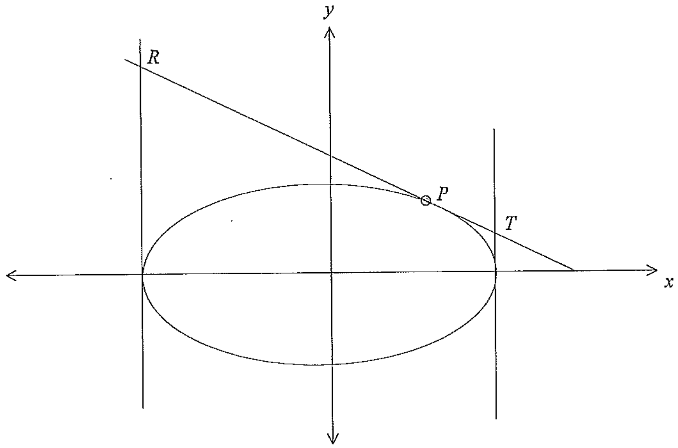
(d) A solid is formed by rotating the region enclosed by the parabola $y^2 = 4ax$, its vertex (0,0) and the line $x = a$, about the x-axis.



Find the volume of this solid using the method of cylindrical shells. **3**

Question 4 (15 marks) **Marks**

(a)



The point P lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b > 0$. The tangent at P meets the tangents at the ends of the major axis at R and T.

(i) Use the parametric representation of an ellipse to show the equation of the tangent is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$. **2**

(ii) Show that RT subtends a right angle at either focus. **3**

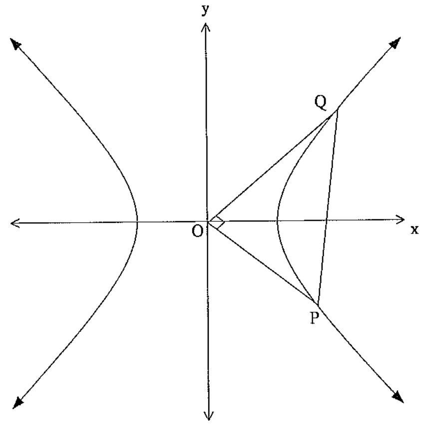
(b) A conical pendulum consists of a body P of mass m kg and a string of length l metres. Point A is fixed and the body P rotates in a horizontal circle of radius r and centre O at a constant angular velocity of ω radians per second. OA is vertical and has a length of h metres. The angle OAP is θ radians. The body, P, is subject to a gravitational force of mg newtons. The tension in the string is T newtons.

(i) Write down an expression for the vertical component of T. **1**

(ii) Show that $\omega^2 = \frac{g}{h}$ by resolving forces. **3**

(iii) What is the period of the motion? **1**

(c)



The diagram shows the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ where $a>b>0$. The points $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\alpha, b\tan\alpha)$ lie on the hyperbola and the chord PQ subtends a right angle at the origin.

(i) Use the parametric representation of the hyperbola to show that **3**

$$\sin\theta\sin\alpha=-\frac{a^2}{b^2}.$$

(ii) Hence show that the gradient of the curve at $P(a\sec\theta, b\tan\theta)$ is **2**

$$\frac{dy}{dx}=-\frac{b^3}{a^3}\sin\alpha$$

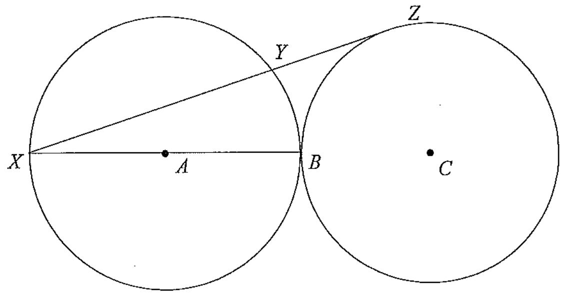
Question 5 (15 marks) **Marks**

(a) (i) Let $I_n=\int_0^x \cos^n t\,dt$, where $0\le x\le\frac{\pi}{2}$. **2**

Show that $I_n=\left(\frac{n-1}{n}\right)I_{n-2}$ with $n\ge 2$.

(ii) Hence, otherwise, find the exact value I_4. **2**

(b)



Two equal circles touch externally at B. XB is a diameter of one circle. XZ is the tangent from X to the other circle and cuts the first circle at Y. **3**
Prove that $2XZ=3XY$

(c) (i) Prove that $\frac{a+b}{2}\ge\sqrt{ab}$ given that $a\ge 0$ and $b\ge 0$. **1**

(ii) Prove that $e^a+e^b\ge 2e^{\frac{a+b}{2}}$ for all real a and b. **2**

(iii) Find the minimum value of $e^{-2x}+e^{-x}+e^x+e^{2x}$ for all real x **2**

(d) The parabola $y=x^3$ is rotated about the y axis $\{x:0\le x\le 2\}$ to form a solid. **3**
Calculate the volume of this solid using the method of slicing.

Question 6 (15 marks) **Marks**

(a) (i) Differentiate $\sin^{-1} x - \sqrt{1-x^2}$ **2**

(ii) Hence show that $\int_0^a \sqrt{\frac{1+x}{1-x}}\,dx = \sin^{-1} a + 1 - \sqrt{1-a^2}$ for $0 < a < 1$ **1**

(b) Solve the equation $x^4 + x^2 + 6x + 4 = 0$ over the complex field given that it has a rational zero of multiplicity 2. **4**

(c) The points $P(cp, \frac{c}{p})$ and Q $(cq, \frac{c}{q})$ lie on the same branch of the hyperbola $xy = c^2$ $(p \neq q)$. The tangents at P and Q meet at the point T.

(i) Show that the equation of the tangent to the hyperbola at P is $x + p^2 y - 2cp = 0$. **2**

(ii) Show that the equation of the normal to the hyperbola at P is $p^3 x - py + c - cp^4 = 0$ **2**

(iii) What are the coordinates of T? **2**

(iv) If P and Q move so that $pq = k$ (where k is a constant), show that the locus of T is a straight line and give its equation in terms of k. **2**

Question 7 (15 marks) **Marks**

(a) A rock is projected vertically upwards from ground level. Assume air resistance is kv, where v is the velocity of the rock and k is a positive constant. The rock falls back to ground level under the influence of g, the acceleration due to gravity. Consider the rock's motion starting from maximum height. Let y be the displacement and t be the time elapsed after the rock has reached maximum height. Assume the rock has a unit mass.

