Student name/number: ____________________

Advanced Mathematical Publications

(Place your crest here)

2001

PRELIMINARY FINAL EXAMINATION

Mathematics

Total marks **(84)**

- Attempt Questions 1 – 7
- All questions are of equal value

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board-approved calculators may be used
- All necessary working should be shown in every question

Question 1 (12 marks) **Start a NEW page** **Marks**

(a) Factorise fully $32y^2 - 2$. **1**

(b) Simplify $\sqrt{32} + \sqrt{50} - \sqrt{8}$ **2**

(c) Express $0.1\dot{8}$ as a simple fraction. **2**

(d) Solve $\frac{8-x}{4} < 5$ **2**

(e) Solve $2 + 5x + x^2 = 0$ expressing the roots in surd form **2**

(f) Rationalise $\frac{3}{3+\sqrt{2}}$ **1**

(g) Calculate correct to 3 significant figures the value of **2**

$4\pi\sqrt{\frac{m}{g}}$ if $m = 15.2$ and $g = 9.8$

Question 2 (12 marks) **Start a NEW page** **Marks**

(a) A function $y = f(x)$ is given by the rule

$$f(x) = \begin{cases} x^2 & \text{if } -1 \le x < 1 \\ 2 - x & \text{if } 1 \le x \le 3 \\ -1 & \text{if } 3 \le x \le 5. \end{cases}$$

(i) Graph the function. **2**

(ii) From the graph or otherwise, find the range and domain of the function. **2**

(iii)Find the value of $f\left(\frac{1}{2}\right) + f(4)$. **1**

(b) Find the largest possible domain for the function **1**

$y = \sqrt{2x - 3}$

(c) Find the exact value of $\sin 45^0 . \cos 30^0$ **1**

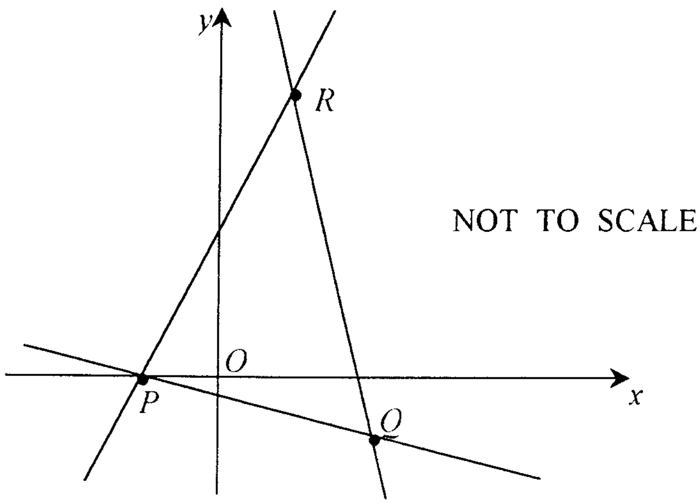
(d) If $\sec\theta = 2$ and $0^0 \le \theta \le 360^0$, **2**

Find the value(s) of θ.

(e) Prove the identity $\dfrac{(\cos\theta + \sin\theta)^2}{\sin\theta\cos\theta} = 2 + \operatorname{cosec}\theta\sec\theta$ **3**

Question 3 (12 marks) **Start a NEW page** **Marks**

The points $P(-1, 0)$, $Q(2, -1)$ and $R(1, 6)$ are shown on the diagram below.



Copy this diagram

(a) Show that the gradient of PR is 3. **1**

(b) Show that the length of PR is $2\sqrt{10}$ units. **1**

(c) Show that the equation of PQ is $x - 3y - 1 = 0$. **2**

(d) Point M is the midpoint of PR. Show that the coordinates of M are $(0, 3)$. Mark point M on your diagram. **1**

(e) Show that PR and PQ are perpendicular **1**

(f) Find the area of $\triangle PMQ$ **2**

(g) On your diagram draw the line l which passes through M and is perpendicular to PR. Label the intersection between l and RQ the point N. Find the equation of the line l. **2**

(h) What type of quadrilateral is $PMNQ$. Give reasons. **2**

Question 4 (12 marks) **Start a NEW page** **Marks**

(a) If $f(x) = 5x^2 - 2x + 2$ and $g(x) = 3x + 2$ **2**
then for what values of x is $f(x) = g(x)$?

