ACE Examinations

Student Name: ______________________

2014

YEAR 11

YEARLY EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time - 5 minutes
- Working time - 1.5 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in Questions 6-8

Total marks - 50

Section 1
5 marks
Attempt Questions 1-5
Allow 10 minutes for this section

Section 2
45 marks
Attempt Questions 6-8
Allow 1 hour and 20 minutes for this section

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n\neq -1;\ x\neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\ a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x>0$

Section I

5 marks
Attempt Questions 1 - 5
Allow about 10 minutes for this section

Use the multiple-choice answer sheet for Questions 1-5

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1 What is the solution to the inequality $(4-x)\geq\frac{3}{x}$?

(A) $x<0$ or $1\leq x\leq 3$

(B) $x\geq 3$ or $0<x\leq 1$

(C) $x>0$ or $-3\leq x\leq -1$

(D) $x\leq -3$ or $-1\leq x<0$

2 $P(x)$ is an odd polynomial. When $P(x)$ is divided by $(x-1)$ the remainder is 3. What is the remainder when $P(x)$ is divided by $(x+1)$?

(A) -3

(B) -1

(C) 1

(D) 3

3 The curves $y=\sqrt{x}+10$ and $y=x^2-4$ meet at the point $(4,12)$. What is the acute angle between the tangents to curves at this point?

(A) 37°

(B) 49°

(C) 69°

(D) 71°

4 What is the equation of the normal to $x=2t$, $y=t^2$ at the point $t=3$?

(A) $x+3y-21=0$

(B) $x+3y-33=0$

(C) $3x-y-9=0$

(D) $3x-y-27=0$

5 Point P is due south of a tower and the angle of elevation from P to the top of the tower is 40°. Another point Q is a bearing 190° from the tower and the angle of elevation from Q to the top of the tower is 43°. The distance PQ is 150 m. Which of the following are the correct expressions for OP and OQ?

(A) $OP=h\tan 40^\circ$ and $OQ=h\tan 43^\circ$

(B) $OP=150\tan 40^\circ$ and $OQ=150\tan 43^\circ$

(C) $OP=\frac{\tan 40^\circ}{h}$ and $OQ=\frac{\tan 43^\circ}{h}$

(D) $OP=\frac{h}{\tan 40^\circ}$ and $OQ=\frac{h}{\tan 43^\circ}$

Section II

45 marks
Attempt Questions 6 – 8
Allow about 1 hour and 20 minutes for this section

Answer each question in the appropriate writing booklet.

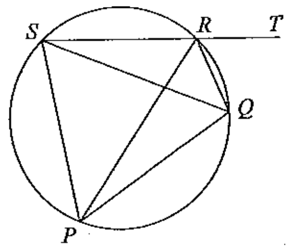
All necessary working should be shown in every question.

Question 6 (15 marks) **Marks**

(a) Find the acute angle between the lines $y=1-3x$ and $2x-3y-4=0$. Answer to the nearest degree. **2**

(b) Solve $\sin\theta+6\cos^2\theta=5$ for $0^\circ\le\theta\le360^\circ$. **2**

(c)



$PQRS$ is a cyclic quadrilateral. SR is produced to T and $\angle PRS=\angle QRT$.

(i) Explain why $\angle PQS=\angle PRS$. **1**

(ii) Hence show that $PS=QS$. **3**

(d) Find the number of ways in which a committee of 3 people can be chosen from 4 teachers and 5 students.

(i) Without restriction. **1**

(ii) Exactly one student. **1**

(iii) Exactly one teacher. **1**

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(e) $A(-4,1)$ and $B(2,4)$ are divided internally by C in the ratio 2:1 and externally by D in the ratio 2:1. $P(x,y)$ is a variable point that moves so that $PA=2PB$.

(i) Find the coordinates of C. **1**

(ii) Find the coordinates of D. **1**

(iii) Show that the locus of P is a circle. **2**

Question 7 (15 marks) **Marks**

(a) Show that the exact value of $\tan 67.5^\circ$ is $1+\sqrt{2}$. **2**

(b) Solve the inequality $\frac{3-2x}{x-3} \le 4$. **3**

(c) The polynomial $x^3 - kx - 2 = 0$ has roots α, α and β.