(i) Explain why $\frac{dv}{dt} = kv - g$ while the rock is in motion. **1**

(ii) Show that $v = \frac{g}{k}(e^{kt} + 1)$ when $t \geq 0$. **3**

(iii) Show that $ky = v + \frac{g}{k}\log_e(kgv - g^2)$ by using $\frac{dv}{dt} = v\frac{dv}{dy}$. **3**

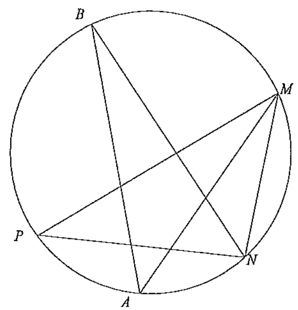
(b) Suppose $2z = \cos\theta + i\sin\theta$ where θ is real.

(i) Find the value of $|z|$. **1**

(ii) Find the imaginary part of the following geometric series. **3**

$$1 + z + z^2 + z^3 + \ldots = \frac{1}{1-z}$$

(c)



The points P, M and N lie on a circle. The chord MN is a fixed chord of a circle and P is any point on the major arc. The chord AM is a bisector of $\angle PMN$ and chord BN is a bisector of $\angle PNM$. Prove that AB is a constant length. **4**

Question 8 (15 marks) **Marks**

(a) (i) Use the binomial theorem **1**

$$(1+x)^n = {}^nC_0 + {}^nC_1x^1 + {}^nC_2x^2 + \ldots + {}^nC_nx^n = \sum_{k=0}^{n} {}^nC_kx^k$$

to show that $(1+\frac{1}{n})^n = \sum_{k=0}^{n} \frac{n(n-1)(n-2)\ldots(n-k+1)}{n^k} \times \frac{1}{k!}$

(ii) Hence show that $\lim_{n\to\infty}(1+\frac{1}{n})^n = 2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots$ **2**

(iii) Prove by induction that $\frac{1}{n!} < \frac{1}{2^{n-1}}$ when $n \geq 3$ and n is an integer. **3**

(iv) Hence show that $\lim_{n\to\infty}(1+\frac{1}{n})^n < 3$ **1**

(b) (i) Prove that $(a+b+c)^2 \geq 3(ab+ac+bc)$ **2**
Where a, b and c are positive integers.

(ii) Hence or otherwise prove that **2**
$a^2b^2 + a^2b^2 + b^2c^2 \geq abc(a+b+c)$

(c) The labor party conducted a survey for the 2010 election. The ratio of the votes in three seats X, Y and Z was 4:3:2 respectively. The percentage of votes for Ms Gillard in these seats was 60%, 30% and 90% respectively. Ten voters were chosen at random, find the probability that Ms Gillard gained at least eight votes. **4**

End of paper

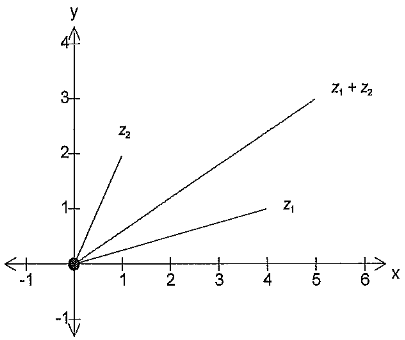
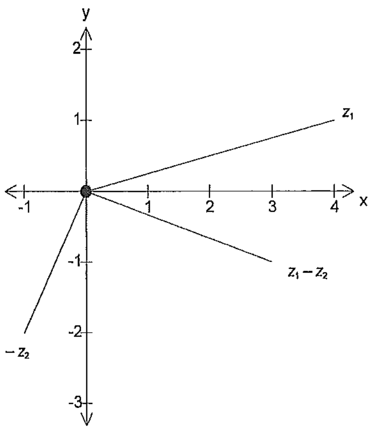
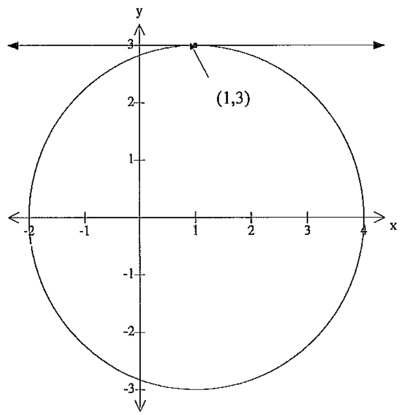
ACE Examination 2011

HSC Mathematics Extension 2 Yearly Examination

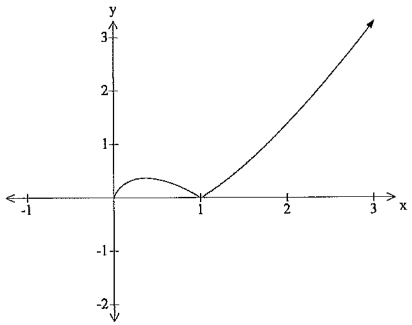
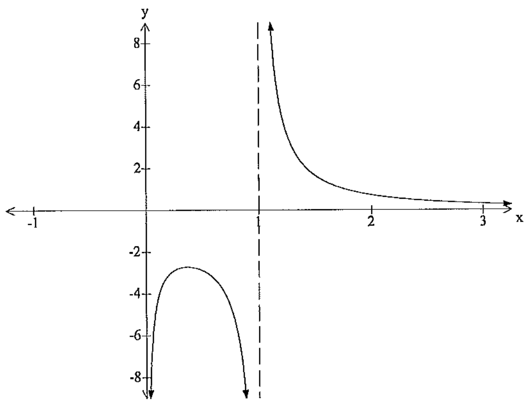
Worked solutions and marking guidelines

