ACE Examinations

Student Name: ____________________

2016

YEAR 12

YEARLY EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time - 5 minutes
- Working time - 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A reference sheet is provided at the back of this paper
- Show relevant mathematical reasoning and/or calculations in Questions 11-16

Total marks - 100

Section I

10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II

90 marks

- Attempt Questions 11-16
- Allow about 2 hour 45 minutes for this section

Section I

10 marks
Attempt Questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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1 What is the value of $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^9$?

(A) -1
(B) 0
(C) 1
(D) 2

2 The cartesian equation of the ellipse is $x^2+4y^2=4$.
What are the parametric equations of this ellipse?

(A) $x=2\cos\theta$ and $y=\sin\theta$
(B) $x=2\sin\theta$ and $y=\cos\theta$
(C) $x=4\cos\theta$ and $y=\sin\theta$
(D) $x=4\sin\theta$ and $y=\cos\theta$

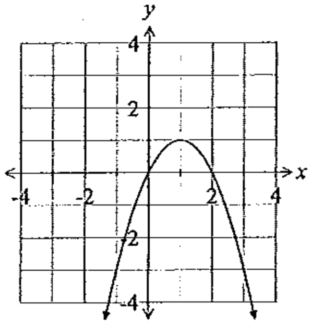
3 Let $z=2-3i$. What is the value of z^{-1}?

(A) $-\frac{1}{5}(2+3i)$
(B) $\frac{1}{13}(2+3i)$
(C) $\frac{1}{5}(2-3i)$
(D) $\frac{1}{13}(2-3i)$

4 A wheel of radius 2 metres rotates at 1200 revolutions per minute.
What is the tangential velocity of a point on the wheel?

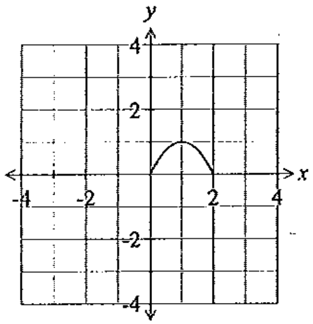
(A) 40 ms^{-1}
(B) 80 ms^{-1}
(C) 251 ms^{-1}
(D) 260 ms^{-1}

5 The diagram below shows the graph of the function $y=f(x)$.

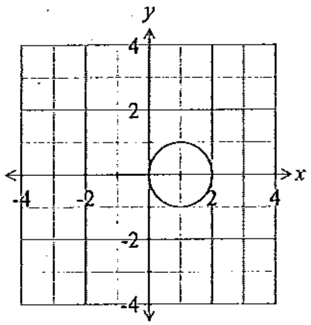


Which of the following is the graph of $y=\sqrt{f(x)}$?

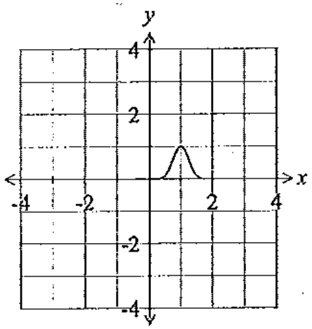
(A)



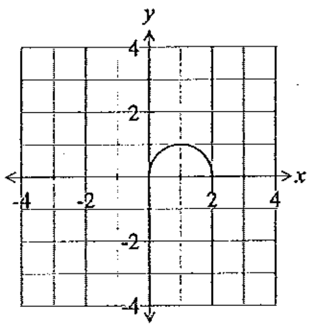
(B)



(C)



(D)



6 Which of the following is the correct expression for $\int \frac{1}{\sqrt{4x^2+36}}\,dx$?

(A) $\sin^{-1}\frac{x}{3}+C$

(B) $\frac{1}{3}\tan^{-1}\frac{x}{3}+C$

(C) $\log_e\left(x+\sqrt{x^2+9}\right)+C$

(D) $\frac{1}{2}\log_e\left(x+\sqrt{x^2+9}\right)+C$

7 For the ellipse with the equation $\frac{(x-2)^2}{5}+\frac{(y+3)^2}{3}=1$. What is the eccentricity?

(A) $\sqrt{\frac{2}{5}}$

(B) $\sqrt{\frac{2}{3}}$

(C) $\sqrt{\frac{5}{3}}$

(D) $\sqrt{\frac{5}{2}}$

8 A solid is formed by rotating the region enclosed by the parabola $y^2=4ax$, its vertex (0,0) and the line $x=a$ about the x-axis. What is the volume of this solid using the method of cylindrical shells?

(A) $\frac{7\pi a^3}{4}$ units^3

(B) $\frac{7\pi a^3}{8}$ units^3

(C) $\frac{7\pi a^3}{16}$ units^3

(D) $2\pi a^3$ units^3

9 What is the value of $\int_0^2 \sqrt{\frac{x}{4-x}}\,dx$ using the substitution $x=4\sin^2\theta$?

(A) 0.75π

(B) $\pi-2$

(C) $\pi+6$

(D) $3\pi-8$

10 What are the values of real numbers p and q such that $1-i$ is a root of the equation $z^3+pz+q=0$?

(A) $p=-2$ and $q=-4$

(B) $p=-2$ and $q=4$

(C) $p=2$ and $q=4$

(D) $p=2$ and $q=4$

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Marks**

(a) Find $\int \frac{\sqrt{1-x^2}}{\sqrt{1-x}}\,dx$. **2**

(b) (i) Show that $(1-2i)^2 = -3-4i$. **1**

(ii) Hence solve the equation $z^2 - 5z + (7+i) = 0$. **2**

(c) Use integration by parts to evaluate $\int xe^{-x}\,dx$. **2**

(d) Let $z = 1 - i\sqrt{3}$

(i) What is the exact value of $|z|$ and $\arg z$? **2**

(ii) Find the exact value of z^6. **2**

(e) Consider the function $y = \cos^{-1}(e^x)$

(i) Find the domain and the range. **2**

(ii) Sketch the graph of $y = \cos^{-1}(e^x)$? **1**

(iii) Hence or otherwise sketch the graph of $y = \left[\cos^{-1}(e^x)\right]^2$. **1**

Question 12 (15 marks) **Marks**

(a) Let two complex numbers be $z_1 = 2(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$ and $z_2 = 2i$.

