19/5/04

Applications of Series

Formulae for Arithmetic Series

1. A series $T_1 + T_2 + T_3 + \ldots + T_n + \ldots$ is an arithmetic series (arithmetic progress – AP) if
$$T_n - T_{n-1} = d \text{ for all } n > 1$$
where d is a constant, called the common difference

2. The n^{th} term of an arithmetic series is
$$T_n = a + (n-1)d$$
where a is the first term (ie $T_1 = a$).

3. Three terms, T_1, T_2, T_3 are in arithmetic progression if $T_2 - T_1 = T_3 - T_2$ $(= d)$

4. The arithmetic mean of 2 numbers, a and b, is $\frac{a+b}{2}$.

5. The sum to n terms of an arithmetic series is given by
$$S_n = \frac{n}{2}(a + l)$$
$a = T_1$ (first term)
$l = T_n$ (last term)

OR
$$S_n = \frac{n}{2}(2a + (n-1)d)$$
d = common difference

Formulae for Geometric Series

1. A series $T_1 + T_2 + T_3 + \ldots + T_n + \ldots$ is a geometric series if $\frac{T_n}{T_{n-1}} = r$ for $n > 1$.
where r is a constant, called the common ratio.

2. The n^{th} term of a geometric series is given by
$$T_n = ar^{n-1}$$ where a is the first term (ie $T_1 = a$)

3. Three terms T_1, T_2, T_3 are in geometric progression if $\frac{T_2}{T_1} = \frac{T_3}{T_2} (= r)$

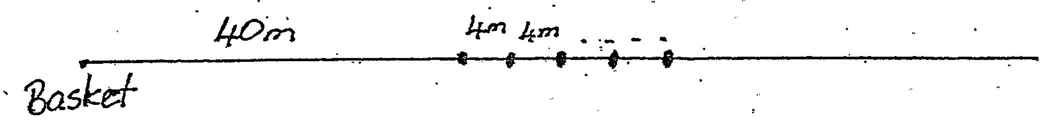
4. The geometric mean of x and y is $\sqrt{xy}$ or $-\sqrt{xy}$.

5. The sum of n terms of a geometric series is
$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{if } r<1$$
OR
$$S_n = \frac{a(r^n-1)}{r-1} \quad \text{if } r>1$$

6. The limiting sum, S_∞ of a geometric series exists only if $-1 < r < 1$ and
$$S_\infty = \frac{a}{1-r}$$

<u>Examples</u>: (1) In a potato race, a runner has to collect 12 potatoes one at a time and place them in a basket. The potatoes and basket are placed in a straight line with the potatoes being 4m apart and the basket being 40 m from the nearest potato.

If the runner starts from the basket, calculate the distance travelled when the last potato has been placed in the basket.



Let $T_1, T_2, \ldots, T_{12}$ be the distances run from the basket to each potato and return.

$$T_1 = 2 \times 40 = 80$$
$$T_2 = 2 \times 44 = 88$$
$$T_3 = 2 \times 48 = 96$$
$$\vdots$$
$$T_{12} = 2 \times \qquad =$$

(2) A games manufacturer agreed to an unusual wag agreement with one of the workers. For the first day the payment is 1c, the 2nd day 2c, the 3rd day 4c, and so on, the amount doubling each day.

Calculate

(a) the worker's pay on the 20^{th} day

(b) the amount earned by the worker in a year if he works 4 days each week for 48 weeks.

YEAR 11 ASSIGNMENT.

1. Which term of the arithmetic series $3+16+29+\cdots$ is 120?

2. Calculate the first term and the common difference for the arithmetic sequence where $T_3 = 19$ and $T_6 = 7$.

3. Why does the series $1+\frac{1}{4}+\frac{1}{16}+\cdots$ have a limiting sum? Find it.

4. Prove that this is a geometric series.
$\sqrt{2}+2+2\sqrt{2}+\cdots$
Find the tenth term.

5. Find the value of p so that $p+5$, $4p+3$, $8p-2$ will form successive terms of an arithmetic sequence.

6. Find the sum of all integers between 200 and 400 that are divisible by 6.

7. Evaluate: a) $\sum_{n=4}^{9}(5n-4)$ (b) $\sum_{n=1}^{\infty} 27 \times \left(\frac{1}{3}\right)^{n-1}$

8. The height of a tree was 10 metres and it increas by 2 metres during the next year. If in each succeeding year the growth is $\frac{2}{3}$ of that in the previous year, find the limiting height?