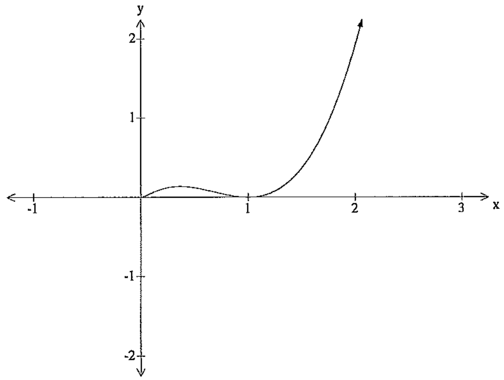
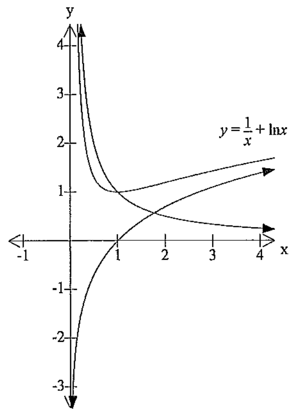
	Solution	Criteria
1(a) (i)	$\int \frac{2}{x^2+4x+13}dx = 2\int \frac{dx}{(x+2)^2+3^2}$ $= \frac{2}{3}\tan^{-1}\frac{(x+2)}{3}+c$	2 Marks: Correct answer. 1 Mark: Correctly completes the square
1(b)	$\int 3xe^x dx = 3\int x\frac{d}{dx}(e^x)dx$ $= 3(xe^x - \int e^x dx)$ $= 3xe^x - 3e^x + c$	2 Marks: Correct answer. 1 Mark: Set up of the integration by parts.
1(c) (i)	$\frac{7x+4}{(x^2+1)(x+2)} = \frac{ax+b}{x^2+1}+\frac{c}{x+2}$ $7x+4 = (ax+b)(x+2)+c(x^2+1)$ Let $x=-2$ and $x=0$ $-10=5c$ $\quad 4=b(0+2)-2(0^2+1)$ $c=-2$ $\quad b=3$ Equating the coefficients of x^2 $0=a-2$ $a=2$ $\therefore a=2,\ b=3$ and $c=-2$	3 Marks: Correct answer. 2 Marks: Calculates two of the variables 1 Mark: Makes some progress in finding a, b or c.
1(c) (ii)	$\int \frac{7x+4}{(x^2+1)(x+2)}dx = \int(\frac{2x+3}{(x^2+1)}-\frac{2}{(x+2)})dx$ $= \int(\frac{2x}{(x^2+1)}+\frac{3}{(x^2+1)}-\frac{2}{(x+2)})dx$ $= \ln(x^2+1)+3\tan^{-1}x-2\ln\|x+2\|+c$	2 Marks: Correct answer. 1 Mark: Correctly finds one of the integrals.
1(d)	$\int_0^1 xe^{-x^2}dx = -\frac{1}{2}\int_0^1 -2xe^{-x^2}dx$ $= -\frac{1}{2}\left[e^{-x^2}\right]_0^1$ $= -\frac{1}{2}(e^{-1}-e^0)$ $= \frac{1}{2}(1-\frac{1}{e})$ $= \frac{e-1}{2e}$	2 Marks: Correct answer. 1 Mark: Integrates correctly

1(e)	$t=\tan\frac{\theta}{2}$ $dt = \frac{1}{2}\sec^2\frac{\theta}{2}d\theta$ $dt = \frac{1}{2}(1+t^2)d\theta$ $d\theta = \frac{2}{1+t^2}dt$ When $\theta=0$ then $t=0$ and when $\theta=\frac{\pi}{2}$ then $t=1$ $\cos\theta+2\sin\theta+3 = \frac{1-t^2+2(2t)+3(1+t^2)}{1+t^2}$ $= \frac{2(t^2+2t+2)}{1+t^2}$ $= \frac{2[1+(t+1)^2]}{1+t^2}$ $\int_0^{\frac{\pi}{2}}\frac{1}{\cos\theta+2\sin\theta+3}d\theta = \int_0^1\frac{1+t^2}{2[1+(t+1)^2]}\times\frac{2}{1+t^2}dt$ $= \int_0^1\frac{1}{1+(t+1)^2}dt$ $= \left[\tan^{-1}(t+1)\right]_0^1$ $= \tan^{-1}2-\frac{\pi}{4}$ $= 0.322$	4 Marks: Correct answer. 3 Marks: Correctly determines the primitive function 2 Marks: Correctly expresses the integral in terms of t 1 Mark: Correctly finds $d\theta$ in terms of dt and determines the new limits.
2(a) (i)	$zw = (1+2i)(-2+i)$ $= -2+i-4i+2i^2$ $= -4-3i$	1 Mark: Correct answer.
2(a) (ii)	$\frac{5}{iw} = \frac{5}{-2i+i^2}$ $= \frac{5}{-1-2i}\times\frac{-1+2i}{-1+2i}$ $= \frac{-5+10i}{1+4}$ $= -1+2i$	1 Mark: Correct answer.

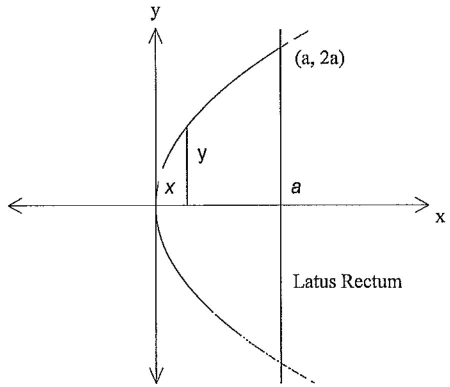
2(b) (i)		1 Mark: Correct answer.
2(b) (ii)		1 Mark: Correct answer.
2(c)		3 Marks: Correct answer. 2 Marks: Correctly graphs one of the inequalities. 1 Mark: Makes progress in graphing one of the inequalities.