(i) On an Argand diagram sketch the vectors OA and OB to represent z_1 and z_2 respectively. **1**

(ii) Draw the vectors $z_1 + z_2$ and $z_1 - z_2$ on the same Argand diagram. **1**

(iii) What are the exact values of $\arg(z_1 + z_2)$ and $\arg(z_1 - z_2)$? **2**

(b) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{2 - \cos x + 2\sin x}\,dx$. **4**

(c) Let $f(x) = (2-x)(x-4)$. Draw separate one-third page sketches of these functions. Indicate clearly any asymptotes and intercepts with the axes.

(i) $y = \frac{1}{f(x)}$ **2**

(ii) $y = e^{f(x)}$ **2**

(d) The parabola $y = x^3$ is rotated about the y axis $\{x : 0 \le x \le 2\}$ to form a solid. Calculate the volume of this solid using the method of slicing. **3**

Question 13 (15 marks) **Marks**

(a) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a>b>0$.

PQ is a diameter of the ellipse. The tangent to the ellipse at P meets the vertical through Q at R and cuts the y-axis at S.

(i) Prove the equation of the tangent is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$. **2**

(ii) Show the ratio of the area of ΔPQR to the area of the ellipse is **3**

$2:\pi|\tan\theta|$

(b) The polynomial equation $x^3+2x+1=0$ has roots α, β and γ.

(i) Find the equation with α^{-1}, β^{-1} and γ^{-1} as the roots. **2**

(ii) Find the equation with α^{-2}, β^{-2} and γ^{-2} as the roots. **2**

(c) (i) Find real numbers a, b, c and d such that: **2**

$$\frac{5x^3-3x^2+2x-1}{x^2(x^2+1)}=\frac{a}{x}+\frac{b}{x^2}+\frac{cx+d}{x^2+1}$$

(ii) Hence evaluate in simplest form **2**

$$\int\frac{5x^3-3x^2+2x-1}{x^2(x^2+1)}dx$$

(d) The polynomial $P(x)=x^4+ax^2+bx+28$ has a double root at $x=2$. **2**
What are the values of a and b?

Question 14 (15 marks) **Marks**

(a) (i) Find the three cubic roots of -1 by solving the equation $z^3+1=0$. **2**

(ii) Let α be a cubic root of -1, where α is not real. **1**
Show that $\alpha^2=\alpha-1$.

(iii) Hence simplify $(1-\alpha)^6$. **1**

(b) Find the equation of the tangent to the curve $x^2y+2x-2xy=0$ at $(1, 2)$. **2**

(c) The tangent at $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the directrix at Q. The focus of the parabola is S.

(i) Find the equation of the tangent at P. **2**

(ii) Find the coordinates of Q. **1**

(iii) Show that PQ subtends a right angle at S. **2**

(d) A solid is formed by rotating about the y-axis the region bounded by the curve $y=\sin x$ and the x-axis between $0\le x\le\pi$. Find the volume of this solid using the method of cylindrical shells. **4**

Question 15 (15 marks) **Marks**

(a) A train of mass 2,000 kg is travelling around a curve of radius 1 km on a track banked so that the outer rail is 3 cm higher than the inner rail, where the rails are 1.2 m apart. The banked track makes an angle θ with the horizontal. When the train has a speed of v ms^{-1}, the normal reaction force exerted by the track on the train is N Newtons and the lateral thrust exerted by one of the rails on the train is F Newtons. Take $g = 10$ ms^{-2}.

(i) By considering the forces acting on the train with F directed up the slope, show that $F = 2\cos\theta(10{,}000\tan\theta - v^2)$. **3**

(ii) Find the magnitude of the lateral thrust when v is 80% of v_0 if the lateral thrust F is zero when $v = v_0$. **2**

(b) Let $P(z) = z^8 - \frac{5}{2}z^4 + 1$. The complex number w is a root of $P(z) = 0$.

(i) Show that iw and $\frac{1}{w}$ are also roots of $P(z) = 0$. **2**

(ii) Find all the roots of $P(z) = 0$ in exact form. **2**

(c) (i) Show that $\sin x + \sin 3x = 2\sin 2x \cos x$ **1**

(ii) Hence or otherwise solve $\sin x + \sin 2x + \sin 3x = 0$ for $0 \le x \le 2\pi$. **2**

(d) A particle of mass m is moving in a straight line under the action of a force. **3**

$$F = \frac{m}{x^3}(6 - 10x)$$

What is the velocity in any position, if the particle starts from rest at $x = 1$?

