ACE Examinations Student Name: ____________________

2015

YEAR 12

YEARLY EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time - 5 minutes
- Working time - 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show relevant mathematical reasoning and/or calculations in Questions 11-16

Total marks - 100

Section I
10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II
90 marks

- Attempt Questions 11-16
- Allow about 2 hour 45 minutes for this section

STUDENT NUMBER/NAME: ....

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Section I

10 marks
Attempt Questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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1 A particle is moving in a circular path of radius r, with a constant angular speed of ω Which of the following is the correct expression for acceleration?

(A) $r\omega$

(B) ωr^2

(C) $r\omega^2$

(D) $(r\omega)^2$

2 What is the value of $\int_0^{\frac{\pi}{6}} \sec 4x \tan 4x dx$?

(A) $-1\frac{1}{2}$

(B) $-\frac{3}{8}$

(C) $\frac{3}{8}$

(D) $1\frac{1}{2}$

3 What is the volume of the solid formed when the region bounded by the curves $y = 2x^3$ and $y = 2\sqrt{x}$ is rotated about the x-axis? Use the method of slicing.

(A) $\frac{5\pi}{14}$ cubic units

(B) $\frac{10\pi}{14}$ cubic units

(C) $\frac{5\pi}{7}$ cubic units

(D) $\frac{10\pi}{7}$ cubic units

4 The normal to the point $P(cp, \frac{c}{p})$ on the rectangular hyperbola $xy = c^2$ has the equation $p^3x - py + c - cp^4 = 0$. The normal cuts the hyperbola at another point $Q(cq, \frac{c}{q})$.

What is the relationship between p and q?

(A) $pq = -1$

(B) $p^2q = -1$

(C) $p^3q = -1$

(D) $p^4q = -1$

5 Which of the following is an expression for $\int \frac{1}{x^2 - 6x + 13} dx$?

(A) $\frac{1}{2}\tan^{-1}\frac{(x-3)}{4} + C$

(B) $\frac{1}{2}\tan^{-1}\frac{(x-3)}{2} + C$

(C) $\frac{1}{4}\tan^{-1}\frac{(x-3)}{4} + C$

(D) $\frac{1}{4}\tan^{-1}\frac{(x-3)}{2} + C$

6 What is the number of asymptotes on the graph of $y = \frac{2x^3}{x^2 - 1}$?

(A) 1

(B) 2

(C) 3

(D) 4

7 What is $(-1+i)^n$ expressed in modulus-argument form? (n is a positive integer)

(A) $\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$

(B) $(\sqrt{2})^n\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$

(C) $\left(\cos\frac{3n\pi}{4} + i\sin\frac{3n\pi}{4}\right)$

(D) $(\sqrt{2})^n\left(\cos\frac{3n\pi}{4} + i\sin\frac{3n\pi}{4}\right)$

8 Let α, β and γ be the roots of the equation $x^3 + 2x^2 + 5 = 0$.

Which of the following polynomial equations have the roots α^2, β^2 and γ^2?

(A) $x^3-4x^2-20x-25=0$

(B) $x^3-4x^2-10x-25=0$

(C) $x^3-4x^2-20x-5=0$

(D) $x^3-4x^2-10x-5=0$

9 What is the derivative of $\sin^{-1}x-\sqrt{1-x^2}$?

(A) $\dfrac{\sqrt{1+x}}{\sqrt{1-x}}$

(B) $\dfrac{\sqrt{1+x}}{1-x}$

(C) $\dfrac{1+x}{\sqrt{1-x}}$

(D) $\dfrac{1+x}{1-x}$

10 Points $P(a\cos\theta,\ b\sin\theta)$ and $Q(a\cos\phi,\ b\sin\phi)$ lie on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

The chord PQ subtends a right angle at $(0,0)$.
Which of the following is the correct expression?

(A) $\tan\theta\tan\phi=-\dfrac{b^2}{a^2}$

(B) $\tan\theta\tan\phi=-\dfrac{a^2}{b^2}$

(C) $\tan\theta\tan\phi=\dfrac{b^2}{a^2}$

(D) $\tan\theta\tan\phi=\dfrac{a^2}{b^2}$

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Marks**

(a) (i) On the Argand diagram sketch the graph of $\left|z-(\sqrt{2}+\sqrt{2}i)\right|=1$. **2**

(ii) Find possible values of $|z|$ and $\arg z$ if z satisfies $\left|z-(\sqrt{2}+\sqrt{2}i)\right|=1$. **2**

(b) (i) Find real numbers a, b and c such that **2**

$$\frac{3x^2-3x+2}{(2x-1)(x^2+1)}=\frac{a}{2x-1}+\frac{bx+c}{x^2+1}$$

(ii) Hence evaluate in simplest form **2**

$$\int\frac{3x^2-3x+2}{(2x-1)(x^2+1)}dx$$

(c) Use the substitution $x=u^2$ $(u>0)$ to evaluate $\displaystyle\int\frac{1}{x(1+\sqrt{x})}dx$. **3**

(d) Let $z_1=\cos\theta_1+i\sin\theta_1$ and $z_2=\cos\theta_2+i\sin\theta_2$, where θ_1 and θ_2 are real.

(i) Show that $\dfrac{1}{z_1}=\cos\theta_1-i\sin\theta_1$. **1**

(ii) Show that $z_1z_2=\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)$. **1**

(e) Let $z_1=\dfrac{a}{1+i}$ and $z_2=\dfrac{b}{1+2i}$, where a and b are real numbers. **2**
What is the value of a and b, if $z_1+z_2=1$?