	$\|z-1\|\le 3$ represents a region with a centre is (1, 0) and radius less than or equal to 3. $\text{Im}(z)\ge 3$ represents a region above the horizontal line $y=3$. The point (1,3) is where the two inequalities hold.	
2(d)	$z=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$ $z^6=\cos(6\times\frac{\pi}{6})+i\sin(6\times\frac{\pi}{6})$ $=\cos\pi+i\sin\pi$ $=-1$	2 Marks: Correct answer. 1 Mark: Uses De Moivre's theorem
2(e) (i)	All the coefficients of $P(z)$ are real. Then any complex roots occur in conjugate pairs. Since $3+i$ is a root then $3-i$ is a root	1 Mark: Correct answer.
2(e) (ii)	Roots are $3+i$, $3-i$ and α $(3+i)(3-i)\alpha=-\frac{10}{1}$ $(9-i^2)\alpha=-10$ $10\alpha=-10$ $\alpha=-1$ $P(z)=(z--1)[z-(3+i)][z-(3-i)]$ $=(z+1)(z^2-6z+10)$	2 Marks: Correct answer. 1 Mark: Makes some progress towards the solution.
2(f)	Let $z=x+iy$ and $\overline{z}=x-iy$ $z^2=i\overline{z}$ $(x+iy)^2=i(x-iy)$ $x^2-y^2+2xyi=y+ix$ Equating the real and imaginary parts $x^2-y^2=y$ (1) $2xy=x$ (2) Rearranging eqn (2) $x(2y-1)=0$ $x=0$ and $y=\frac{1}{2}$ Substitute $x=0$ into eqn (1) $-y^2=y$ $y(y+1)=0$ $y=0$ and $y=-1$	3 Marks: Correct answer. 2 Marks: Determines one possible solution to the equation. 1 Mark: Correctly expresses the equation in terms of x and y

	Substitute $y=\frac{1}{2}$ into eqn (1) $x^2-\frac{1}{4}=\frac{1}{2}$ $x^2=\frac{3}{4}$ $x=\pm\frac{\sqrt{3}}{2}$ Solution is $(0,0)$, $(0,-1)$, $(\frac{\sqrt{3}}{2},\frac{1}{2})$ and $(-\frac{\sqrt{3}}{2},\frac{1}{2})$	
3(a) (i)		1 Mark: Correct answer.
3(a) (ii)		2 Marks: Correct answer. 1 Mark: Calculates both vertical asymptotes.

3(a) (iii)		2 Marks: Correct answer. 1 Mark: Graph is above the x-axis with a minimum at (1,0).
3(b) (i) and (ii)		3 Marks: Correct answer. 2 Marks: Correctly draws two graphs 1 Mark: Correct draws one graph
3(c) (i)	The equation $x^3-3x^2-x+2=0$ has roots α,β,γ. $\alpha+\beta+\gamma=3$ $x=2\alpha+\beta+\gamma=\alpha+3$ $x=\alpha+2\beta+\gamma=\beta+3$ $x=\alpha+\beta+2\gamma=\gamma+3$ $\alpha=x-3$ satisfies $x^3-3x^2-x+2=0$ $(x-3)^3-3(x-3)^2-(x-3)+2=0$ $x^3-12x^2+44x-49=0$ is the equation	2 Marks: Correct answer. 1 Mark: Makes significant progress towards the solution.
3(c) (ii)	$x=\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ $\alpha=\frac{1}{x}$ satisfies $x^3-3x^2-x+2=0$ $\left(\frac{1}{x}\right)^3-3\left(\frac{1}{x}\right)^2-\frac{1}{x}+2=0$ $1-3x-x^2+2x^3=0$	2 Marks: Correct answer. 1 Mark: Makes significant progress towards the solution.

3(d)



Cylindrical shells radius is y and height $a - x = a - \dfrac{y^2}{4a}$

$$V = \lim_{\delta y \to 0} \sum_{y=0}^{2a} 2\pi y(a - \frac{y^2}{4a})\delta y$$

$$= 2\pi \int_0^{2a} y(a - \frac{y^2}{4a})dy$$

$$= 2\pi \int_0^{2a} (ya - \frac{y^3}{4a})dy$$

$$= 2\pi \left[\frac{ay^2}{2} - \frac{y^4}{16a} \right]_0^{2a}$$

$$= 2\pi \left[(\frac{4a^3}{2} - \frac{16a^4}{16a}) - 0 \right]$$

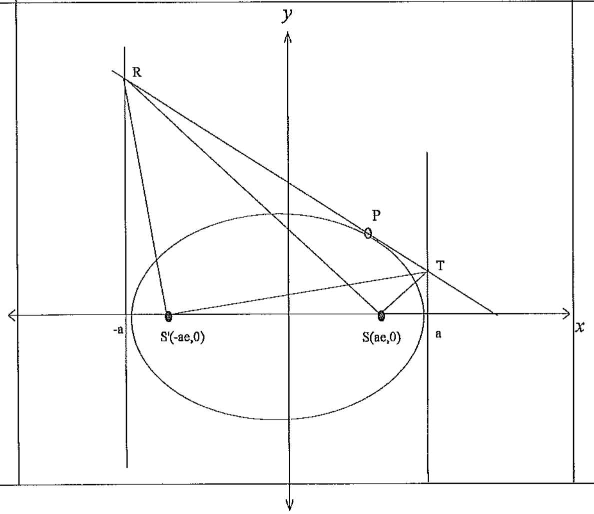
$$= 2\pi a^3$$

3 Marks: Correct answer.

2 Marks: Correct integral for the volume of the solid.

1 Mark: Determined that the cylindrical shells have a radius of y and a height $a - x$

4(a) (i)



To find the equation of tangent through P

$x = a\cos\theta$ $\quad y = b\sin\theta$

$\dfrac{dx}{d\theta} = -a\sin\theta$ $\quad \dfrac{dy}{d\theta} = b\cos\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= b\cos\theta \times \frac{1}{-a\sin\theta}$$

$$= \frac{-b\cos\theta}{a\sin\theta}$$

Equation of the tangent

$$y - y_1 = m(x - x_1)$$

$$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$$

$$bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta)$$

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

2 Marks: Correct answer

1 Mark: Correctly calculates the gradient

4(a) (ii)