Question 16 (15 marks) **Marks**

(a) ΔABC has sides of length a, b and c. If $a^2 + b^2 + c^2 = ab + bc + ca$ show that ΔABC is an equilateral triangle. **3**

(b) (i) Let $I_n = \int_0^1 (1 - x^r)^n \, dx$, where $r > 0$, for $n = 0, 1, 2, 3, \ldots$

Show that $I_n = \frac{nr}{nr+1} I_{n-1}$. **3**

(ii) Hence or otherwise, find the value of $I_n = \int_0^1 (1 - x^{\frac{3}{2}})^3 \, dx$. **2**

(c) (i) Use the binomial theorem $(1+x)^n = \sum_{k=0}^{n} {}^nC_k x^k$ to show that **1**

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{n(n-1)(n-2)\ldots(n-k+1)}{n^k} \times \frac{1}{k!}$$

(ii) Hence show that $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$ **2**

(iii) Prove by induction that $\frac{1}{n!} < \frac{1}{2^{n-1}}$ when $n \ge 3$ and n is an integer. **3**

(iv) Hence show that $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n < 3$. **1**

End of paper

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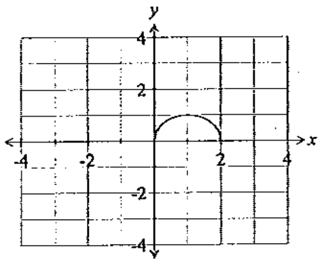
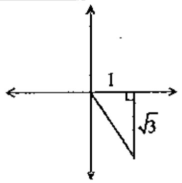
Worked solutions and marking guidelines

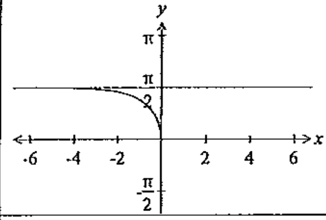
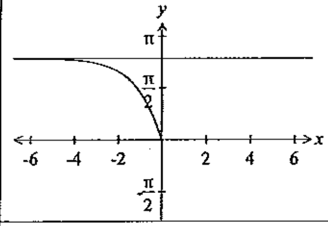
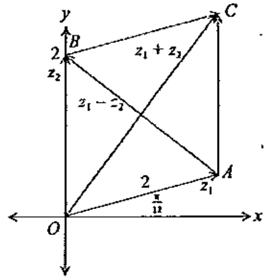
Section I

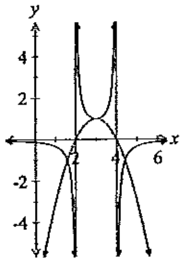
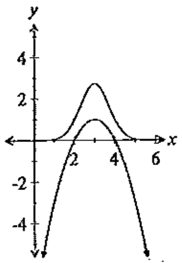
	Solution	Criteria
1	$\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^9=\cos\left(9\times\frac{\pi}{3}\right)+i\sin\left(9\times\frac{\pi}{3}\right)$ $=-1+i\times 0=-1$	1 Mark: A
2	$x^2+4y^2=4$, $\frac{x^2}{4}+\frac{y^2}{1}=1$ Hence $a=2$ and $b=1$. Parametric equations are $x=2\cos\theta$ and $y=\sin\theta$	1 Mark: A
3	$z^{-1}=\frac{1}{2-3i}\times\frac{2+3i}{2+3i}$ $=\frac{2+3i}{4+9}$ $=\frac{1}{13}(2+3i)$	1 Mark: B
4	$\omega=1200$ rpm $=\frac{1200\times 2\pi}{60}=40\pi$ radians per second $v=rw$ $=2\times 40\pi=80\pi\approx 251\text{ ms}^{-1}$	1 Mark: C
5		1 Mark: D
6	$\int\frac{1}{\sqrt{4x^2+36}}dx=\frac{1}{2}\int\frac{1}{\sqrt{x^2+9}}dx$ $=\frac{1}{2}\log_e\left(x+\sqrt{x^2+9}\right)+C$	1 Mark: D

7	$b^2=a^2(1-e^2)$ $3=5(1-e^2)$ $(1-e^2)=\frac{3}{5}$ or $e^2=\frac{2}{5}$ or $e=\sqrt{\frac{2}{5}}$	1 Mark: A
8	Cylindrical shells radius is y and height $a-x=a-\frac{y^2}{4a}$ $V=\lim_{\delta y\to 0}\sum_{y=0}^{2a}2\pi y(a-\frac{y^2}{4a})\delta y$ $=2\pi\int_0^{2a}y(a-\frac{y^2}{4a})dy$ $=2\pi\int_0^{2a}(ya-\frac{y^3}{4a})dy$ $=2\pi\left[\frac{ay^2}{2}-\frac{y^4}{16a}\right]_0^{2a}$ $=2\pi\left[(\frac{4a^3}{2}-\frac{16a^4}{16a})-0\right]=2\pi a^3$	1 Mark: D
9	$x=4\sin^2\theta$ and $dx=8\sin\theta\cos\theta d\theta$ When $x=0$ then $\theta=0$ and when $x=2$ then $\theta=\frac{\pi}{4}$ $\int_0^2\sqrt{\frac{x}{4-x}}dx=\int_0^{\frac{\pi}{4}}\sqrt{\frac{4\sin^2\theta}{4-4\sin^2\theta}}\times 8\sin\theta\cos\theta d\theta$ $=\int_0^{\frac{\pi}{4}}\tan\theta\times 8\sin\theta\cos\theta d\theta$ $=\int_0^{\frac{\pi}{4}}8\sin^2\theta d\theta$ $=8\int_0^{\frac{\pi}{4}}\frac{1}{2}(1-\cos 2\theta)d\theta$ $=4\left[x-\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}=\pi-2$	1 Mark: B
10	Using the conjugate root theorem $1+i$ and $1-i$ are both roots of the equation $z^3+pz+q=0$. $(1+i)+(1-i)+\alpha=0$ (sum of the roots) $\alpha=-2$ $(1+i)\times(1-i)\times -2=-q$ (product of the roots) $(1+1)\times -2=-q$ $q=4$ $(1+i)(1-i)+(1-i)\times -2+(1+i)\times -2=p$ $p=-2$ Therefore $p=-2$ and $q=4$.	1 Mark: B