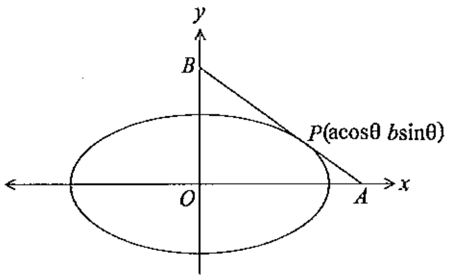
Question 12 (15 marks) **Marks**

(a) Sketch the graph of $\arg\left(\frac{z-2}{z+2i}\right)=\frac{\pi}{2}$. **2**

(b) Let $z=1+i$ be a root of the polynomial $z^2-biz+c=0$ where b and c are real numbers. Find the value of b and c. **2**

(c) The parabola $y=4-x^2$ is rotated about the line $y=4$ $\{x:0\le x\le 2\}$ to form a solid. Use the method of slicing to find the volume of the solid. **3**

(d) The point P lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a>b>0$.



(i) Use the parametric representation of an ellipse to show that the equation of the tangent is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$. **2**

(ii) The tangent at P cuts the x-axis at A, y-axis at B and C is the foot of the perpendicular from P to the y-axis. Show that $OC\times OB=b^2$. **2**

(e) Consider the function $f(x)=x^4-4x^3$.

(i) Sketch the graph of $y=f(x)$. **3**

(ii) Hence or otherwise find the number of real roots of the equation $x^4-4x^3=kx$, where k is a positive real number. **1**

Question 13 (15 marks) **Marks**

(a) The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ has a focus S on the positive x-axis and the corresponding directrix cuts the asymptotes to the hyperbola at the points P and Q in the first and fourth quadrants respectively.

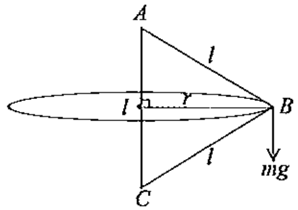
(i) Show that PS is perpendicular to the asymptote through P. **2**

(ii) Show that $PS=b$. **1**

(iii) A circle with centre S touches the asymptotes of the hyperbola. Deduce that the point of contact are the points P and Q. **1**

(iv) The circle with centre S touches the asymptotes of the hyperbola and cuts the hyperbola at the points R and T. Show that RT is a diameter of the circle if $a=b$. **2**

(b) Two light inextensible strings AB and BC each of length l are attached to a particle of mass m at B. The other ends A and C are fixed in a vertical line such that AC is also the length l. The particle describes a horizontal circle with constant angular velocity ω. T_1 and T_2 are tensions in the strings AB and BC respectively.



(i) Find the tensions in the strings. **4**

(ii) What is the least value of ω in order for the strings to be taut? **1**

(c) Let $f(x)=\frac{x^2}{x^2-1}$. Draw separate one-third page sketches of these functions.

(i) $y=|f(x)|$ **2**

(ii) $y=\log_2[f(x)]$ **2**

Question 14 (15 marks) **Marks**

(a) (i) Show that $z\overline{z} = |z|^2$ for any complex number z. **1**

(ii) A sequence of complex numbers z_n is given by the rule $z_1 = w$ and $z_n = v\overline{z}_{n-1}$ where w is a given complex number and v is a complex number with modulus 1. Show that $z_3 = w$. **2**

(b) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$, where n is positive integer.

(i) Show that $I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x\,dx$ when $n \geq 2$. **2**

(ii) Prove that $I_n = \frac{(n-1)}{n} I_{n-2}$ when $n \geq 2$. **2**

(iii) What is the value of I_4? **1**

(c) A solid is formed by rotating about the y-axis the region bounded by the curve $y = \log_e x$ and the x-axis between $1 \leq x \leq e$. Find the volume of this solid using the method of cylindrical shells. **4**

(d) Show that if $x \neq 1$ then $1 + x + x^2 + ... + x^n = \frac{x^{n+1} - 1}{x - 1}$ for $n \geq 1$. **3**

HSC Mathematics Extension 2

Question 15 (15 marks) **Marks**

(a) (i) The polynomial $P(x)$ has a double root at $x = \alpha$. Prove that $P'(x)$ has a root at $x = \alpha$. **2**

(ii) The polynomial $P(x) = x^5 - ax^2 + b$ has a double root at $x = \alpha$. Find the values of a and b. **2**

(b) A and B are on the curve $y = x^4 + 4x^3$ at $x = \alpha$ and $x = \beta$ respectively. The line $y = mx + b$ is a tangent to the curve at both points A and B.

(i) The zeros of the equation $x^4 + 4x^3 - mx - b = 0$ are α, α, β and β. Explain this result. **1**

(ii) Find the values for m and b. **3**

(c) A rock is dropped under gravity (g) from rest at the top of a cliff. The air resistance is proportional to the velocity (v) of the rock.

(i) Explain why $\frac{dv}{dt} = g - kv$. **1**

(ii) Show that $v = \frac{g}{k}(1 - e^{-kt})$ when $t \geq 0$. **3**

(iii) Show that $x = -\frac{1}{k}v - \frac{g}{k^2}\log_e\left(\frac{g}{g - kv}\right)$ by using $\frac{dv}{dt} = v\frac{dv}{dx}$. **3**

Question 16 (15 marks) **Marks**

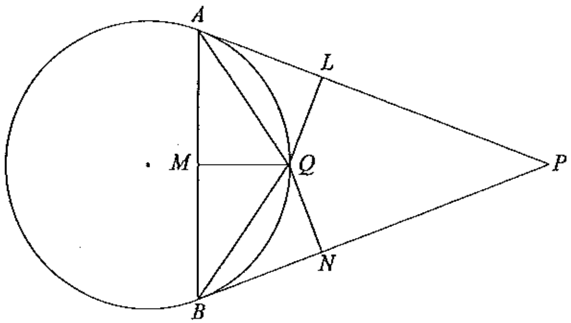
(a) Let $z = r(\cos\theta + i\sin\theta)$ where $z \neq 0$.