(b) Find the equation of the tangent to the curve $y = \sqrt{x+1}$ at the point $(3,2)$. **3**

(c) Differentiate the following **4**

(i) $y = (x^3 + 5)^4$

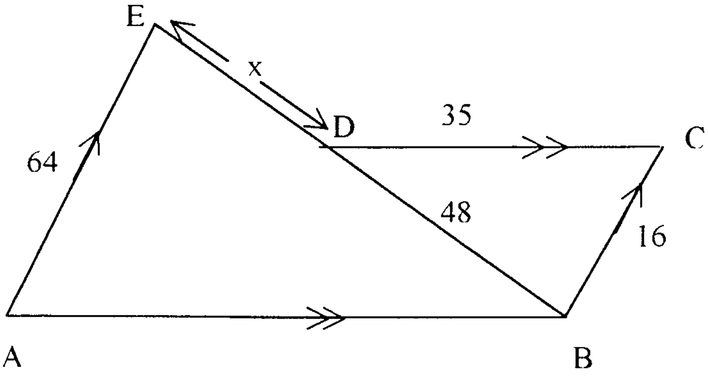
(ii) $y = \dfrac{x+1}{x-2}$

(d) Find the values of c for which the function **3**

$y = x^2 - 5x + c$ is positive definite.

Question 5 (12 marks) **Start a NEW page** **Marks**

(a) Using the diagram below:



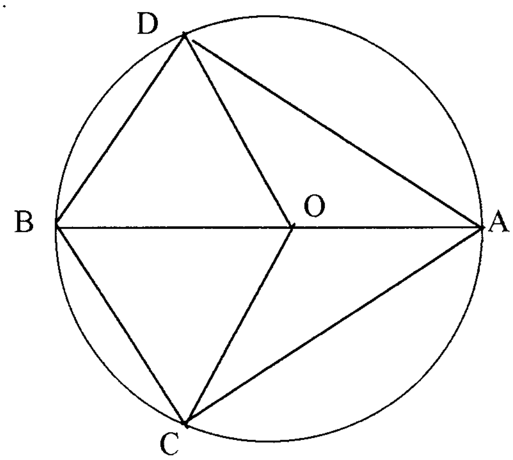
Given that AE || BC, and DC || AB.

(i) Prove $\Delta ABE ||| \Delta CDB$. **3**

(ii) Hence, calculate the length of ED. **2**

(b) Find the gradient of the normal to the curve $y = \frac{1}{x^2}$ at the point where $x = 2$. **2**

(c) AOB is the diameter of the given circle, centre O. **5**

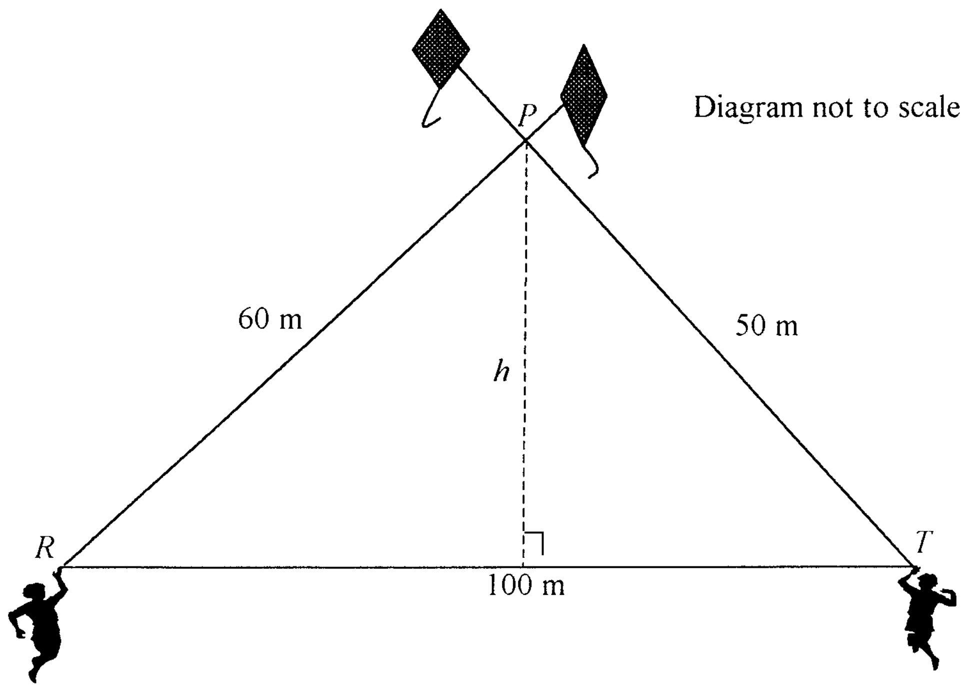


(i) If BC = BD, prove that $\Delta COB \equiv \Delta ODB$.