Section II

	Solution	Criteria
11(a)	$\int \frac{\sqrt{1-x^2}}{\sqrt{1-x}}dx = \int \frac{\sqrt{1-x}\times\sqrt{1+x}}{\sqrt{1-x}}dx$ $= \int \sqrt{1+x}dx$ $= \frac{2}{3}(1+x)^{\frac{3}{2}} + C$	2 Marks: Correct answer. 1 Mark: Simplifies the integrand.
11(b) (i)	$(1-2i)^2 = 1-4i-4$ $= -3-4i$	1 Mark: Correct answer.
11(b) (ii)	$z = \frac{-(-5)\pm\sqrt{(-5)^2-4\times1\times(7+i)}}{2\times1}$ $= \frac{5\pm\sqrt{-3-4i}}{2} = \frac{5\pm(1-2i)}{2}$ $\therefore z = 3-i$ or $z = 2+i$	2 Marks: Correct answer. 1 Mark: Shows some understanding of the problem.
11(c)	$\int xe^{-x}dx = \int x\frac{d}{dx}(e^{-x})dx$ $= -xe^{-x} + \int e^{-x}dx$ $= -xe^{-x} - e^{-x} + C$	2 Marks: Correct answer. 1 Mark: Sets up the integration by parts.
11(d) (i)	$\lvert z\rvert = \sqrt{1^2+(\sqrt{3})^2}$ $= \sqrt{4}$ $= 2$ $\arg z = -\frac{\pi}{3}$  	2 Marks: Correct answer. 1 Mark: Determines $\lvert z\rvert$ or arg z.
11(d) (ii)	$z = 1-i\sqrt{3}$ $= 2\left[\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right]$ $z^6 = 2^6\left[\cos\left(-\frac{6\pi}{3}\right)+i\sin\left(-\frac{6\pi}{3}\right)\right]$ $= 2^6\left[\cos(-2\pi)+i\sin(-2\pi)\right]$ $= 2^6(1-i\times0)$ $= 64$	2 Marks: Correct answer. 1 Mark: Makes some progress towards the solution.
11(e) (i)	$\cos^{-1}(e^x): -1\le e^x\le 1$ but $e^x>0$ for all x hence $0<e^x\le 1$ Domain: $x\le 0$ Range: $0\le y<\frac{\pi}{2}$	2 Marks: Correct answer. 1 Mark: Finds the domain or range.

11(e) (ii)	 	1 Mark: Correct answer.
11(e) (iii)	Horizontal asymptote at $y = \frac{\pi^2}{4}$  	1 Mark: Correct answer.
12(a) (i)	 	1 Mark: Correct answer.
12(a) (ii)	See Argand diagram above.	1 Mark: Correct answer.
12(a) (iii)	Vectors OC and AB form a parallelogram. However $OA = OB = 2$. Hence $OBCA$ is a rhombus. $\angle AOC = \frac{1}{2}\times\left(\frac{\pi}{2}-\frac{\pi}{12}\right) = \frac{5\pi}{24}$ (diagonals of a rhombus bisect the angles through which they pass) $\arg(z_1+z_2) = \frac{\pi}{12}+\frac{5\pi}{24}$ $= \frac{7\pi}{24}$ $AB \perp OC$ (diagonals of a rhombus intersect at right angles) $\arg(z_1-z_2) = \frac{\pi}{2}+\frac{7\pi}{24}$ $= \frac{19\pi}{24}$	2 Marks: Correct answer. 1 Mark: Finds the value of $\arg(z_1+z_2)$ or $\arg(z_1-z_2)$ or shows some understanding.

12(b)	$t=\tan\frac{x}{2}$ $dt=\frac{1}{2}\sec^2\frac{x}{2}dx$ or $dt=\frac{1}{2}(1+t^2)dx$ $dx=\frac{2}{1+t^2}dt$ When $x=0$ then $t=0$ and when $x=\frac{\pi}{2}$ then $t=1$ $2-\cos x+2\sin x=\frac{2(1+t^2)-(1-t^2)+4t}{1+t^2}$ $=\frac{3t^2+4t+1}{1+t^2}=\frac{(3t+1)(t+1)}{1+t^2}$ $\int_0^{\frac{\pi}{2}}\frac{1}{2-\cos x+2\sin x}dx=\int_0^1\frac{1+t^2}{(3t+1)(t+1)}\times\frac{2}{1+t^2}dt$ $=\int_0^1\frac{3}{(3t+1)}-\frac{1}{(t+1)}dt$ $=\left[\log_e(3t+1)-\log_e(t+1)\right]_0^1$ $=(\log_e4-\log_e2)-(\log_e1-\log_e1)$ $=\log_e2$	4 Marks: Correct answer 3 Marks: Correctly determines the primitive function. 2 Marks: Correctly expresses the integral in terms of t. 1 Mark: Correctly finds $d\theta$ in terms of dt and determines the new limits.
12(c) (i)	$y=\frac{1}{f(x)}=\frac{1}{(2-x)(x-4)}$ Vertical asymptote at $x=2$ & $x=4$ $\lim_{x\to\infty}\frac{1}{(2-x)(x-4)}=0^-$ Horizontal asymptote $y=0$ 	2 Marks: Correct answer. 1 Mark: Determines the asymptotes or shows some understanding.
12(c) (ii)	$\lim_{x\to\infty}e^{(2-x)(x-4)}=0$ Horizontal asymptote $y=0$ When $x=2$ then $y=e^0=1$ When $x=4$ then $y=e^0=1$ When $x=3$ then $y=e^1=e$ 	2 Marks: Correct answer. 1 Mark: Determines the asymptotes or shows some understanding.
12(d)	Area of the slice is a circle radius is x and height y $A=\pi x^2=\pi(y^{\frac{1}{3}})^2$ $=\pi y^{\frac{2}{3}}$	3 Marks: Correct answer.