(i) Use De Moivre's theorem to show that $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$ for positive integers $n \geq 1$. **2**

(ii) Expand $\left(z - \dfrac{1}{z}\right)^5$ and show that $\sin^5\theta = \dfrac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$ **3**

(b) Show that $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$. **2**

(c) Tangents *PA* and *PB* are drawn to a circle. Point *Q* is on the minor arc *AB*. Perpendiculars *QL*, *QM* and *QN* are drawn from *Q* to *PA*, *AB* and *PB* respectively.



(i) Show that $\Delta BNQ \,|||\, \Delta AMQ$ and $\Delta ALQ \,|||\, \Delta BMQ$. **3**

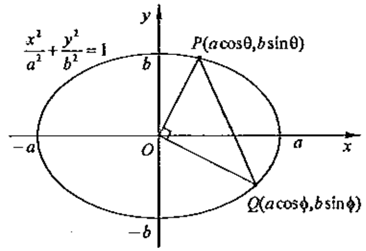
(ii) Hence show that *QN*, *QM* and *QL* form a geometric sequence. **2**

(d) Show that $1 + x + \dfrac{x^2e^x}{2} > e^x$ for $x > 0$. **3**

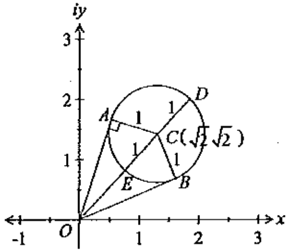
End of paper

ACE Examination 2015
HSC Mathematics Extension 2 Yearly Examination
Worked solutions and marking guidelines

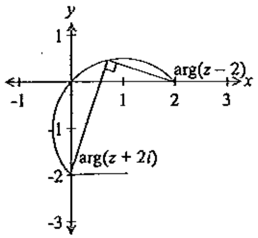
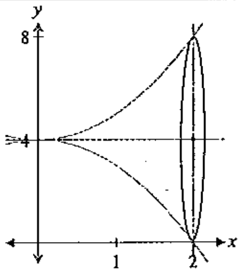
Section I		
	Solution	**Criteria**
1	Acceleration for uniform circular motion. $a = r\omega^2$	1 Mark: C
2	$\int_0^{\frac{\pi}{6}} \sec 4x \tan 4x dx = \left[\frac{1}{4}\sec 4x\right]_0^{\frac{\pi}{6}}$ $= \frac{1}{4}\left[\sec\frac{2\pi}{3} - \sec 0\right] = -1\frac{1}{2}$	1 Mark: A
3	Slices are taken perpendicular to the axis of rotation (x-axis). The base is an annulus. $A = \pi(r_2^2 - r_1^2) = \pi((2\sqrt{x})^2 - (2x^3)^2)$ $= \pi(4x - 4x^6) = 4\pi(x - x^6)$ $V = \lim_{\delta x \to 0} \sum_{x=0}^{1} 4\pi(x - x^6)\delta x$ $= \int_0^1 4\pi(x - x^6)dx = 4\pi\int_0^1 (x - x^6)dx$ $= 4\pi\left[\frac{x^2}{2} - \frac{x^7}{7}\right]_0^1 = 4\pi\left[(\frac{1}{2} - \frac{1}{7}) - 0\right] = \frac{10\pi}{7}$	1 Mark: D
4	$Q(cq, \frac{c}{q})$ is on the normal and satisfies the equation. $p^3cq - p\frac{c}{q} + c - cp^4 = 0$ $p^3q^2 - p + q - qp^4 = 0$ $p^3q(q-p) = -(q-p)$ or $p^3q = -1$	1 Mark: C
5	$\int \frac{1}{x^2 - 6x + 13}dx = \int \frac{dx}{(x-3)^2 + 2^2} = \frac{1}{2}\tan^{-1}\frac{(x-3)}{2} + C$	1 Mark: B
6	$x \neq 1$ or $x \neq -1$ $y = \frac{2x(x^2-1)}{(x^2-1)} + \frac{2x}{(x^2-1)} = 2x + \frac{2x}{(x^2-1)}$ $\quad x \to \infty \; \frac{2x}{(x^2-1)} \to 0$ As $x \to \infty$ then $y \to 2x$ $\therefore x = 1$, $x = -1$ and $y = 2x$ are the asymptotes of the graph.	1 Mark: C

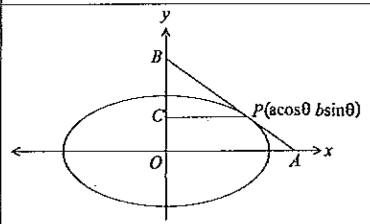
7	$\tan\theta = \frac{1}{-1}$ or $\theta = \frac{3\pi}{4}$ $\quad r^2 = x^2 + y^2 = 1^2 + 1^2$ or $r = \sqrt{2}$ $-1 + i = \sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$ $(-1+i)^n = (\sqrt{2})^n\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)^n$ $= (\sqrt{2})^n\left(\cos\frac{3n\pi}{4} + i\sin\frac{3n\pi}{4}\right)$	1 Mark: D
8	If α, β and γ are zeros of $x^3 + 2x^2 + 5 = 0$ then the polynomial equation with roots α^2, β^2 and γ^2 is: $(\sqrt{x})^3 + 2(\sqrt{x})^2 + 5 = 0$ $(\sqrt{x})^3 = -(2x+5)$ $x^3 = 4x^2 + 20x + 25$ $x^3 - 4x^2 - 20x - 25 = 0$	1 Mark: A
9	$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times -2x$ $= \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} = \frac{1+x}{\sqrt{1-x^2}}$ $= \frac{1+x}{\sqrt{(1+x)(1-x)}} = \frac{\sqrt{1+x}}{\sqrt{1-x}}$ $\quad$ Result defined for $-1 \le x \le 1$	1 Mark: A
10	 POQ is a right-angled triangle. Therefore $OP^2 + OQ^2 = PQ^2$. $a^2\cos^2\theta + b^2\sin^2\theta + a^2\cos^2\varphi + b^2\sin^2\varphi$ $= a^2(\cos\theta - \cos\varphi)^2 + b^2(\sin\theta - \sin\varphi)^2$ $a^2(\cos^2\theta + \cos^2\varphi) + b^2(\sin^2\theta + \sin^2\varphi)$ $= a^2(\cos\theta - \cos\varphi)^2 + b^2(\sin\theta - \sin\varphi)^2$ $0 = -2a^2\cos\theta\cos\varphi - 2b^2\sin\theta\sin\varphi$ $2b^2\sin\theta\sin\varphi = -2a^2\cos\theta\cos\varphi$ Hence $\frac{\sin\theta\sin\varphi}{\cos\theta\cos\varphi} = \frac{-2a^2}{2b^2}$ or $\tan\theta\tan\phi = -\frac{a^2}{b^2}$	1 Mark: B