At T $x = a$ then $\dfrac{a}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$

$$\frac{y}{b}\sin\theta = 1 - \cos\theta$$

$$y = \frac{b(1-\cos\theta)}{\sin\theta}$$

At R $x = -a$ then similarly $y = \dfrac{b(1+\cos\theta)}{\sin\theta}$

Gradients of lines at the focus $S(ae,0)$

Gradient RS $\times$ Gradient TS

$$= \frac{\frac{b(1+\cos\theta)}{\sin\theta} - 0}{-a - ae} \times \frac{\frac{b(1-\cos\theta)}{\sin\theta} - 0}{a - ae}$$

$$= \frac{b(1+\cos\theta)}{-a(1+e)\sin\theta} \times \frac{b(1-\cos\theta)}{a(1-e)\sin\theta}$$

$$= \frac{b^2(1-\cos^2\theta)}{-a^2(1-e^2)\sin^2\theta}$$

$$= -1$$

Gradients of lines at the focus S'$(-ae,0)$

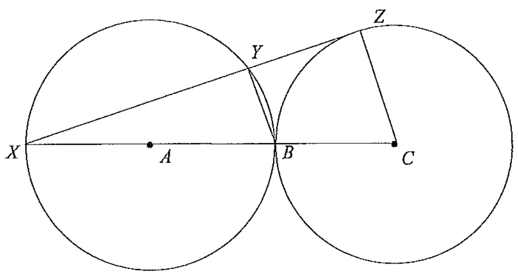
3 Marks: Correct answer

2 Marks: Made progress in proving the lines are perpendicular to the focus.

1 Mark: Determined the coordinates at T and R.

	Gradient RS' × Gradient TS' $= \dfrac{\frac{b(1+\cos\theta)}{\sin\theta}-0}{-a+ae} \times \dfrac{\frac{b(1-\cos\theta)}{\sin\theta}-0}{a+ae}$ $= \dfrac{b(1+\cos\theta)}{-a(1-e)\sin\theta}\times\dfrac{b(1-\cos\theta)}{a(1+e)\sin\theta}$ $= \dfrac{b^2(1-\cos^2\theta)}{-a^2(1-e^2)\sin^2\theta}$ $=-1$	
4(b) (i)	The vertical component of T is $T\cos\theta$	1 Mark: Correct answer.
4(b) (ii)	Body moving in a horizontal circle. $T\cos\theta - mg = 0$ (1) $T\sin\theta = mr\omega^2$ (2) Eqn (2) divided by Eqn (1) $\dfrac{T\sin\theta}{T\cos\theta}=\dfrac{mr\omega^2}{mg}$ $\tan\theta = \dfrac{r\omega^2}{g}$ However $\tan\theta = \dfrac{r}{h}$ Therefore $\dfrac{r\omega^2}{g}=\dfrac{r}{h}$ $\omega^2 = \dfrac{g}{h}$	3 Marks: Correct answer. 2 Marks: Solves the two equations of motion. 1 Mark: Correctly states the two equations of motion
4(b) (iii)	Period $=\dfrac{2\pi}{\omega}=\dfrac{2\pi}{\sqrt{\frac{g}{h}}}=\dfrac{2\pi\sqrt{h}}{\sqrt{g}}$	1 Mark: Correct answer.
4(c) (i)	Gradient of PO $m=\dfrac{y_2-y_1}{x_2-x_1}$ $=\dfrac{b\tan\theta-0}{a\sec\theta-0}$ $=\dfrac{b\frac{\sin\theta}{\cos\theta}}{a\frac{1}{\cos\theta}}$ $=\dfrac{b\sin\theta}{a}$ Gradient of QO $m=\dfrac{y_2-y_1}{x_2-x_1}$ $=\dfrac{b\tan\alpha-0}{a\sec\alpha-0}$ $=\dfrac{b\frac{\sin\alpha}{\cos\alpha}}{a\frac{1}{\cos\alpha}}$ $=\dfrac{b\sin\alpha}{a}$ PO and QO are at right angles $m_1m_2=-1$	3 Marks: Correct answer. 2 Marks: Makes significant progress. 1 Mark: Correctly calculates the gradient of PO or QO

	$\dfrac{b\sin\theta}{a}\times\dfrac{b\sin\alpha}{a}=-1$ $\sin\theta\sin\alpha=-\dfrac{a^2}{b^2}$	
4(c) (ii)	To find the gradient of curve through P $x=a\sec\theta$ $=\dfrac{a}{\cos\theta}$ $y=b\tan\theta$ $\dfrac{dy}{d\theta}=b\sec^2\theta$ $\dfrac{dx}{d\theta}=\dfrac{\cos\theta.0-a-\sin\theta}{\cos^2\theta}$ $=\dfrac{a\sin\theta}{\cos^2\theta}$ $\dfrac{dy}{dx}=\dfrac{dy}{d\theta}\times\dfrac{d\theta}{dx}$ $=b\sec^2\theta\times\dfrac{\cos^2\theta}{a\sin\theta}$ $=\dfrac{b}{a\sin\theta}$ Now from c(i) $\dfrac{dy}{dx}=\dfrac{b}{a\times-\dfrac{a^2}{b^2\sin\alpha}}$ $=-\dfrac{b^3}{a^3\sin\alpha}$	2 Marks: Correct answer. 1 Mark: Calculates the gradient as $\dfrac{b}{a\sin\theta}$
5(a) (i)	$I_n=\int_0^{\pi}\cos^n x dx$ $=\int_0^{\frac{\pi}{2}}\cos^n x dx$ Integration by parts $I_n=\int_0^{\frac{\pi}{2}}\cos^{n-1}t\cos t dt$ $=\left[\cos^{n-1}t\sin t\right]_0^{\frac{\pi}{2}}+(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}t\sin^2 t dt$ $=(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}t\sin^2 t dt$ $=(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}t(1-\cos^2 t)dt$ $=(n-1)\int_0^{\frac{\pi}{2}}(\cos^{n-2}t-\cos^n t)dt$ $=(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}t dt-(n-1)\int_0^{\frac{\pi}{2}}\cos^n t dt$ Using the original integral	2 Marks: Correct answer. 1 Mark: Correctly using integration by parts.