(ii) Hence prove AC = AD.

Question 6 (12 marks) **Start a NEW page** **Marks**

(a) Ramish and Tran have their kites tangled at point P in a kite flying exhibition. Point P is 60 metres from Ramish at R and 50 metres from Tran at T. Ramish and Tran are 100 metres apart.



(i) Show that the angle (to the nearest degree) between the two tangled kites is 131^0. **3**

(ii) Find the height h (to the nearest metre). **3**

(b) (i) Shade the region bounded by the inequations **3**

$$y \geq 0,\ x + y < 1,\ y < 4x + 6$$

neatly on the same number plane.

(ii) Find the area of the shaded region in part (i). **1**

(c) Find the values of p and q if $3x^2 - 12x + 7 \equiv 3(x+p)^2 + q$ **2**

Question 7 (12 marks) **Start a NEW page** **Marks**

(a) Solve $x^2 - 2x \leq 8$ **2**

(b) Solve the equation $4^x - 2^x - 56 = 0$ **3**

(c) For what values of k does the quadratic equation $(k + 2)x^2 - 2kx + 1 = 0$ have real roots ? **3**

(d) Given the parabola $x^2 - 6x + 3y + 6 = 0$ find the **4**

(i) coordinates of the vertex
(ii) focal length
(iii) equation of the directrix

End of examination

Question 1 (12 Marks)

(a) $2(16y^2-1)$
$=2(4y+1)(4y-1)$ ✓

(b) $4\sqrt{2}+5\sqrt{2}-2\sqrt{2}$ ✓
$=7\sqrt{2}$ ✓

(c) Let $x=0.1888...$
$10x=1.888....$ (1)
$100x=18.888....$ (2)
(2) – (1) $90x=17$ ✓
$\therefore\ x=\dfrac{17}{90}$ ✓

(d) $\dfrac{8-x}{4}<5$
$\therefore\ 8-x<20$ ✓
$\therefore\ -x<12$
$\therefore\ x>-12$ ✓

(e) $x=\dfrac{-5\pm\sqrt{5^2-4\times1\times2}}{2\times1}$ ✓

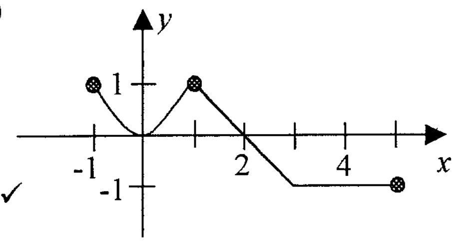
$x=\dfrac{-5\pm\sqrt{17}}{2}$ ✓

(f) $\dfrac{3}{3+\sqrt{2}}\times\dfrac{3-\sqrt{2}}{3-\sqrt{2}}$
$=\dfrac{9-3\sqrt{2}}{5}$ ✓

(g) $4\times\pi\sqrt{\dfrac{15.2}{9.8}}$
$=15.65015417$ ✓
$=15.7$ (to 3 s.f.) ✓

Question 2 (12 Marks)

(a) (i)



✓✓

(ii) $D: -1\le x\le 5$ ✓
$R: -1\le y\le 1$ ✓

(iii) $f\left(\dfrac{1}{2}\right)+f(4)=\left(\dfrac{1}{2}\right)^2-1$
$=-\dfrac{3}{4}$ ✓

(b) $2x-3\ge 0$
$x\ge\dfrac{3}{2}$ ✓

(c) $\dfrac{1}{\sqrt{2}}\times\dfrac{\sqrt{3}}{2}=\dfrac{\sqrt{3}}{2\sqrt{2}}$ or $\dfrac{\sqrt{6}}{4}$ ✓

(d) $\cos\theta=\dfrac{1}{2}$
$\therefore\ \theta=60^0, 300^0$ ✓✓

(e) L.H.S. $=\dfrac{\cos^2\theta+\sin^2\theta+2\cos\theta\sin\theta}{\sin\theta\cos\theta}$

$=\dfrac{1+2\sin\theta\cos\theta}{\sin\theta\cos\theta}$ $\{\because\ \cos^2\theta+\sin^2\theta=1\}$ ✓

$=\dfrac{1}{\sin\theta\cos\theta}+\dfrac{2\sin\theta\cos\theta}{\sin\theta\cos\theta}$ ✓

$\left(\because\ \text{cosec}\theta=\dfrac{1}{\sin\theta};\sec\theta=\dfrac{1}{\cos\theta}\right)$ ✓

$=\text{cosec}\theta\sec\theta+2$ = R.H.S.