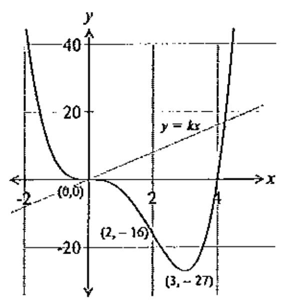
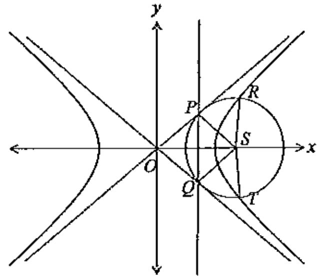
Section II

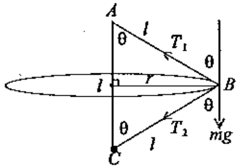
	Solution	Criteria
11(a) (i)	$\left\|z-(\sqrt{2}+\sqrt{2}i)\right\|=1$ Represents a circle with centre $(\sqrt{2},\sqrt{2})$ and radius of 1 unit. 	2 Marks: Correct answer. 1 Mark: Draws a circle or states the radius or centre.
11(a) (ii)	$OC=\sqrt{(\sqrt{2})^2+(\sqrt{2})^2}=2$ $\therefore OE=1$ and $OD=3$ and therefore $1\le\|z\|\le 3$ Arg $OC=\frac{\pi}{4}$ $\sin\angle AOC=\frac{1}{2}$, $\angle AOC=\frac{\pi}{6}$ $\sin\angle BOC=\frac{1}{2}$, $\angle BOC=\frac{\pi}{6}$ $\frac{\pi}{4}-\frac{\pi}{6}\le\arg z\le\frac{\pi}{4}+\frac{\pi}{6}$ or $\frac{\pi}{12}\le\arg z\le\frac{5\pi}{12}$	2 Marks: Correct answer. 1 Mark: Finds $\|z\|$ or arg z or shows some understanding.
11(b) (i)	$\frac{3x^2-3x+2}{(2x-1)(x^2+1)}=\frac{a}{2x-1}+\frac{bx+c}{x^2+1}$ $3x^2-3x+2=a(x^2+1)+(bx+c)(2x-1)$ Let $x=\frac{1}{2}$ and $x=0$ $\frac{5}{4}=a\times\frac{5}{4}$ or $a=1$ $2=a+c(-1)$ $c=-1$ Equating the coefficients of x^2 $3=a+2b$ or $b=1$ $\therefore a=1,\ b=1$ and $c=-1$	2 Marks: Correct answer. 1 Mark: Makes some progress in finding a,b or c.
11(b) (ii)	$\int\frac{3x^2-3x+2}{(2x-1)(x^2+1)}dx=\int\left(\frac{1}{2x-1}+\frac{x-1}{x^2+1}\right)dx$ $=\int\left(\frac{1}{2x-1}+\frac{x}{x^2+1}-\frac{1}{x^2+1}\right)dx$ $=\frac{1}{2}\log_e\|2x-1\|+\frac{1}{2}\log_e\|x^2+1\|-\tan^{-1}x+C$ $=\frac{1}{2}\log_e\left[\|2x-1\|(x^2+1)\right]-\tan^{-1}x+C$	2 Marks: Correct answer. 1 Mark: Correctly finds one of the integrals.

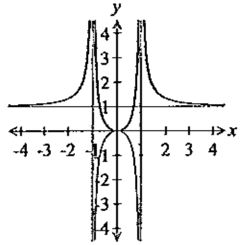
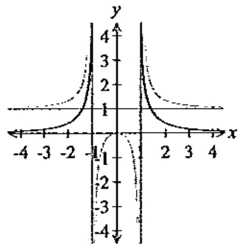
11(c)	Let $x=u^2$ then $\frac{dx}{du}=2u$ or $dx=2udu$ $\int\frac{1}{x(1+\sqrt{x})}dx=\int\frac{1}{u^2(1+u)}2udu$ $=2\int\frac{1}{u(1+u)}du$ $=2\int\left(\frac{1}{u}-\frac{1}{1+u}\right)dx$ $=2\left[\log_e u-\log_e(1+u)\right]$ $=2\log_e\left\|\frac{u}{1+u}\right\|+C$ $=2\log_e\frac{\sqrt{x}}{1+\sqrt{x}}+C$	3 Marks: Correct answer. 2 Marks: Finds the primitive function. 1 Mark: Sets up the integral in terms of u
11(d) (i)	$\frac{1}{z_1}=(\cos\theta_1+i\sin\theta_1)^{-1}$ $=\cos(-\theta_1)+i\sin(-\theta_1)$ De Moivre's theorem $=\cos\theta_1-i\sin\theta_1$	1 Mark: Correct answer.
11(d) (ii)	$z_1z_2=(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)$ $=\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2+i(\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)$ $=\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)$	1 Mark: Correct answer.
11(e)	$\frac{a}{1+i}+\frac{b}{1+2i}=1$ $\frac{a}{1+i}\times\frac{1-i}{1-i}+\frac{b}{1+2i}\times\frac{1-2i}{1-2i}=1$ $\frac{a-ia}{2}+\frac{b-2ib}{5}=1$ $\left(\frac{a}{2}+\frac{b}{5}\right)-i\left(\frac{a}{2}+\frac{2b}{5}\right)=1$ Comparing real and imaginary parts. $\frac{a}{2}+\frac{b}{5}=1$ and $\frac{a}{2}+\frac{2b}{5}=0$ $5a+2b=10$ (1) $5a+4b=0$ (2) Equation (1) – (2) $-2b=10$ or $b=-5$ Substitute $b=-5$ into equation (2) $5a+4\times-5=0$ or $a=4$ Therefore $a=4$ and $b=-5$.	2 Marks: Correct answer. 1 Mark: Substitutes into $z_1+z_2=1$ and uses the conjugate.