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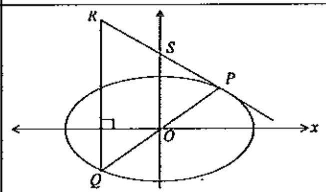
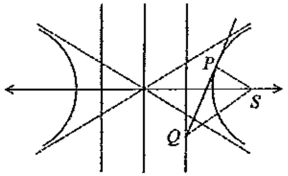
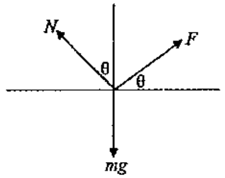
12(d)	$\delta V=\delta A.\delta y \quad V=\lim_{\delta y\to0}\sum_{y=0}^{8}\pi y^{\frac{2}{3}}\delta y$ $=\int_0^8\pi y^{\frac{2}{3}}dy$ $=\pi\left[\frac{3}{5}y^{\frac{5}{3}}\right]_0^8=\frac{3\pi}{5}\times8^{\frac{5}{3}}=\frac{96\pi}{5}$ cubic units	2 Marks: Correct integral for the volume of the solid. 1Mark: Sets up the area of the slice.
13(a) (i)	To find the equation of tangent through P $x=a\cos\theta \qquad y=b\sin\theta$ $\frac{dx}{d\theta}=-a\sin\theta \qquad \frac{dy}{d\theta}=b\cos\theta$ $\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}=b\cos\theta\times\frac{1}{-a\sin\theta}=\frac{-b\cos\theta}{a\sin\theta}$ Equation of the tangent $y-y_1=m(x-x_1)$ $y-b\sin\theta=\frac{-b\cos\theta}{a\sin\theta}(x-a\cos\theta)$ $ay\sin\theta-ab\sin^2\theta=-bx\cos\theta+ab\cos^2\theta$ $bx\cos\theta+ay\sin\theta=ab(\sin^2\theta+\cos^2\theta)$ $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$	2 Marks: Correct answer 1 Mark: Correctly calculates the gradient.
13(a) (ii)	 OS □ QR and O is the midpoint of PQ (diameter). Therefore $\Delta PQR \lvert\lvert\lvert \Delta POS$ Area $\Delta PQR=4\times$ Area ΔPOS (enlargement factor of 2) $S\left(0,\frac{b}{\sin\theta}\right)$ and $P(a\cos\theta,b\sin\theta)$ Area $\Delta POS=\frac{1}{2}\times\frac{b}{\lvert\sin\theta\rvert}\times a\lvert\cos\theta\rvert=\frac{1}{2}\times\frac{ab}{\lvert\tan\theta\rvert}$ Area $\Delta PQR=4\times\frac{1}{2}\times\frac{ab}{\lvert\tan\theta\rvert}=\frac{2ab}{\lvert\tan\theta\rvert}$ Area of the ellipse is πab $\frac{\text{Area }\Delta PQR}{\text{Area of ellipse}}=\frac{\frac{2ab}{\lvert\tan\theta\rvert}}{\pi ab}=\frac{2}{\pi\lvert\tan\theta\rvert}=2:\pi\lvert\tan\theta\rvert$	3 Marks: Correct answer. 2 Marks: Finds the area of ΔPOS. 1 Mark: Recognises similar triangles and the enlargement factor.

13(b) (i)	The equation $x^3+2x+1=0$ has roots α,β,γ. $x=\alpha^{-1},\beta^{-1},\gamma^{-1}$ satisfies $\left(\frac{1}{x}\right)^3+2\left(\frac{1}{x}\right)+1=0$ $1+2x^2+x^3=0$ $\therefore x^3+2x^2+1=0$ has roots α^{-1}, β^{-1} and γ^{-1}	2 Marks: Correct answer. 1 Mark: Writes an equation in $\frac{1}{x}$ with roots α^{-1}, β^{-1} and γ^{-1}
13(b) (ii)	$x=\alpha^{-2},\beta^{-2},\gamma^{-2}$, $\alpha=\frac{1}{\sqrt{x}}$ satisfies $\left(\frac{1}{\sqrt{x}}\right)^3+2\left(\frac{1}{\sqrt{x}}\right)+1=0$ $x^0+2x^1+x^{\frac{3}{2}}=0$ $x^{\frac{3}{2}}=-2x-1$ $x^3=(-2x-1)^2$ $x^3-4x^2-4x-1=0$ $x^3-4x^2-4x-1=0$ has roots α^{-2}, β^{-2} and γ^{-2}.	2 Marks: Correct answer. 1 Mark: Writes an equation in $\frac{1}{\sqrt{x}}$ with roots α^{-2}, β^{-2} and γ^{-2}
13(c) (i)	$\frac{5x^3-3x^2+2x-1}{x^2(x^2+1)}=\frac{a}{x}+\frac{b}{x^2}+\frac{cx+d}{x^2+1}$ $5x^3-3x^2+2x-1=a(x)(x^2+1)+b(x^2+1)+(cx+d)(x^2)$ $=ax^3+ax+bx^2+b+cx^3+dx^2$ $a+c=5$ (coefficients x^3) $b+d=-3$ (coefficients x^2) $a=2$ (coefficients x) $b=-1$ (constant) Therefore $c=3$ and $d=-2$ $\therefore a=2,\ b=-1,\ c=3$ and $d=-2$	2 Marks: Correct answer. 1 Mark: Makes some progress in finding a,b,c or d.
13(c) (ii)	$\int\frac{5x^3-3x^2+2x-1}{x^2(x^2+1)}dx=\int\frac{2}{x}-\frac{1}{x^2}+\frac{3x-2}{x^2+1}dx$ $=\int\frac{2}{x}-\frac{1}{x^2}+\frac{3x}{x^2+1}-\frac{2}{x^2+1}dx$ $=2\log_e x+\frac{1}{x}+\frac{3}{2}\log_e(x^2+1)-2\tan^{-1}x+C$	2 Marks: Correct answer. 1 Mark: Correctly finds one of the integrals.
13(d)	$P(x)=x^4+ax^2+bx+28$ $P'(x)=4x^3+2ax+b$ Root at $x=2$ $P(2)=2^4+a\times2^2+b\times2+28=0$ $44+4a+2b=0$ (1) $P'(2)=4\times2^3+2a\times2+b=0$ $32+4a+b=0$ (2)	2 Marks: Correct answer. 1 Mark: Finds either a or b. Alternatively uses the double root by differentiating the polynomial.