(i) Find the value of $2\alpha + \beta$. **1**

(ii) Find the value of $\alpha^2\beta$. **1**

(iii) What is the value of k? **2**

(d) Prove that $y = mx + b$ is a tangent to $x^2 = 4ay$ if $am^2 + b = 0$. **3**

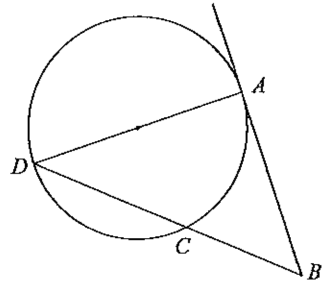
(e) (i) State the remainder theorem for polynomials. **1**

(ii) Find a if $x^2 + 4x + 5a$ is divided by x and the remainder is -10. **1**

(iii) Find b if $x - 1$ is a factor of $2x^3 + x^2 + 2x + b$. **1**

Question 8 (15 marks) **Marks**

(a) AD is a diameter of the circle. A tangent at A meets a chord DC produced to B. Prove that $AB^2 = BD \times BC$ **3**



(b) Use the t-method to solve the equation $\sin\theta + 2\cos\theta + 2 = 0$ for $0 \le \theta \le 360$. **3**

(c) The point $P(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$ with a focus S. The line SP is produced to point R such that P is the midpoint of SR.

(i) Show that R has the coordinates of $(4ap, 2ap^2 - a)$ **1**

(ii) Hence show that the locus of R as P moves on the parabola is $x^2 = 8a(y+a)$. Find the focal length and vertex of this parabola. **3**

(d) Express $P(x) = x^4 - 3x^3 - 2x^2 - 5x$ in the form $P(x) = x(x-4)Q(x) + R(x)$. **2**

(e) (i) Express $\sqrt{3}\sin 2\theta + \cos 2\theta$ in the form $A\sin(2\theta + \alpha)$ for $0 \le \theta \le 360^\circ$ **2**

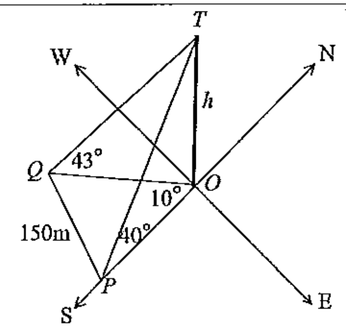
(ii) Hence or otherwise find the maximum value of $\sqrt{3}\sin 2\theta + \cos 2\theta$. **1**

End of paper

ACE Examination 2014

Year 11 Mathematics Extension 1 Yearly Examination

Worked solutions and marking guidelines

Section I		
	Solution	**Criteria**
1	$(4-x)\ge\frac{3}{x}$ $\quad x\ne 0$ $x^2\times(4-x)\ge\frac{3}{x}\times x^2$ $x^2(4-x)\ge 3x$ $x^2(4-x)-3x\ge 0$ $x[x(4-x)-3]\ge 0$ $x(x^2-4x+3)\le 0$ $x(x-3)(x-1)\le 0$ Critical points are 0, 1 and 3 Test values in each region $x<0$ and $1\le x\le 3$	1 Mark: A
2	Remainder theorem $P(1)=3$ Odd function $P(-x)=-P(x)$ Therefore $P(-1)=-P(1)=-3$	1 Mark: A
3	$y=\sqrt{x}+10$ $\quad y=x^2-4$ $\frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$ $\quad \frac{dy}{dx}=2x$ At (4,12) then $m_1=\frac{1}{4}$ and $m_2=8$ $\tan\theta=\left\lvert\frac{m_1-m_2}{1+m_1m_2}\right\rvert$ $=\left\lvert\frac{\frac{1}{4}-8}{1+\frac{1}{4}\times 8}\right\rvert$ $=\frac{31}{12}$ $\theta=68.8387401...\approx 69^\circ$	1 Mark: C
4	Parabola is $x^2=4y$ To find the gradient of the tangent $y=\frac{1}{4}x^2$ and $\frac{dy}{dx}=\frac{1}{2}x$ At $(6,9)$ $\frac{dy}{dx}=\frac{1}{2}\times 6=3$ Gradient of the normal is $-\frac{1}{3}$ Equation of the normal at $(6,9)$ $y-y_1=m(x-x_1)$ $y-9=-\frac{1}{3}(x-6)$ $3y-27=-x+6$ $x+3y-33=0$	1 Mark: B
5	 In ΔTOP $\quad$ In ΔTOQ $\tan 40^\circ=\frac{h}{OA}$ $\quad \tan 43^\circ=\frac{h}{OQ}$ $OP=\frac{h}{\tan 40^\circ}$ $\quad OQ=\frac{h}{\tan 43^\circ}$	1 Mark: D