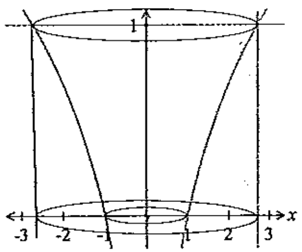
12(a)	$\arg\left(\frac{z-2}{z+2i}\right)=\arg(z-2)-\arg(z+2i)=\frac{\pi}{2}$ Angle in a semicircle.  	2 Marks: Correct answer. 1 Mark: Shows some understanding of the problem
12(b)	$z=1+i$ satisfies the polynomial $z^2-biz+c=0$ $(1+i)^2-bi(1+i)+c=0$ $1+2i-1-bi+b+c=0$ $(b+c)+(2-b)i=0$ Equating real and imaginary parts Therefore $b=2$ and $c=-2$	2 Marks: Correct answer. 1 Mark: Uses the factor theorem.
12(c)	 Same volume as $y=x^2$ rotated about the x-axis. Area of the slice is a circle radius is y and height x $A=\pi y^2$ $=\pi x^4$ $\delta V=\delta A.\delta x$ $V=\lim_{\delta x\to 0}\sum_{x=0}^{2}\pi x^4\delta x$ $=\int_0^2 \pi x^4 dx$ $=\pi\left[\frac{1}{5}x^5\right]_0^2$ $=\frac{\pi}{5}\times 2^5=\frac{32\pi}{5}$ cubic units	3 Marks: Correct answer. 2 Marks: Correct integral for the volume of the solid. 1 Marks: Sets up the area of the slice

12(d) (i)	To find the equation of tangent through P $y=b\sin\theta$ $\frac{dy}{d\theta}=b\cos\theta$ $x=a\cos\theta$ $\frac{dx}{d\theta}=-a\sin\theta$ $\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}$ $=b\cos\theta\times\frac{1}{-a\sin\theta}=\frac{-b\cos\theta}{a\sin\theta}$ Equation of the tangent $y-y_1=m(x-x_1)$ $y-b\sin\theta=\frac{-b\cos\theta}{a\sin\theta}(x-a\cos\theta)$ $ay\sin\theta-ab\sin^2\theta=-bx\cos\theta+ab\cos^2\theta$ $bx\cos\theta+ay\sin\theta=ab(\sin^2\theta+\cos^2\theta)$ $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$	2 Marks: Correct answer 1 Mark: Correctly calculates the gradient
12(d) (ii)	 At B $x=0$ and $\frac{0}{a}\cos\theta+\frac{y}{b}\sin\theta=1$ or $y=b\text{cosec}\theta$ Point B is $(0,b\text{cosec}\theta)$ and Point C is $(0,b\sin\theta)$ $OC\times OB=b\sin\theta\times b\text{cosec}=b^2$	2 Marks: Correct answer 1 Mark: Finds the coordinates or B or C.
12(e) (i)	$f(x)=x^4-4x^3, f'(x)=4x^3-12x^2, f''(x)=12x^2-24x$ Stationary points $f'(x)=0$ $4x^3-12x^2=0$ or $4x^2(x-3)=0$ or $x=0$ or 3 $f''(0)=0$ possible point of inflection. $f''(3)=36>0$ $(3,-27)$ is a Minima Points of inflection $f''(x)=0$ $12x^2-24x=0$ or $12x(x-2)$ or $x=0$ or $x=2$ $f''(0^-)>0$ and $f''(0^+)<0$ Hence $(0,0)$ is a point of inflection $f''(2^-)<0$ and $f''(2^+)>0$ Hence $(2,-16)$ is a point of inflection	3 Marks: Correct answer. 2 Marks: Makes significant progress. 1 Mark: Finds stationary point or shows some understanding.

		
12(e) (ii)	The real solution of $x^4-4x^3=kx$ is given by the x values where $y=x^4-4x^3$ and $y=kx$ intersect. If $k>0$ then from the graph there are 2 real roots.	1 Mark: Correct answer.
13(a) (i)	 Focus $S(ae,0)$ and at P (directrix $x=\frac{a}{e}$ and asymptote $y=\frac{b}{a}x$). At P $x=\frac{a}{e}$ and $y=\frac{b}{a}\times\frac{a}{e}=\frac{b}{e}$ $\quad\therefore P\left(\frac{a}{e},\frac{b}{e}\right)$ Gradient $PS=\frac{\frac{b}{e}}{\frac{a}{e}-ae}=\frac{b}{a(1-e^2)}$ $\quad$ Gradient $OP=\frac{b}{a}$ $\therefore m_1m_2=\frac{b}{a(1-e^2)}\times\frac{b}{a}=\frac{b^2}{a^2(1-e^2)}=\frac{b^2}{-b^2}=-1$ Hence PS is perpendicular to OP.	2 Marks: Correct answer. 1 Mark: Finds the coordinates of P or shows some understanding of the problem.
13(a) (ii)	$PS^2=\left(\frac{a}{e}-ae\right)^2+\left(\frac{b}{e}\right)^2$ $=\frac{1}{e^2}\left[a^2(1-e^2)^2+b^2\right]$ $=\frac{1}{e^2}\left[-b^2(1-e^2)+b^2\right]$ $=\frac{1}{e^2}\left[b^2e^2\right]$ $PS=b$	1 Mark: Correct answer.