	$\int_0^{\frac{\pi}{2}} \cos^n t\,dt = (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} t\,dt - n\int_0^{\frac{\pi}{2}} \cos^n t\,dt + \int_0^{\frac{\pi}{2}} \cos^n t\,dt$ $n\int_0^{\frac{\pi}{2}} \cos^n t\,dt = (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} t\,dt$ $\int_0^{\frac{\pi}{2}} \cos^n t\,dt = \frac{(n-1)}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} t\,dt$ $I_n = \frac{(n-1)}{n} I_{n-2}$	
5(a) (ii)	$I_n = \frac{(n-1)}{n} I_{n-2}$ $I_4 = \frac{(4-1)}{4} I_{4-2}$ $= \frac{3}{4}\int_0^{\frac{\pi}{2}} \cos^2 t\,dt$ $= \frac{3}{4}\int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2t)\,dt$ $= \frac{3}{8}\left[(x+\frac{\sin 2t}{2})\right]_0^{\frac{\pi}{2}}$ $= \frac{3}{8}\left[(\frac{\pi}{2}+\frac{\sin 0}{2}) - (0+\frac{\sin 0}{2})\right]$ $= \frac{3\pi}{16}$	2 Marks: Correct answer. 1 Mark: Using the result from (a)(i) to obtain the definite integral.
5(b)	 Construction: Join BY, produce XB to C, join CZ. Proof: $\angle XYB = 90^0$ (angle in a semicircle is a right angle) $\angle XZC = 90^0$ (angle in a semicircle is a right angle) $BY \square CZ$ (corresponding angles are equal) $\Delta XYB \vert\vert\vert \Delta XZC$ (equiangular) $\frac{XY}{XZ} = \frac{XB}{XC}$ (corresponding sides of similar triangles)	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the proof. 1 Mark: States a relevant circle theorem property or equivalent statement.

	However $\frac{XB}{XC} = \frac{2}{3}$ $(BC = \frac{1}{2}XB)$ $\frac{XY}{XZ} = \frac{2}{3}$ $\therefore 2XZ = 3XY$	
5(c) (i)	If a and b are greater than 0 $(\sqrt{a}-\sqrt{b})^2 \geq 0$ $a - 2\sqrt{a}\sqrt{b} + b \geq 0$ $a + b \geq 2\sqrt{ab}$ $\frac{a+b}{2} \geq \sqrt{ab}$	1 Mark: Correct answer.
5(c) (ii)	If a and b are real then $e^a > 0$ and $e^b > 0$ Hence from part (i) $e^a + e^b \geq 2\sqrt{e^a e^b}$ $e^a + e^b \geq 2e^{\frac{a}{2}}e^{\frac{b}{2}}$ $e^a + e^b \geq 2e^{\frac{a+b}{2}}$	2 Marks: Correct answer. 1 Mark: Correctly uses part (i) expression.
5(c) (iii)	Using part (ii) $e^{-2x} + e^{-x} + e^x + e^{2x} = (e^x + e^{-x}) + (e^{2x} + e^{-2x})$ $\geq 2e^{\frac{x+-x}{2}} + 2e^{\frac{2x+-2x}{2}}$ $\geq 2+2$ ≥ 4	2 Marks: Correct answer. 1 Mark Correctly uses part (ii) expression.
5(d)	Area of the slice is a circle radius is x and height y $A = \pi x^2$ $= \pi(y^{\frac{1}{3}})^2$ $= \pi y^{\frac{2}{3}}$ $\delta V = \delta A.\delta y$ $V = \lim_{\delta y \to 0}\sum_{y=0}^{8} \pi y^{\frac{2}{3}}\delta y$ $= \int_0^8 \pi y^{\frac{2}{3}}\,dy$ $= \pi\left[\frac{3}{5}y^{\frac{5}{3}}\right]_0^8$ $= \frac{3\pi}{5} \times 8^{\frac{5}{3}}$ $= \frac{96\pi}{5}$ cubic units	3 Marks: Correct answer. 2 Marks: Correct integral for the volume of the solid. 1Mark: Correct expression for the volume of the solid.

6(a) (i)	$y=\sin^{-1}x-\sqrt{1-x^2}$ $\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{2}(1-x^2)^{-\frac{1}{2}}\times -2x$ $=\frac{1}{\sqrt{1-x^2}}+\frac{x}{\sqrt{1-x^2}}$ $=\frac{1+x}{\sqrt{1-x^2}}$ $=\frac{1+x}{\sqrt{(1+x)(1-x)}}$ $=\frac{\sqrt{1+x}}{\sqrt{1-x}}$ Result defined for $-1\le x\le 1$	2 Marks: Correct answer. 1 Mark Correctly differentiates the function.
6(a) (ii)	$\int_0^a\sqrt{\frac{1+x}{1-x}}\,dx=\left[\sin^{-1}x-\sqrt{1-x^2}\right]_0^a$ $=(\sin^{-1}a-\sqrt{1-a^2})-(\sin^{-1}0-\sqrt{1})$ $=\sin^{-1}a-\sqrt{1-a^2}+1$ $=\sin^{-1}a+1-\sqrt{1-a^2}$	1 Mark: Correct answer
6(b)	$P(x)=x^4+x^2+6x+4$ $P'(x)=4x^3+2x+6$ $=2(2x^3+x+3)$ To determine the roots of $2x^3+x+3$ $P'(-1)=2(2(-1)^3+(-1)+3)$ $=0$ Therefore -1 is a zero of multiplicity 2 of $P(x)$ $P(x)=x^4+x^2+6x+4$ $=(x+1)^2(x^2+bx+c)$ $=(x^2+2x+1)(x^2+bx+c)$ $=x^4+bx^3+cx^2+2x^3+2bx^2+2cx+x^2+bx+c$ $=x^4+(b+2)x^3+(c+2b+1)x^2+(b+2c)x+4$ Hence $c=4$, $b=-2$ Therefore $P(x)=x^4+x^2+6x+4$ $=(x+1)^2(x^2-2x+4)$ $=(x+1)^2((x-1)^2-1+4)$ $=(x+1)^2((x-1)^2+3)$ $=(x+1)^2(x-1+\sqrt{3}i)(x-1-\sqrt{3}i)$ Zeros of $P(x)$ are -1, $1+\sqrt{3}i$ and $1-\sqrt{3}i$	4 Marks: Correct answer. 3 Marks: Factorises the polynomial 2 Marks: Recognises $(x+1)^2$ as a factor of the polynomial. 1 Mark: Calculates the derivative and finds its zeros.