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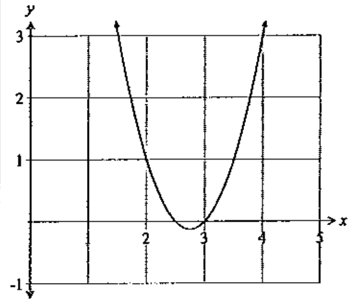
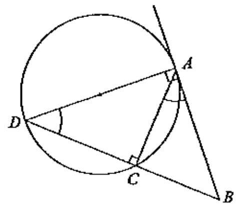
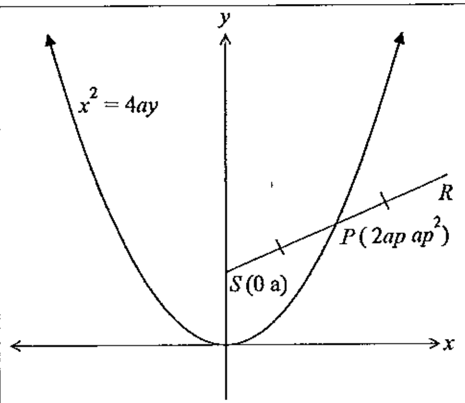
Section II		
6(a)	For $y=1-3x$ then $m_1=-3$ For $2x-3y-4=0$ then $m_2=\frac{2}{3}$ $\tan\theta=\left\|\frac{m_1-m_2}{1+m_1m_2}\right\|$ $=\left\|\frac{-3-\frac{2}{3}}{1+-3\times\frac{2}{3}}\right\|=\left\|\frac{-\frac{11}{3}}{-1}\right\|=\frac{11}{3}$ $\theta=74.7448813..\approx 75^\circ$	2 Marks: Correct answer. 1 Mark: Uses the angle between two lines formula and one correct gradient.
6(b)	$\sin\theta+6\cos^2\theta=5$ $\sin\theta+6(1-\sin^2\theta)=5$ $6\sin^2\theta-\sin\theta-1=0$ $(2\sin\theta-1)(3\sin\theta+1)=0$ $2\sin\theta=1$ or $3\sin\theta=-1$ $\sin\theta=\frac{1}{2}$ (1st & 2nd quad) $\sin\theta=-\frac{1}{3}$ (3rd & 4th quad) $\theta=30^\circ,150^\circ$ $\theta=199^\circ,341^\circ$	2 Marks: Correct answer. 1 Mark: Finds and factorises the quadratic equation.
6(c) (i)	$\angle PQS=\angle PRS$ (angles in the same segment standing on the same arc are equal).	1 Mark: Correct answer.
6(c) (ii)	$\angle QRT=\angle QPS$ (exterior angle of a cyclic quadrilateral is equal to the interior opposite angle). $\angle PRS=\angle QRT$ (given). Hence $\angle QPS=\angle PRS$ (both equal to $\angle QRT$) $\angle QPS=\angle PQS$ (from part (i)) $\therefore \Delta PSQ$ is an isosceles triangle (two angles are equal) $\therefore PS=QS$ (sides opposite the equal angles in ΔPSQ)	3 Marks: Correct answer. 2 Marks: Makes significant progress. 1 Mark: States the exterior angle of a cyclic quadrilateral theorem.
6(d) (i)	Unordered selection of 3 from 9. ${}^9C_3=84$	1 Mark: Correct answer.
6(d) (ii)	Unordered selection of 2 teachers from 4. Unordered selection of 1 student from 5. ${}^4C_2\times{}^5C_1=30$	1 Mark: Correct answer.
6(d) (iii)	Unordered selection of 1 teacher from 4. Unordered selection of 2 students from 5. ${}^4C_1\times{}^5C_2=40$	1 Mark: Correct answer.