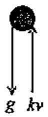
13(a) (iii)	Perpendicular distance from S to P is b (from parts (i) and (ii)). Tangent to a circle is perpendicular to the radius through the point of contact. Therefore P is the point of contact of a circle with centre S and radius b. Similarly, by symmetry Q is the point of contact of a circle with centre S and radius b.	1 Mark: Correct answer.
13(a) (iv)	If $a=b$ then $b^2=a^2(e^2-1)$ $b^2=b^2(e^2-1)$ $e^2=2$ or $e=\sqrt{2}$ Hence $S(a\sqrt{2},0)$ Using the locus definition of a hyperbola with $SR=ST=b$ $\dfrac{b}{x-\frac{a}{e}}=e$ $b=e(x-\frac{a}{e})$ $x=\dfrac{a+b}{e}=\dfrac{a+a}{\sqrt{2}}=a\sqrt{2}$ Therefore, if $a=b$, R and T have the same x coordinate ($a\sqrt{2}$) as S. Hence R, S and T are collinear and RT is the diameter of the circle with centre S.	2 Marks: Correct answer. 1 Mark: Finds the eccentricity or the x-coordinate of R or T in terms of a, b and e.
13(b) (i)	 Resolving the forces vertically and horizontally at B $T_1\cos\theta-T_2\cos\theta-mg=0$ $\quad$ $T_1\sin\theta+T_2\sin\theta=mr\omega^2$ But $\cos\theta=\dfrac{\frac{1}{2}l}{l}=\dfrac{1}{2}$ $\quad$ But $\sin\theta=\dfrac{r}{l}$ $T_1\times\dfrac{1}{2}-T_2\times\dfrac{1}{2}-mg=0$ $\quad$ $T_1\dfrac{r}{l}+T_2\dfrac{r}{l}=mr\omega^2$ $T_1-T_2=2mg$ (1) $\quad$ $T_1+T_2=ml\omega^2$ (2) Adding equations (1) and (2) $2T_1=2mg+ml\omega^2$ or $T_1=m\left(\dfrac{l\omega^2}{2}+g\right)$ Then from equation (2) $m\left(\dfrac{l\omega^2}{2}+g\right)+T_2=ml\omega^2$ or $T_2=m\left(\dfrac{l\omega^2}{2}-g\right)$ Therefore $T_1=m\left(\dfrac{l\omega^2}{2}+g\right)$ and $T_2=m\left(\dfrac{l\omega^2}{2}-g\right)$	4 Marks: Correct answer. 3 Marks: Makes significant progress towards the solution. 2 Marks: Resolves forces and finds expressions for $\sin\theta$ and $\cos\theta$ 1 Mark: Resolves forces in the vertical and horizontal directions a B.

13(b) (ii)	Least value of ω for the strings to be taut: $T_1>T_2$ and $T_2>0$ $T_2=m\left(\dfrac{l\omega^2}{2}-g\right)>0$ $\dfrac{l\omega^2}{2}>g$ $\omega^2>\dfrac{2g}{l}$ or $\omega>\sqrt{\dfrac{2g}{l}}$	1 Mark: Correct answer.
13(c) (i)	$y=\left\|\dfrac{x^2}{x^2-1}\right\|=\left\|\dfrac{x^2}{(x+1)(x-1)}\right\|$ (asymptote at $x=\pm1$) $y=\left\|\dfrac{x^2}{x^2-1}\right\|=\left\|\dfrac{1}{1-\frac{1}{x^2}}\right\|$ $x\to\pm\infty$, $y\to1$ (asymptote at $y=1$) 	2 Marks: Correct answer. 1 Mark: Determines the asymptotes or shows some understanding.
13(c) (ii)		2 Marks: Correct answer. 1 Mark: Determines the asymptotes or shows some understanding.
14(a) (i)	Let $z=a+ib$ where a and b are real. $z\bar{z}=(a+ib)(a-ib)$ $=a^2-i^2b^2$ $=a^2+b^2=\|z\|^2$	1 Mark: Correct answer.
14(a) (ii)	Now $z_1=w$, $z_2=v\bar{z}_1=v\bar{w}$ and $\|v\|=1$ $z_3=v\bar{z}_2$ $=v\times\overline{v\bar{w}}$ $=v\bar{v}w$ $=\|v\|^2w$ from (i) $=w$	2 Marks: Correct answer. 1 Mark: Uses the formula to obtain an expression for z_3.

<table>
<tr><td>14(b)
(i)</td><td>Integration by parts
$I_n=\int_0^{\frac{\pi}{2}}\sin^{n-1}x\sin x\,dx$
$=-\left[\sin^{n-1}x\cos x\right]_0^{\frac{\pi}{2}}+(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\cos^2 x\,dx$
$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\cos^2 x\,dx$</td><td>2 Marks: Correct answer.
1 Mark: Sets up the integration and shows some understanding.</td></tr>
<tr><td>14(b)
(ii)</td><td>$I_n=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\cos^2 x\,dx$
$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x(1-\sin^2 x)dx$
$=(n-1)\int_0^{\frac{\pi}{2}}(\sin^{n-2}x-\sin^n x)dx$
$=(n-1)\left[I_{n-2}-I_n\right]=(n-1)I_{n-2}-nI_n+I_n$
$nI_n=(n-1)I_{n-2}$
$I_n=\frac{(n-1)}{n}I_{n-2}$</td><td>2 Marks: Correct answer.
1 Mark: Makes some progress towards the solution.</td></tr>
<tr><td>14(b)
(iii)</td><td>$I_4=\frac{(4-1)}{4}I_2$
$=\frac{3}{4}\times\frac{(2-1)}{2}I_0$
$=\frac{3}{8}\times\int_0^{\frac{\pi}{2}}1dx=\frac{3\pi}{16}$</td><td>1 Mark: Correct answer.</td></tr>
<tr><td>14(c)</td><td>