13(d)	Equation (1) – (2) $12+b=0$ $b=-12$ Substitute $b=-12$ into Equation (2) $32+4a-12=0$ $a=-5$ Therefore $a=-5$ and $b=-12$.	
14(a) (i)	$z^3+1=0$ $(z+1)(z^2-z+1)=0$ $\therefore z=-1$ or $z=\frac{1\pm\sqrt{(-1)^2-4\times1\times1}}{2}$ $=\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$	2 Marks: Correct answer. 1 Mark: Finds one root.
14(a) (ii)	The two non-real cubic roots of -1 are $\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$. These roots satisfy the equation $(z^2-z+1)=0$. However since α is not real it also satisfies the equation. $\therefore\alpha^2-\alpha+1=0$ $\alpha^2=\alpha-1$	1 Mark: Correct answer.
14(a) (iii)	$(1-\alpha)^6=(-\alpha^2)^6$ $=\alpha^{12}$ $=(\alpha^3)^4$ $=(-1)^4=1$	1 Mark: Correct answer.
14(b)	$x^2y+2x-2xy=0$ $2xy+x^2\frac{dy}{dx}+2-2y-2x\frac{dy}{dx}=0$ (Implicit differentiation) $\frac{dy}{dx}(x^2-2x)=2y-2xy-2$ $\frac{dy}{dx}=\frac{2y-2xy-2}{x^2-2x}$ At (1, 2) $\frac{dy}{dx}=\frac{2\times2-2\times1\times2-2}{1^2-2\times1}=\frac{-2}{-1}=2$ $y-y_1=m(x-x_1)$ $y-2=2(x-1)$ $2x-y=0$	2 Marks: Correct answer. 1 Mark: Finds the derivative using implicit differentiation.

14(c) (i)	 $\frac{d}{dx}\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=\frac{d}{dx}(1)$ $\frac{2x}{a^2}-\frac{2y\times\frac{dy}{dx}}{b^2}=0$ or $\frac{dy}{dx}=\frac{b^2x}{a^2y}$ Gradient at $P(a\sec\theta, b\tan\theta)=\frac{b^2\times a\sec\theta}{a^2\times b\tan\theta}=\frac{b}{a\sin\theta}$ $y-b\tan\theta=\frac{b}{a\sin\theta}(x-a\sec\theta)$ $a\sin\theta y-ab\tan\theta\sin\theta=bx-ab\sec\theta$ $bx-a\sin\theta y=ab(\sec\theta-\tan\theta\sin\theta)$ $=ab\left(\frac{1}{\cos\theta}-\frac{\sin^2\theta}{\cos\theta}\right)$ $bx-a\sin\theta y=ab\cos\theta$	2 Marks: Correct answer. 1 Mark: Finds the gradient of the curve or shows some understanding.
14(c) (ii)	Q is intersection of the directrix $x=\frac{a}{e}$ and the tangent. $b\times\frac{a}{e}-a\sin\theta y=ab\cos\theta$ $y=\frac{b(1-e\cos\theta)}{e\sin\theta}$ $\therefore Q$ is $\left(\frac{a}{e},\frac{b(1-e\cos\theta)}{e\sin\theta}\right)$	1 Mark: Correct answer.
14(c) (iii)	Focus $S(ae,0)$ and Q is $\left(\frac{a}{e},\frac{b(1-e\cos\theta)}{e\sin\theta}\right)$ Gradient $QS=\frac{\frac{b(1-e\cos\theta)}{e\sin\theta}}{\frac{a}{e}-ae}=\frac{b(1-e\cos\theta)}{a(1-e^2)\sin\theta}$ Gradient $PS=\frac{b\tan\theta}{a\sec\theta-ae}=\frac{b\sin\theta}{a(1-e\cos\theta)}$ $\therefore m_1m_2=\frac{b(1-e\cos\theta)}{a(1-e^2)\sin\theta}\times\frac{b\sin\theta}{a(1-e\cos\theta)}$ $=\frac{b^2}{a^2(1-e^2)}=\frac{b^2}{-b^2}=-1$ Hence QS is perpendicular to PS.	2 Marks: Correct answer. 1 Mark: Finds the gradient of QS or the gradient of PS.