9. The sum of n terms of a sequence is given by $S_n = n^2+3n$. Find the first 3 terms and show that it is an arithmetic sequence.

10. How many terms of the series $23+19+15+\cdots$ must b added to give a sum of 50?

11. Find the first term of the sequence $\frac{1}{2}, \frac{3}{2}, \frac{9}{2}\cdots$ which is greater than 300.

YR 11 ASSIGNMENT

1) AP : $3+16+29+\dots$

$\therefore a=3,\ d=13$ ✓

$\therefore$ AP is $a+(n-1)d$

ie. $3+13(n-1)$

ie. $13n-10$

when $T_n=120$,

$13n-10=120$ ✓

ie. $n=10$

$\therefore 120$ is the 10^{th} term. ✓

2) $T_3=19$, $T_6=7$

ie. $a+2d=19$ —— ①

ie. $a+5d=7$ —— ② ✓

② − ① : $3d=-12$

$\therefore d=-4$ ✓

Sub $d=-4$ in ①: $a=27$ ✓

$\therefore d=-4,\ a=27$.

$\therefore$ first term of AP is 27, common difference is -4. ✓

3) limiting sum occurs only if $-1<r<1$ ✓

For the series $1+\frac{1}{4}+\frac{1}{16}+\dots$

$\frac{1}{16}\div\frac{1}{4}=\frac{1}{4}\div 1=r$

$\therefore r=\frac{1}{4}$ ✓

$\therefore -1<r<1$

$\therefore$ GP has limiting sum, S_∞

$S_\infty=\frac{a}{1-r}\quad (a=1,\ r=\frac{1}{4})$

$\therefore S_\infty=\frac{1}{3/4}=\frac{4}{3}=1\frac{1}{3}$ ✓

$\therefore$ Limiting sum is $1\frac{1}{3}$.

4) $\sqrt{2}+2+2\sqrt{2}+\dots$

$\frac{2\sqrt{2}}{\cancel{2}}=\frac{2}{\sqrt{2}}=r$ ✓

$\therefore r=\sqrt{2}$

$\therefore$ Geometric series with $a=\sqrt{2}$, $r=\sqrt{2}$ ✓

ar^{n-1}

ie. $\sqrt{2}\times(\sqrt{2})^{n-1}$

ie. $\sqrt{2}^{\,n}$

$\therefore T_{10}=(\sqrt{2})^{10}$

$=32$ ✓

$\therefore$ Tenth term is 32. ✓

5) $p+5,\ 4p+3,\ 8p-2$

for an AP: $(8p-2)-(4p+3)=(4p+3)-(p+5)$

$(8p-2)-(4p+3)=4p-5$ —— ①

$(4p+3)-(p+5)=3p-2$ —— ②

① = ②

$\therefore 4p-5=3p-2$

$\therefore p=3$ ✓

$4(3)+3=15$

when $p=3$, AP is $8,\ \cancel{35},\ \cancel{62}$ $8(3)-2=22$

$\therefore$ common difference is $\cancel{27}$

$\therefore$ sequence is an AP, when $p=3$

6) $6, 12, 18, 24, \dots$

AP : $a+(n-1)d$

ie. $6+6(n-1)$

ie. $6n$

when $6n<400$

$n<66\frac{2}{3}$

$\therefore n=66$

Too long! Try $\frac{200}{6}\approx 33.33$

$\therefore 34\times 6=204$

$\frac{400}{6}\approx 66.6$

$\therefore 66\times 6=396$ *

$\therefore$ there are 66 multiples of 6 which are less than 400.

when $6n<200$

Unnecessarily long!