Question 3 (12 Marks)

(a) $P(-1, 0)$ and $R(1, 6)$

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$m = \dfrac{6-0}{1+1}$

$m = 3$ ✓

(b) $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$PR = \sqrt{(1+1)^2 + (6-0)^2}$

$PR = \sqrt{40}$

$PR = 2\sqrt{10}$ ✓

(c) $P(-1, 0), Q(2, -1)$

$\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$

$\dfrac{y-0}{x+1} = \dfrac{-1-0}{2+1}$ ✓

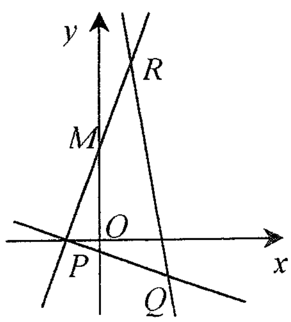
$3y = -x - 1$

$x + 3y + 1 = 0$ ✓

(d) $M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

$M = \left(\dfrac{-1+1}{2}, \dfrac{0+6}{2}\right)$

$M = (0, 3)$ ✓



(e) PQ ; $x + 3y + 1 = 0.$

$m = -\frac{a}{b}$ $\therefore m_{pq} = -\frac{1}{3}$

but $m_{pr} = 3$

$\therefore m_{pq} \times m_{pr} = -1$

hence PR and PQ are perpendicular. ✓

(f) $PQ = \sqrt{(2+1)^2 + (-1-0)^2}$

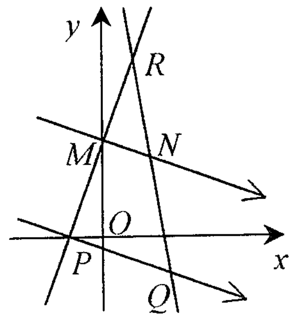
$PQ = \sqrt{10}$

$PM = \frac{1}{2} \times 2\sqrt{10} = \sqrt{10}$ ✓

Area $\triangle PMQ = \frac{1}{2} \times \sqrt{10} \times \sqrt{10}$

$= 5$ units^2 ✓

(g)



$m_{pq} = m_{mn} = -\frac{1}{3}$ ✓

equation of the line l

$y - y_1 = m(x - x_1)$

$y - 3 = -\frac{1}{3}(x - 0)$

$3y - 9 = -x$

$\therefore x + 3y - 9 = 0$ ✓

(h) $PMNQ$ is a trapezium ✓

proof; $\angle MPQ = 90°$

(PR and PQ are perpendicular, proven above in (e))

$\angle PMN = 90°$ (MN constructed perpendicular to PR)

$\therefore MN$ is parallel to PQ

(co-interior angles are supplementary)

$\therefore PMNQ$ is a trapezium as it has one pair of opposite sides parallel ✓

Question 4 (12 Marks)

(a) $5x^2 - 2x + 2 = 3x + 2$

$5x^2 - 5x = 0$ ✓

$5x(x-1) = 0$

$x = 0$ or $x = 1$ ✓

(b) $\frac{dy}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}}$ ✓

at (3,2) $\frac{dy}{dx} = \frac{1}{2}(3+1)^{-\frac{1}{2}} = \frac{1}{4}$ ✓

equation of tangent

$y - 2 = \frac{1}{4}(x-3)$

$4y - 8 = x - 3$

$x - 4y + 5 = 0$ ✓

(c) (i) $\frac{dy}{dx} = 4(x^3+5)^3 \times 3x^2$ ✓

$= 12x^2(x^3+5)$ ✓

(ii) $\frac{dy}{dx} = \frac{(x-2).1-(x+1).1}{(x-2)^2}$ ✓

$= \frac{x-2-x-1}{(x-2)^2}$

$= \frac{-3}{(x-2)^2}$ ✓

(d) For positive definite $\Delta < 0$ and $a > 0$.

$a = 1 > 0$ ✓

$\therefore \ \Delta = b^2 - 4ac$

$= 25 - 4 \times 1 \times c$ ✓

$\therefore \ 25 - 4c < 0$

$\therefore \ c > 6\frac{1}{4}$ or $\frac{25}{4}$ ✓

Question 5 (12 Marks)

(a) (i) In ΔABE and ΔBCD

$\angle AEB = \angle DBC$ (alternate $\angle$s, AE || BC)