14(d)	Cylindrical shell – inner radius x, outer radius $x+\delta x$, height y. $\delta V=\pi\left[(x+\delta x)^2-x^2\right]y$ $=\pi\left[2x\delta x+\delta x^2\right]y$ $=\pi(2x+\delta x)(\sin x)\delta x$ $V=\lim_{\delta x\to 0}\sum_{x=0}^{\pi}\pi(2x+\delta x)\sin x\delta x$ $=2\pi\int_0^{\pi}x\sin x\,dx$ $=2\pi\left(\left[-x\cos x\right]_0^{\pi}\right)-2\pi\int_0^{\pi}-\cos x\,dx$ $=2\pi(\pi-0)+2\pi\left[\sin x\right]_0^{\pi}$ $=2\pi^2$	4 Marks: Correct answer. 3 Marks: Correct integral for the volume of the solid. 2 Marks: Correct expression for δV. 1 Mark: Determines the radius or height of the cylindrical shell.
15(a) (i)	 Resolving the forces vertically and horizontally $N\cos\theta+F\sin\theta-mg=0$ (1) $N\sin\theta-F\cos\theta=\frac{mv^2}{r}$ (2) Equation (1) $\times\sin\theta$ – Equation (2) $\times\cos\theta$ $F(\sin^2\theta+\cos^2\theta)=mg\sin\theta-\frac{mv^2\cos\theta}{r}$ $F=\frac{m\cos\theta}{r}(gr\tan\theta-v^2)$ $F=\frac{2000\times\cos\theta}{1,000}(10\times1,000\tan\theta-v^2)$ $=2\cos\theta(10,000\tan\theta-v^2)$	3 Marks: Correct answer. 2 Marks: Eliminates N to give an expression for F. 1 Mark: Resolves the forces vertically and horizontally.
15(a) (ii)	Now $F=0$ when $v=v_0$. $F=2\cos\theta(10,000\tan\theta-v^2)$ $0=2\cos\theta(10,000\tan\theta-{v_0}^2)$ ${v_0}^2=10,000\tan\theta$ Now $v=0.80\times v_0$ $v^2=0.64\times{v_0}^2$ $=0.64\times10,000\tan\theta=6,400\tan\theta$	2 Marks: Correct answer. 1 Mark: Finds the correct expression for v^2 or shows some understanding.

15(a) (ii)	$F = 2\cos\theta(10{,}000\tan\theta - v^2)$ $= 2\cos\theta(10{,}000\tan\theta - 6{,}400\tan\theta)$ $= 2\cos\theta \times 3{,}600\tan\theta$ $= 7{,}200\sin\theta$ Also $\sin\theta = \frac{0.03}{1.2}$ $= 0.025$ $F = 7{,}200 \times 0.025$ $= 180\ N$ Therefore the lateral thrust is $180\ N$	
15(b) (i)	The complex number w is a root of $P(z) = 0$ Therefore $P(w) = w^8 - \frac{5}{2}w^4 + 1 = 0$ Now $P(iw) = (iw)^8 - \frac{5}{2}(iw)^4 + 1$ $= i^8w^8 - \frac{5}{2}i^4w^4 + 1$ $= w^8 - \frac{5}{2}w^4 + 1 = 0$ Also $P(\frac{1}{w}) = (\frac{1}{w})^8 - \frac{5}{2}(\frac{1}{w})^4 + 1$ $= \frac{1}{w^8}\left(1 - \frac{5}{2}w^4 + w^8\right)$ $= \frac{1}{w^8} \times 0 = 0$ Therefore iw and $\frac{1}{w}$ are roots of $P(z) = 0$.	2 Marks: Correct answer. 1 Mark: Shows iw or $\frac{1}{w}$ are roots of the polynomial or alternatively shows some understanding.
15(b) (ii)	$z^8 - \frac{5}{2}z^4 + 1 = 0$ $2z^8 - 5z^4 + 2 = 0$ $(2z^4 - 1)(z^4 - 2) = 0$ $\therefore z^4 = \frac{1}{2}$ or $z^4 = 2$ Roots are $z = \pm\sqrt[4]{2}$ or $z = \pm\frac{1}{\sqrt[4]{2}}$.	2 Marks: Correct answer. 1 Mark: Factorises the polynomial.

15(c) (i)	$\text{LHS} = \sin x + \sin 3x$ $= \sin(2x - x) + \sin(x + 2x)$ $= \sin 2x\cos x - \cos 2x\sin x + \sin x\cos 2x + \cos x\sin 2x$ $= 2\sin 2x\cos x$ $= \text{RHS}$	1 Mark: Correct answer.
15(c) (ii)	$\sin x + \sin 2x + \sin 3x = 0$ $\sin 2x + 2\sin 2x\cos x = 0$ from part (i) $\sin 2x(2\cos x + 1) = 0$ $\sin 2x = 0$ or $2\cos x + 1 = 0$ $x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$	2 Marks: Correct answer. 1 Mark: Uses the result in part (i) and factorises.
15(d)	$F = \frac{m}{x^3}(6 - 10x)$ $ma = \frac{m}{x^3}(6 - 10x)$ $v\frac{dv}{dx} = \frac{6}{x^3} - \frac{10}{x^2}$ $\int v\,dv = \int(\frac{6}{x^3} - \frac{10}{x^2})dx$ $\frac{1}{2}v^2 = (\frac{6x^{-2}}{-2} - \frac{10x^{-1}}{-1}) + C$ $\frac{1}{2}v^2 = (\frac{-3}{x^2} + \frac{10}{x}) + C$ When $v = 0$ and $x = 1$ $\frac{1}{2}0^2 = (\frac{-3}{1^2} + \frac{10}{1}) + C$ $C = -7$ $\frac{1}{2}v^2 = (\frac{-3}{x^2} + \frac{10}{x}) - 7$ $v^2 = (\frac{-6}{x^2} + \frac{20}{x}) - 14$ $= \frac{-6 + 20x - 14x^2}{x^2}$ $v = \pm\frac{1}{x}\sqrt{2(-3 + 10x - 7x^2)}$	3 Marks: Correct answer. 2 Marks: Integrates and uses the initial conditions to find an expression for $\frac{1}{2}v^2$. 1 Mark: Uses $v\frac{dv}{dx}$ for acceleration.