$n<33\frac{1}{3}$

$\therefore n=33$

$\therefore$ there are 33 mutiples of 6 less than 200.

$\therefore$ no. of multiples bet. 200 and 400 is $66-33=33$

$\therefore n=33$. ✓

* $\therefore S_n=\frac{n}{2}(a+l)$ $[a=204,\ l=396,\ n=33]$

$\therefore S_n=\frac{33}{2}(204+396)=9900$ ✓

$\therefore$ sum of all ...

200 & 400 that are divisible by 6 is 9900.

7a). $\sum_{n=4}^{9}(5n-4) = 16+21+26+31+36+41$

$= 171$ ✓

b) $\sum_{n=1}^{\infty} 27\times\left(\frac{1}{3}\right)^{n-1}$

$ar^{n-1} \quad (a=27, r=\frac{1}{3})$

$S_\infty = \frac{a}{1-r} = \frac{27}{2/3} = 40\frac{1}{2}$

$\therefore \sum_{n=1}^{\infty} 27\times\left(\frac{1}{3}\right)^{n-1} = 40\frac{1}{2}.$ ✓

8) $10+2+2\times\frac{2}{3}+2\times\left(\frac{2}{3}\right)^2+\ldots$

$\therefore 10 +$ GP with $a=2,\ r=\frac{2}{3}$

$\therefore S_\infty = 10+\frac{a}{1-r}$ ✓

$= 10+\frac{2}{1-2/3} = 10+\frac{2}{\frac{1}{3}}$

$= 16$ ✓

$\therefore$ The limiting height of the tree is 16m

9). $S_n = n^2+3n$

$T_n = S_n - S_{n-1}$

$= n^2+3n-\left[(n-1)^2+3(n-1)\right]$

$= n^2+3n-(n^2-2n+1+3n-3)$

$= \cancel{n^2}+\cancel{3n}-\cancel{n^2}+2n-1-\cancel{3n}+3$ ✓

$= 2n+2$

$\therefore T_1, T_2, T_3 = 4, 6, 8$ ✓

$\therefore$ Arithmetic sequence with $a=4,\ d=2.$ ✓

10). $23+19+15+\ldots$

$\therefore$ AP with $a=23,\ d=-4.$

$S_n = 50$

$S_n = \frac{n}{2}(2a+(n-1)d)$

ie. $50 = \frac{n}{2}(2\times23-4(n-1))$

$50 = \frac{n}{2}(46-4n+4)$

$50 = \frac{n}{2}(50-4n)$

$100 = n(50-4n)$ ✓

$100 = 50n-4n^2$

$\therefore 4n^2-50n+100=0$

$\therefore 2n^2-25n+50=0$ ✓

$\therefore 2n^2-20n-5n+50=0$

$\therefore 2n(n-10)-5(n-10)=0$

$\therefore (2n-5)(n-10)=0$

$\therefore n = 10, \cancel{\frac{5}{2}}$

$\therefore$ 10 terms are needed to give a sum of 50. ✓

11) $\frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \ldots$

$\therefore$ GP with $a=\frac{1}{2},\ r=3.$

ar^{n-1}

ie. $\frac{1}{2}\times3^{(n-1)} = T_n$

when $T_n > 300$,

$\frac{1}{2}\times3^{(n-1)} > 300$ ✓

ie. $3^{(n-1)} > 600$

$\therefore (n-1)\log_{10}3 > \log_{10}600$

$\therefore n-1 > \frac{\log_{10}600}{\log_{10}3}$ ✓

$\therefore n > \frac{\log_{10}600}{\log_{10}3}+1$

$\therefore n > 6.82273\ldots$

$\therefore n = 7$ ✓

Sub $n=7$ in $\frac{1}{2}\times3^{(n-1)}$

ie. $\frac{1}{2}\times3^6$

$= 364\frac{1}{2}$

$\therefore$ First term in sequence which is greater than 300 is $364\frac{1}{2}$. ✓