$\angle CDB = \angle ABE$ (alternate $\angle$s, CD || AB) ✓

$\angle EAB = \angle DCB$ (Angle sum of Δ) ✓

$\therefore \Delta ABE \ ||| \ \Delta CDB$ (equiangular) ✓

(ii) $\frac{x+48}{48} = \frac{64}{16} = 4$ ✓

$x + 48 = 192$

$x = 144$ ✓

(b) $\frac{dy}{dx} = \frac{-2}{x^3}$

when $x = 2$, $\frac{dy}{dx} = -\frac{2}{8} = -\frac{1}{4}$ ✓

$\therefore$ gradient of the normal is 4 ✓

(c) (i) In ΔCOB, ΔDOB

BC=BD (given)

OD=OC (radii) ✓

OB is common

$\therefore \Delta COB \equiv \Delta ODB$ (S.S.S test) ✓

(ii) $\angle BOC = \angle BOD$ (corresp. $\angle$s in congruent Δs)

$\therefore \angle COA = \angle DOA$ ($\angle$s on straight line) ✓

$\therefore$ in ΔCOA, ΔDOA

$\angle COA = \angle DOA$ (proven)

OC=OD (radii)

OA is common

$\therefore \Delta COA \equiv \Delta DOA$ (SAS) ✓

$\therefore$ AC=AD (corresp. sides in congruent Δs) ✓

Question 6 (12 Marks)

(a) (i) Using Cosine rule in ΔPTR

$100^2 = 50^2 + 60^2 - 2 \times 50 \times 60 \times \cos \angle P$ ✓

$\therefore \ \cos \angle P = \frac{2500 + 3600 - 10000}{6000}$ ✓

$= -0.65$

$\therefore \ \angle P = \cos^{-1}(-0.65) \approx 131^0$ as required. ✓

(ii) In ΔPTR, using the sine rule

$$\frac{\sin\angle T}{60}=\frac{\sin\angle P}{100} \checkmark$$

$$\sin\angle T=\frac{60\times\sin\angle P}{100}$$

$$\therefore\ \angle P=27^0 8' \checkmark$$

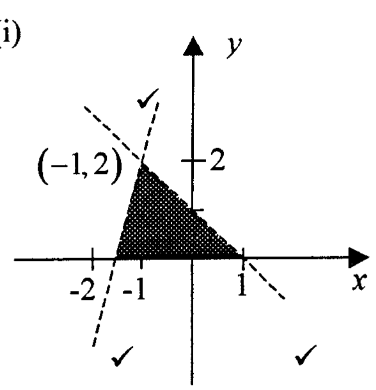
In ΔPST, using sine ratio

$$\sin 27^0 8'=\frac{h}{50}$$

$$\therefore\ h=50\sin 27^0 8'$$

$= 23$ m (to the nearest metre). ✓

(b) (i)



(ii) $A=\frac{1}{2}bh$

$=\frac{1}{2}\times\frac{5}{2}\times 2$

$=2\frac{1}{2}$ sq. units ✓

(c) L.H.S. $=3\left(x^2-4x+4\right)+17-12$

$=3\left(x-2\right)^2+5$ ✓

$=3\left(x+p\right)^2+q$

$\therefore\ p=-2,\ q=5$ ✓

Question 7 (12 Marks)

(a)
$$x^2-2x\le 8$$
$$x^2-2x-8\le 0$$
$$(x-4)(x+2)\le 0 \checkmark$$
$$\therefore -2\le x\le 4 \checkmark$$

(b)
$$4^x-2^x-56=0$$
$$2^{2x}-2^x-56=0$$
let $m=2^x$
$$\therefore\ m^2-m-56=0$$
$$(m+7)(m-8)=0 \checkmark$$
$\therefore\ m=-7$ or $m=8$ but $m=2^x$

$\therefore 2^x=-7$ or $2^x=8$ ✓

No Soln $\quad x=3$

$\therefore$ solution is $x=3$ ✓

(c)
$$b^2-4ac\ge 0$$
$$(2k)^2-4(k+2).1\ge 0 \checkmark$$
$$4k^2-4k+8\ge 0$$
$$k^2-k+2\ge 0 \checkmark$$
$$(k+1)(k-2)\ge 0$$
$$\therefore\ k\le -1 \text{ or } k\ge 2 \checkmark$$

(d) (i)
$$x^2-6x+3y+6=0$$
$$x^2-6x+9+3y+6=9$$
$$x^2-6x+9=-3y+3$$
$$(x-3)^2=-3(y-1) \checkmark$$
$\therefore$ vertex is (3, 1) ✓

(ii) focal length $4a=3$

$a=\frac{3}{4}$ ✓

(iii) directrix $y=p+a$ (concave down, vertex (p, q))

$y=1\frac{3}{4}$ ✓

End of examination