Cylindrical shell – inner radius x, outer radius $x+\delta x$, height y.
$\delta V=\pi\left[(x+\delta x)^2-x^2\right]y$
$=\pi\left[2x\delta x+\delta x^2\right]y=\pi(2x+\delta x)(\log_e x)\delta x$
$V=2\times\lim_{\delta x\to 0}\sum_{x=1}^{e}\pi(2x+\delta x)\log_e x\delta x=2\pi\int_1^e(x\log_e x)dx$
$=2\pi\left(\left[\log_e x\times\frac{1}{2}x^2\right]_1^e-\int_1^e\left(\frac{1}{2}x^2\times\frac{1}{x}\right)dx\right)$
$=2\pi\left[\frac{1}{2}e^2-\frac{1}{2}\int_1^e xdx\right]=\pi\left(e^2-\left[\frac{x^2}{2}\right]_1^e\right)=\frac{\pi}{2}(e^2+1)$</td><td>4 Marks: Correct answer.
3 Marks: Correct integral for the volume of the solid.
2 Marks: Correct expression for δV.
1 Mark: Determines the radius or height of the cylindrical shell.</td></tr>
</table>

<table>
<tr><td>14(d)</td><td>Step 1: To prove the statement true for $n=1$
LHS $=1+x$ and RHS $=\frac{x^{1+1}-1}{x-1}=\frac{(x+1)(x-1)}{(x-1)}=x+1$
Result is true for $n=1$
Step 2: Assume the result true for $n=k$
$1+x+x^2+...+x^k=\frac{x^{k+1}-1}{x-1}$
To prove the result is true for $n=k+1$
$1+x+x^2+...+x^k+x^{k+1}=\frac{x^{k+2}-1}{x-1}$
LHS $=1+x+x^2+...+x^k+x^{k+1}$
$=\frac{x^{k+1}-1}{x-1}+x^{k+1}$
$=\frac{x^{k+1}-1}{x-1}+\frac{x^{k+1}(x-1)}{x-1}$
$=\frac{x^{k+1}-1+x^{k+2}-x^{k+1}}{x-1}=\frac{x^{k+2}-1}{x-1}=$ RHS
Result is true for $n=k+1$ if true for $n=k$
Step 3: Result true by principle of mathematical induction.</td><td>3 Marks: Correct answer.
2 Marks: Proves the result true for $n=1$ and attempts to use the result of $n=k$ to prove the result for $n=k+1$.
1 Mark: Proves the result true for $n=1$</td></tr>
<tr><td>15(a)
(i)</td><td>$P(x)=(x-\alpha)^2Q(x)$
$P'(x)=(x-\alpha)^2Q'(x)+2(x-\alpha)Q(x)$
$=(x-\alpha)\left[(x-\alpha)Q'(x)+2Q(x)\right]$
Therefore $P'(\alpha)=0$ and $x=\alpha$ is a root of $P'(x)$.</td><td>2 Marks: Correct answer.
1 Mark: Finds $P'(x)$</td></tr>
<tr><td>15(a)
(ii)</td><td>$P(x)=x^5-ax^2+b$ has a root $x=\alpha$
$P(\alpha)=\alpha^5-a\alpha^2+b=0$ (1)
$P'(x)=5x^4-2ax$
$P'(\alpha)=5\alpha^4-2a\alpha=0$ or $a=\frac{5}{2}\alpha^3=2.5\alpha^3$
Substituting $a=2.5\alpha^3$ into equation (1)
$\alpha^5-\frac{5}{2}\alpha^3\times\alpha^2+b=0$
$b=\frac{3}{2}\alpha^5=1.5\alpha^5$
$\therefore a=2.5\alpha^3$ and $b=1.5\alpha^5$</td><td>2 Marks: Correct answer.
1 Mark: Shows some understanding of the problem.</td></tr>
<tr><td>15(b)
(i)</td><td>Solving the equations $y=x^4+4x^3$ and $y=mx+b$ simultaneously has two solutions ($x=\alpha$ and $x=\beta$).
$\therefore x^4+4x^3-mx-b=0$ is degree 4 with multiple roots at $x=\alpha$ and $x=\beta$. Zeros are α,α,β and β.</td><td>1 Mark: Correct answer.</td></tr>
</table>

15(b) (ii)	$\alpha+\alpha+\beta+\beta=-\frac{b}{a}$ $2(\alpha+\beta)=-4$ or $\alpha+\beta=-2$ $\alpha\alpha+\alpha\beta+\alpha\beta+\alpha\beta+\alpha\beta+\beta\beta=\frac{c}{a}$ $\alpha^2+\beta^2+4\alpha\beta=0$ $(\alpha+\beta)^2+2\alpha\beta=0$ $\alpha\beta=-2$ $\alpha\alpha\beta+\alpha\alpha\beta+\alpha\beta\beta+\alpha\beta\beta=-\frac{d}{a}$ and $\alpha\alpha\beta\beta=\frac{e}{a}$ $2\alpha\beta(\alpha+\beta)=m$ and $(\alpha\beta)^2=-b$ $m=8$ and $b=-4$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Uses the relationships between the roots and coefficients.
15(c) (i)	Newton's second law: $\ddot{x}=g-kv$ $\frac{dv}{dt}=g-kv$  	1 Mark: Correct answer.
15(c) (ii)	$\frac{dv}{dt}=g-kv$ $\frac{dt}{dv}=\frac{1}{g-kv}$ $t=-\frac{1}{k}\log_e(g-kv)+C$ Initial conditions $t=0$ and $v=0$ $0=-\frac{1}{k}\log_e(g)+C$ $C=\frac{1}{k}\log_e g$ $t=-\frac{1}{k}\log_e(g-kv)+\frac{1}{k}\log_e g$ $=\frac{1}{k}\log_e\left(\frac{g}{g-kv}\right)$ $kt=\log_e\left(\frac{g}{g-kv}\right)$ $e^{kt}=\frac{g}{g-kv}$ $ge^{kt}-kve^{kt}=g$ $kve^{kt}=ge^{kt}-g$ $v=\frac{g}{k}(1-e^{-kt})$	3 Marks: Correct answer. 2 Marks: Correctly substitutes the initial conditions into the expression for t 1 Mark: Finds the correction expression for t.