6(c) (i)	To find the gradient of the tangent. $xy=c^2$ $x\frac{dy}{dx}+y=0$ $\frac{dy}{dx}=-\frac{y}{x}$ At $P(cp,\frac{c}{p})$ $\frac{dy}{dx}=-\frac{\frac{c}{p}}{cp}$ $=-\frac{1}{p^2}$ Equation of the tangent at P $y-\frac{c}{p}=-\frac{1}{p^2}(x-cp)$ $p^2y-cp=-x+cp$ $x+p^2y-2cp=0$	2 Marks: Correct answer. 1 Mark: Correctly calculates the gradient of the tangent to the hyperbola
6(c) (ii)	At $P(cp,\frac{c}{p})$ Gradient of the tangent is $-\frac{1}{p^2}$ Gradient of the normal is p^2 $(m_1m_2=-1)$ Equation of the normal at P $y-\frac{c}{p}=p^2(x-cp)$ $py-c=p^3x-cp^4$ $p^3x-py+c-cp^4=0$	2 Marks: Correct answer. 1 Mark: Correctly calculates the gradient of the normal to the hyperbola
6(c) (iii)	Equation of tangent at Q is $x+q^2y-2cq=0$ Point *T* is the point of intersection of these tangents. Solve equations simultaneously $x+p^2y-2cp=0$ (1) $x+q^2y-2cq=0$ (2) Eqn (1) – Eqn (2) $p^2y-q^2y=2cp-2cq$ $y(p+q)(p-q)=2c(p-q)$ $y(p+q)=2c$ $y=\frac{2c}{(p+q)}$ To find *x* substitute the value for *y* into eqn (1)	2 Marks: Correct answer. 1 Mark: Correctly finds one of the coordinates or finds correct *x* from incorrect *y*

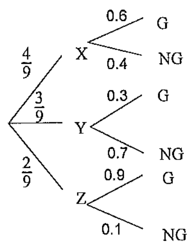
	$x+p^2\frac{2c}{p+q}-2cp=0$ $x=2cp(1-\frac{p}{p+q})$ $x=2cp(\frac{p+q-p}{p+q})$ $x=\frac{2cpq}{p+q}$ Therefore the coordinates for T are $(\frac{2cpq}{p+q},\frac{2c}{p+q})$	
6(c) (iv)	Given that $pq=k$ and the coordinates of *T*. $x=\frac{2cpq}{p+q}$ $=pq\times\frac{2c}{p+q}$ $=pq\times y$ $=ky$ $y=\frac{x}{k}$	2 Marks: Correct answer. 1 Mark: Using the coordinates of *T* and substituting *k* for *pq*
7(a) (i)	Newton's second law: $\ddot{y}=kv-g$ $\frac{dv}{dt}=kv-g$	1 Mark: Correct answer.
7(a) (ii)	$\frac{dt}{dv}=\frac{1}{kv-g}$ $\int\frac{dt}{dv}dv=\int\frac{1}{kv-g}dv$ $t=\frac{1}{k}\log_e(kv-g)+c$ Initial conditions $t=0$ and $v=0$ $0=\frac{1}{k}\log_e(-g)+c$ $c=-\frac{1}{k}\log_e g$ $t=\frac{1}{k}\log_e(kv-g)-\frac{1}{k}\log_e g$ $=\frac{1}{k}\log_e(\frac{kv-g}{g})$ $kt=\log_e(\frac{kv-g}{g})$	3 Marks: Correct answer. 2 Marks: Correctly substitutes the initial conditions into the expression for *t* 1 Mark: Finds the correction expression for *t*.

	$e^{kt}=\frac{kv-g}{g}$ $=\frac{kv}{g}-1$ $v=\frac{g}{v}(e^{kt}+1)$	
7(a) (iii)	$\frac{dv}{dt}=v\frac{dv}{dy}$ $kv-g=v\frac{dv}{dy}$ $\frac{dv}{dy}=\frac{kv-g}{v}$ $\frac{dy}{dv}=\frac{v}{kv-g}$ $=\frac{1}{k}\times\frac{kv}{kv-g}$ $=\frac{1}{k}\times\frac{kv-g+g}{kv-g}$ $=\frac{1}{k}\times(1+\frac{g}{kv-g})$ $ky=(v+\frac{g}{k}\log_e(kv-g))+c$ Initially $y=0,\ v=0$ $0=0+\frac{g}{k}\log_e(-g)+c$ $c=\frac{g}{k}\log_e g$ $ky=v+\frac{g}{k}\log_e(kv-g)+\frac{g}{k}\log_e g$ $ky=v+\frac{g}{k}(\log_e(kv-g)+\log_e g)$ $=v+\frac{g}{k}\log_e(kgv-g^2)$	3 Marks: Correct answer. 2 Marks: Finds the correction expression for *ky* 1 Mark: Uses results for part (i) to determine an expression for $\frac{dv}{dy}$

7(b) (i)	$2z=\cos\theta+i\sin\theta$ $z=\frac{1}{2}(\cos\theta+i\sin\theta)$ $\lvert z\rvert=\frac{1}{2}\lvert\cos\theta+i\sin\theta\rvert$ $=\frac{1}{2}\times 1$ $=\frac{1}{2}$	1 Mark: Correct answer
7(b) (ii)	$\frac{1}{1-z}=\frac{1}{1-\frac{1}{2}(\cos\theta+i\sin\theta)}$ $=\frac{2}{2-\cos\theta-i\sin\theta}$ $=\frac{2}{2-\cos\theta-i\sin\theta}\times\frac{2-\cos\theta+i\sin\theta}{2-\cos\theta+i\sin\theta}$ $=\frac{4-2\cos\theta+i2\sin\theta}{(2-\cos\theta)^2-(i\sin\theta)^2}$ $=\frac{4-2\cos\theta+i2\sin\theta}{4-4\cos\theta+\cos^2\theta+\sin^2\theta}$ $=\frac{2(2-\cos\theta)+i2\sin\theta}{5-4\cos\theta}$ Imaginary part is $\frac{2\sin\theta}{5-4\cos\theta}$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Multiplies by the conjugate.
7(c)	Given: $\angle AMN=\angle AMP$ and $\angle PNB=\angle BNM$. Aim: To prove AB is a constant length. Proof: Let $\angle AMN=\angle AMP=x$, $\angle BNM=\angle BNP=y$ $\angle NPM$ is a constant angle (angles in the same segment standing on the same arc (MN) are equal). Now $\angle NPM+x+x+y+y=180$ (angle sum of a triangle is 180). $\angle NPM=180-2(x+y)$ Therefore $(x+y)$ must be constant. $\angle PMB=\angle PNB=y$ (angles in the same segment standing on the same arc (PB) are equal). $\angle AMB=\angle AMP+\angle PMB$ $=x+y$ Therefore $\angle AMB$ is a constant angle. Therefore AB must be a constant length.	4 Marks: Correct answer. 3 Marks: Shows that $\angle NPM$ is a constant angle. 2 Marks: Correctly applying the angle in the same segment thm 1 Mark: Drawing a diagram and inserting information about the bisectors.