16(a)	$(a-b)^2=a^2+b^2-2ab$ $(b-c)^2=b^2+c^2-2bc$ $(c-a)^2=c^2+a^2-2ca$ $2\left[a^2+b^2+c^2-(ab+bc+ca)\right]=(a-b)^2+(b-c)^2+(c-a)^2$ Now a, b and c are side lengths of the triangle and are all positive real numbers. $\therefore (a-b)^2\ge 0$ and $(a-b)^2=0$ only if $a=b$ Hence if $a^2+b^2+c^2=ab+bc+ca$ (given) then $(a-b)^2+(b-c)^2+(c-a)^2=0$ $\therefore (a-b)^2=(b-c)^2=(c-a)^2=0$ $\therefore a=b=c$ Therefore ΔABC is an equilateral triangle.	3 Marks: Correct answer. 2 Marks: Makes significant progress. 1 Mark: Uses the sum of binomial square to obtain the required expression.
16(b)(i)	$I_n=\int_0^1(1-x^r)^n\,dx$ $=\left[x(1-x^r)^n\right]_0^1-n\int_0^1 x(1-x^r)^{n-1}(-rx^{r-1})dx$ $=0-nr\int_0^1\left[(1-x^r)-1\right](1-x^r)^{n-1}dx$ $=0-nr\int_0^1(1-x^r)^n-(1-x^r)^{n-1}dx$ $=nr(-I_n+I_{n-1})$ $(nr+1)I_n=nrI_{n-1}$ $I_n=\dfrac{nr}{nr+1}I_{n-1}$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Sets up the integration and shows some understanding.
16(b)(ii)	For $r=\frac{3}{2}$ and $n=3$ $I_3=\dfrac{3\times\frac{3}{2}}{3\times\frac{3}{2}+1}\times I_2,\ I_2=\dfrac{2\times\frac{3}{2}}{2\times\frac{3}{2}+1}\times I_1,\ I_1=\dfrac{1\times\frac{3}{2}}{1\times\frac{3}{2}+1}\times I_0$ But $I_0=\int_0^1(1-x^r)^0dx=\int_0^1 1dx=1$ $I_3=\dfrac{3\times\frac{3}{2}}{3\times\frac{3}{2}+1}\times\dfrac{2\times\frac{3}{2}}{2\times\frac{3}{2}+1}\times\dfrac{1\times\frac{3}{2}}{1\times\frac{3}{2}+1}\times 1$ $=\dfrac{81}{220}$	2 Marks: Correct answer. 1 Mark: Uses the result from (b)(i) to obtain an expression for $(a-b)^2=a^2+b^2-2ab$
16(c)(i)	$(1+\frac{1}{n})^n=\sum_{k=0}^{n}{}^nC_k(\frac{1}{n})^k$ $=\sum_{k=0}^{n}\dfrac{n!}{(n-k)!k!}(\frac{1}{n})^k$ $=\sum_{k=0}^{n}\dfrac{n(n-1)(n-2)...(n-k+1)}{n^k}\dfrac{1}{k!}$	1 Mark: Correct answer.

16(c)(ii)	$(1+\frac{1}{n})^n=\sum_{k=0}^{n}\dfrac{n(n-1)(n-2)...(n-k+1)}{n^k}\dfrac{1}{k!}$ $=\sum_{k=0}^{n}\dfrac{n}{n}\times\dfrac{(n-1)}{n}\times\dfrac{(n-2)}{n}\times...\times\dfrac{(n-k+1)}{n}\dfrac{1}{k!}$ As $n\to\infty$ then $\dfrac{n-1}{n}\to 1$, $\dfrac{n-2}{n}\to 1$, $\dfrac{n-3}{n}\to 1$.... $\lim_{n\to\infty}(1+\frac{1}{n})^n=\sum_{k=0}^{n}\dfrac{1}{k!}=1+1+\dfrac{1}{2!}+\dfrac{1}{3!}+\dfrac{1}{4!}+...=2+\dfrac{1}{2!}+\dfrac{1}{3!}+\dfrac{1}{4!}+...$	2 Marks: Correct answer. 1 Mark: Splitting the expression into fractions.
16(c)(iii)	Test the result for $n=3$ $\dfrac{1}{3!}<\dfrac{1}{2^{3-1}}$ or $\dfrac{1}{6}<\dfrac{1}{4}$. Therefore the result is true for $n=3$. Assume the result is true for $n=k$. $\dfrac{1}{k!}<\dfrac{1}{2^{k-1}}$ To prove the result is true for $n=k+1$. $\dfrac{1}{(k+1)!}<\dfrac{1}{2^{(k+1)-1}}<\dfrac{1}{2^k}$ LHS $=\dfrac{1}{(k+1)!}$ $=\dfrac{1}{(k+1)k!}$ $<\dfrac{1}{(k+1)2^{k-1}}$ Assumption for $n=k$ $<\dfrac{1}{2\times 2^{k-1}}$ $k+1>2$ as $n\ge 3$ $=\dfrac{1}{2^k}=$ RHS Thus if the result is true for $n=k$, it is true for $n=k+1$. It has been shown true for $n=3$, hence true for $n=4$ and so on.	3 Marks: Correct answer. 2 Marks: Proves the result true for $n=k+1$ 1 Mark: Tests the result for $n=3$
16(c)(iv)	From part (ii) $\lim_{n\to\infty}(1+\frac{1}{n})^n=2+\dfrac{1}{2!}+\dfrac{1}{3!}+\dfrac{1}{4!}+...$ $=2+\dfrac{1}{2}+\sum_{k=3}^{\infty}\dfrac{1}{k!}$ $<2+\dfrac{1}{2}+\sum_{k=3}^{\infty}\dfrac{1}{2^{k-1}}$ $<2+\dfrac{1}{2}+(\dfrac{1}{2^2}+\dfrac{1}{2^3}+\dfrac{1}{2^4}+...)$ $<2+\dfrac{1}{2}+(\dfrac{\frac{1}{2^2}}{1-\frac{1}{2}})$ Limiting sum of GP $<2+\dfrac{1}{2}+\dfrac{1}{2}<3$	1 Mark: Correct answer.