6(e) (i)	$x=\frac{mx_2+nx_1}{m+n}$ $y=\frac{my_2+ny_1}{m+n}$ $=\frac{2\times2+1\times-4}{2+1}=0$ $=\frac{2\times4+1\times1}{2+1}=3$ The coordinates of point are C are $(0,3)$	1 Mark: Correct answer.
6(e) (ii)	$x=\frac{mx_2+nx_1}{m+n}$ $y=\frac{my_2+ny_1}{m+n}$ $=\frac{-2\times2+1\times-4}{-2+1}=8$ $=\frac{-2\times4+1\times1}{-2+1}=7$ The coordinates of point are D are $(8,7)$	1 Mark: Correct answer.
6(e) (iii)	$PA=2PB$ $\sqrt{(x--4)^2+(y-1)^2}=2\times\sqrt{(x-2)^2+(y-4)^2}$ $(x+4)^2+(y-1)^2=4\left[(x-2)^2+(y-4)^2\right]$ $x^2+8x+16+y^2-2y+1=4\left[x^2-4x+4+y^2-8y+16\right]$ $x^2+8x+y^2-2y+17=4x^2-16x+4y^2-32y+80$ $3x^2-24x-3y^2-30y+63=0$ $x^2-8x-y^2-10y+21=0$ $(x-4)^2-16-(y-5)^2-25+21=0$ $(x-4)^2-(y-5)^2=20$ $(x-4)^2-(y-5)^2=4\sqrt{5}$ Locus of P is a circle with centre (4,5) and radius $4\sqrt{5}$	2 Marks: Correct answer. 1 Mark: Uses distance formula correctly and makes some progress towards the solution.
7(a)	Let $\theta=67.5^\circ$ $\tan2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$ $\tan135^\circ=\frac{2\tan67.5^\circ}{1-\tan^2 67.5^\circ}$ $-1=\frac{2\tan67.5^\circ}{1-\tan^2 67.5^\circ}$ $-1+\tan^2 67.5=2\tan67.5^\circ$ $\tan^2 67.5-2\tan67.5^\circ-1=0$ $\tan67.5^\circ=\frac{2\pm\sqrt{2^2-4\times1\times-1}}{2}$ $=\frac{2\pm\sqrt{8}}{2}=\frac{2\pm2\sqrt{2}}{2}$ $=1\pm\sqrt{2}$ $=1+\sqrt{2}$ (67.5° is acute)	2 Marks: Correct answer. 1 Mark: Uses the double angle formula or shows some understanding.

7(b)	$(x-3)^2\times\frac{3-2x}{x-3}\le 4\times(x-3)^2 \quad x\ne 3$ $(x-3)(3-2x)\le 4(x-3)^2$ $(x-3)(3-2x)-4(x-3)^2\le 0$ $(x-3)[(3-2x)-4(x-3)]\le 0$ $(x-3)(15-6x)\le 0$ $3(x-3)(2x-5)\ge 0$ Critical points are $2\frac{1}{2}$ and 3 $(x\ne 3)$  Test values in each region Solution is $x\le 2\frac{1}{2}$ or $x>3$	3 Marks: Correct answer. 2 Marks: Finds one correct region or makes significant progress. 1 Mark: Multiplies both sides of the inequality by $(x-3)^2$ or finds a critical point.
7(c) (i)	$\alpha+\alpha+\beta=-\frac{b}{a}=\frac{0}{1}$ $2\alpha+\beta=0 \quad (1)$	1 Mark: Correct answer.
7(c) (ii)	$\alpha\beta\gamma=-\frac{d}{a}=-\frac{-2}{1}=2$ $\alpha^2\beta=2 \quad (2)$	1 Mark: Correct answer.
7(c) (iii)	From eqn(1) $\beta=-2\alpha$ and substitute into eqn(2) $\alpha^2\times -2\alpha=2$ $\alpha^3=-1$ or $\alpha=-1$ Substituting $\alpha=-1$ into eqn(2) then $\beta=2$ $\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}=\frac{-k}{1}$ $\alpha^2+2\alpha\beta=-k$ $(-1)^2+2\times -1\times 2=-k$ $k=3$	2 Marks: Correct answer. 1 Mark: Shows some understanding of the problem.