15(c) (iii)	$\frac{dv}{dt}=v\frac{dv}{dx}$ $v\frac{dv}{dx}=g-kv$ $\frac{dv}{dx}=\frac{g-kv}{v}$ $\frac{dx}{dv}=\frac{v}{g-kv}$ $x=\int\frac{-\frac{1}{k}(g-kv)+\frac{g}{k}}{g-kv}dv$ $=-\frac{1}{k}v-\frac{g}{k^2}\log_e(g-kv)+C$ When $x=0$ and $v=0$ $0=-\frac{1}{k}\times 0-\frac{g}{k^2}\log_e(g-k\times 0)+C$ $C=\frac{g}{k^2}\log_e g$ $x=-\frac{1}{k}v-\frac{g}{k^2}\log_e(g-kv)+\frac{g}{k^2}\log_e g$ $=-\frac{1}{k}v-\frac{g}{k^2}\log_e\left(\frac{g}{g-kv}\right)$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Uses results for part (i) to determine an expression for $\frac{dx}{dv}$
16(a) (i)	$z^n=[\cos\theta+i\sin\theta]^n$ $=\cos n\theta+i\sin n\theta$ $\frac{1}{z^n}=[\cos\theta+i\sin\theta]^{-n}$ $=\cos n\theta-i\sin n\theta$ $z^n-\frac{1}{z^n}=\cos n\theta+i\sin n\theta-\cos n\theta+i\sin n\theta$ $=2i\sin n\theta$	2 Marks: Correct answer. 1 Mark: Uses De Moivre's theorem
16(a) (ii)	$\left(z-\frac{1}{z}\right)^5=z^5+5z^4\left(-\frac{1}{z}\right)+10z^3\left(-\frac{1}{z}\right)^2+10z^2\left(-\frac{1}{z}\right)^3$ $+5z\left(-\frac{1}{z}\right)^4+\left(-\frac{1}{z}\right)^5$ $=\left(z^5-\frac{1}{z^5}\right)-5\left(z^3-\frac{1}{z^3}\right)+10\left(z-\frac{1}{z}\right)$ $(2i\sin\theta)^5=2i\sin 5\theta-10i\sin 3\theta+20i\sin\theta$ $\sin^5\theta=\frac{1}{16}(\sin 5\theta-5\sin 3\theta+10\sin\theta)$	3 Marks: Correct answer. 2 Marks: Makes significant progress. 1 Mark: Writes the binomial expansion.

16(b)	$(a+b+c)^2 \le 3(a^2+b^2+c^2)$ $3(a^2+b^2+c^2)-(a+b+c)^2 \ge 0$ $3(a^2+b^2+c^2)-\left[a^2+2ab+2ac+b^2+2bc+c^2\right] \ge 0$ $2(a^2+b^2+c^2-ab-ac-bc) \ge 0$ $(a-b)^2+(a-c)^2+(b-c)^2 \ge 0$ True	2 Marks: Correct answer. 1 Mark: Makes some progress towards the solution.
16(c) (i)	Consider ΔBNQ and ΔAMQ. $\angle QBN = \angle QAM$ (angle between a tangent and a chord equals the angle in the alternate segment) $\angle QNB = \angle QMA = 90^\circ$ (perpendiculars from Q) $\therefore \Delta BNQ \,\vert\vert\vert\, \Delta AMQ$ (Two angles of one triangle are respectively equal to two angles of another triangle) $\Delta ALQ \,\vert\vert\vert\, \Delta BMQ$ is a similar proof. Consider ΔALQ and ΔBMQ. $\angle QAL = \angle QBM$ (angle between a tangent and a chord equals the angle in the alternate segment) $\angle QLA = \angle QMB = 90^\circ$ (perpendiculars from Q) $\therefore \Delta ALQ \,\vert\vert\vert\, \Delta BMQ$ (Two angles of one triangle are respectively equal to two angles of another triangle)	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Applies a relevant circle theorem.
16(c) (ii)	$\frac{QN}{QM} = \frac{QB}{QA}$ (matching sides in similar triangles $\Delta BNQ \,\vert\vert\vert\, \Delta AMQ$) $\frac{QM}{QL} = \frac{QB}{QA}$ (matching sides in similar triangles $\Delta ALQ \,\vert\vert\vert\, \Delta BMQ$) $\therefore \frac{QN}{QM} = \frac{QM}{QL}$ This represents a geometric sequence $QN, QM, QL, \ldots$	2 Marks: Correct answer. 1 Mark: Matches the sides in the similar triangles.
16(d)	Let $f(x) = 1 + x + \frac{x^2e^x}{2} - e^x$ $f'(x) = 1 + \frac{1}{2}(x^2e^x + e^x 2x) - e^x = 1 + xe^x + \frac{x^2e^x}{2} - e^x$ $f'(0) = 0$ $f''(x) = xe^x + e^x + \frac{1}{2}(x^2e^x + e^x 2x) - e^x$ $= 2xe^x + \frac{x^2e^x}{2} > 0$ for $x > 0$ Therefore $f'(x) > 0$ (increasing) for $x > 0$ and $f(0) = 0$ $\therefore 1 + x + \frac{x^2e^x}{2} - e^x > 0$ $\therefore 1 + x + \frac{x^2e^x}{2} > e^x$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Sets up $f(x)$ and uses calculus.