8(a) (i)	$(1+\frac{1}{n})^n=\sum_{k=0}^{n}{}^nC_k(\frac{1}{n})^k$ $=\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}(\frac{1}{n})^k$ $=\sum_{k=0}^{n}\frac{n(n-1)(n-2)...(n-k+1)}{n^k}\frac{1}{k!}$	1 Mark: Correct answer.
8(a) (ii)	$(1+\frac{1}{n})^n=\sum_{k=0}^{n}\frac{n(n-1)(n-2)...(n-k+1)}{n^k}\frac{1}{k!}$ $=\sum_{k=0}^{n}\frac{n}{n}\times\frac{(n-1)}{n}\times\frac{(n-2)}{n}\times...\times\frac{(n-k+1)}{n}\frac{1}{k!}$ As $n\to\infty$ then $\frac{n-1}{n}\to 1$, $\frac{n-2}{n}\to 1$, $\frac{n-3}{n}\to 1$.... $(1+\frac{1}{n})^n=\sum_{k=0}^{n}\frac{1}{k!}$ $=1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+...$ $=2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+...$	2 Marks: Correct answer. 1 Mark: Splitting the expression into fractions.
8(a) (iii)	Test the result for $n=3$ $\frac{1}{3!}<\frac{1}{2^{3-1}}$ $\frac{1}{6}<\frac{1}{4}$ Therefore the result is true for $n=3$ Assume the result is true for $n=k$ $\frac{1}{k!}<\frac{1}{2^{k-1}}$ To prove the result is true for $n=k+1$ $\frac{1}{(k+1)!}<\frac{1}{2^{(k+1)-1}}$ $<\frac{1}{2^k}$	3 Marks: Correct answer. 2 Marks: Proves the result true for $n=k+1$ 1 Mark: Tests the result for $n=3$

	$\text{LHS} = \frac{1}{(k+1)!}$ $= \frac{1}{(k+1)k!}$ $< \frac{1}{(k+1)2^{k-1}}$ Assumption for $n=k$ $< \frac{1}{2\times 2^{k-1}}$ $k+1>2$ as $n\geq 3$ $= \frac{1}{2^k}$ $= \text{RHS}$ Therefore the result holds true for $n=k+1$ Thus if the result is true for $n=k$, it is true for $n=k+1$. It has been shown that the result is $n=1$, hence it is correct for $n=2$, then $n=3$ and so on.	
8(a) (iv)	From part (ii) $\lim_{n\to\infty}(1+\frac{1}{n})^n = 2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots$ $= 2+\frac{1}{2}+\sum_{k=3}^{\infty}\frac{1}{k!}$ $< 2+\frac{1}{2}+\sum_{k=3}^{\infty}\frac{1}{2^{k-1}}$ $< 2+\frac{1}{2}+(\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\ldots)$ $< 2+\frac{1}{2}+(\frac{\frac{1}{2^2}}{1-\frac{1}{2}})$ Limiting sum of GP $< 2+\frac{1}{2}+\frac{1}{2}$ < 3	1 Mark: Correct answer.
8(b) (i)	Now $(a-b)^2\geq 0$ $a^2+b^2\geq 2ab$ (1) Also $(a-c)^2\geq 0$ $a^2+c^2\geq 2ac$ (2) Also $(b-c)^2\geq 0$ $b^2+c^2\geq 2bc$ (3) Eqn(1)+(2)+(3)	2 Marks: Correct answer. 1 Mark: Makes some progress towards the solution such as using $a^2+b^2\geq 2ab$

	$2(a^2+b^2+c^2)\geq 2(ab+ac+bc)$ $a^2+b^2+c^2\geq (ab+ac+bc)$ $a^2+b^2+c^2+2(ab+ac+bc)\geq (ab+ac+bc)+2(ab+ac+bc$ $a^2+b^2+c^2+2(ab+ac+bc)\geq 3(ab+ac+bc)$ $(a+b+c)^2\geq 3(ab+ac+bc)$	
8(b) (ii)	Using c (i) $a^2+b^2+c^2\geq (ab+ac+bc)$ Let $a=xy$, $b=xz$ and $c=yz$ $x^2y^2+x^2z^2+y^2z^2\geq (xyxz+xyyz+xzyz)$ $\geq xyz(x+y+z)$ Replace $x=a, y=b$ and $z=c$ $a^2b^2+a^2c^2+b^2c^2\geq abc(a+b+c)$	2 Marks: Correct answer. 1 Mark: Makes some progress towards the solution.
8(c)	Consider one voter.  The probability that Ms Gillard will gain this vote. $P=\frac{4}{9}\times 0.6+\frac{3}{9}\times 0.3+\frac{2}{9}\times 0.9$ $=0.6$ The probability that Ms Gillard will receive at least eight votes. $P(\text{at least 8 votes})$ $= {}^{10}C_8(0.6)^8(0.4)^2+{}^{10}C_9(0.6)^9(0.4)^1+{}^{10}C_{10}(0.6)^{10}(0.4)^0$ $=0.1672897536$	4 Marks: Correct answer. 3 Marks: Mostly correct solution 2 Marks: Calculates probability of one voter. 1 Mark: Draws a tree diagram with probabilities for each outcome.