7(d)	A tangent to a parabola intersects at one point. Substitute $mx+b$ for y into $x^2=4ay$ $x^2=4a(mx+b)$ $x^2-4amx-4ab=0$ Quadratic equation has one solution if $\Delta=0$ $\Delta=b^2-4ac$ $=(-4am)^2-4\times 1\times -4ab$ $=16a(am^2+b)=0$ Therefore $am^2+b=0$ $(a\ne 0$ otherwise no parabola)	3 Marks: Correct answer. 2 Marks: Makes significant progress. 1 Mark: Solves the equations simultaneously.
7(e) (i)	The remainder theorem states that when a $P(x)$ is divided by $(x-a)$ the remainder is $P(a)$.	1 Mark: Correct answer.
7(e) (ii)	$P(x)=x^2+4x+5a$ $P(0)=0^2+4\times 0+5a=-10$ ($(x-0)$ has a remainder of -10) $a=-2$	1 Mark: Correct answer.
7(e) (iii)	$P(x)=2x^3+x^2+2x+b$ $P(1)=2\times 1^3+1^2+2\times 1+b=0$ ($(x-1)$ is a factor of $P(x)$) $2+1+2+b=0$ $b=-5$	1 Mark: Correct answer.
8(a) (i)	 $\angle ACD=90^\circ$ (angle in a semi-circle) $\angle BCA+\angle ACD=180^\circ$ (straight angle) $\angle BCA+90^\circ=180^\circ$ or $\angle BCA=90^\circ$ $\angle DAB=90^\circ$ (tangent to a circle is perpendicular to the radius drawn to the point of contact) Consider ΔABD and ΔABC $\angle DAB=\angle BCA$ (from above) $\angle CDA=\angle CAB$ (angle between a tangent and a chord is equal to the angle in the alternate segment) ΔABD is similar to ΔABC (equiangular) $\frac{AB}{BD}=\frac{BC}{AB}$ $AB^2=BD\times BC$	3 Marks: Correct answer. 2 Marks: Makes significant progress. 1 Mark: Applies a relevant circle theorem.

8(b)	$\sin\theta+2\cos\theta+2=0$ $\sin\theta+2\cos\theta+2=\frac{2t}{1+t^2}+2\times\frac{1-t^2}{1+t^2}+2$ $=\frac{2t+2(1-t^2)+2(1+t^2)}{1+t^2}$ $=\frac{2t+2-2t^2+2+2t^2}{1+t^2}$ $=\frac{2(t+2)}{1+t^2}=0$ Hence $t=-2$ $\tan\frac{\theta}{2}=-2$ $\frac{\theta}{2}=116.5652\ldots$ or $296.656\ldots$ $\theta=233.130\ldots=233°8'$ Testing boundaries $\theta=180°$ is a solution. There $\theta=180°$ and $233°8'$	3 Marks: Correct answer. 2 Marks: Finds $\theta=233°8'$ or makes significant progress. 1 Mark: Uses 't' results correctly or recognises that 180° is a solution.
8(c) (i)	 Let R be the coordinates (x,y). Use the midpoint formula. $x=\frac{x_1+x_2}{2}$ $\quad y=\frac{y_1+y_2}{2}$ $2ap=\frac{0+x}{2}$ $\quad ap^2=\frac{a+y}{2}$ $x=4ap$ $\quad y=2ap^2-a$	1 Mark: Correct answer.

8(c) (ii)	Eliminate p to find the locus of R. $y=2ap^2-a$ $=2a\times\left(\frac{x}{4a}\right)^2-a$ $=\frac{x^2}{8a}-a$ $\frac{x^2}{8a}=y+a$ $x^2=8a(y+a)$ Parabola with a focal length of $2a$, and vertex $(0,-a)$	3 Marks: Correct answer. 2 Marks: Makes significant progress. 1 Mark: Eliminates p or finds the focal length and vertex of the given parabola.
8(d)	$x-4\overline{)x^4-3x^3-2x^2-5x}$ quotient x^3+x^2+2x x^4-4x^3 $1x^3-2x^2$ $1x^3-4x^2$ $2x^2-5x$ $2x^2-8x$ $3x$ $P(x)=(x-4)(x^3+x^2+2x)+3x$ $=x(x-4)(x^2+x+2)+3x$	2 Marks: Correct answer. 1 Mark: Completes a division showing some understanding.
8(e) (i)	$\sqrt{3}\sin 2\theta+\cos 2\theta$ $A\sin 2\theta\cos\alpha+A\cos 2\theta\sin\alpha=A\sin(2\theta+\alpha)$ Hence $A\cos\alpha=\sqrt{3}$ and $A\sin\alpha=1$ Dividing these equations $\tan\alpha=\frac{1}{\sqrt{3}}$ $\alpha=30°$ Squaring and adding the equations $A^2=(\sqrt{3})^2+1^2$ $A=2$ $\sqrt{3}\sin 2\theta+\cos 2\theta=2\sin(\theta+30°)$	2 Marks: Correct answer. 1 Mark: Finds α or A. Alternatively shows some understanding of the problem.
8(e) (ii)	Maximum value of the sine function is 1. Maximum value of $2\sin(\theta+30°)$ is 2. Maximum value of $\sqrt{3}\sin 2\theta+\cos 2\theta$ is 2.	1 Mark: Correct answer.