Centre Number

Student Number

SCEGGS Darlinghurst

2007
Higher School Certificate
Assessment Task 2

Mathematics-Extension I

Task Weighting: 35%

Outcomes Assessed: HE3, HE4, HE6 & HE7

General Instructions

- Time allowed – 60 minutes
- **Start each question on a new page.**
- Attempt **all** questions and show all necessary working.
- Answer **Question 1 (c)** on the answer sheet provided
- Write your student number at the top of each page.
- Marks can be deducted for careless or badly arranged work
- Mathematical templates, geometrical equipment and scientific calculators may be used.

Question	Reasoning	Communication	Total
1	/2	/4	/12
2	/7	/1	/13
3	/3	/3	/13
4	/7	/1	/12
Total	/19	/9	/50

Average:_________ St. Dev.:_________ Rank:______

Parent's Signature____________________

Start A New Page

Marks

Question 1: (12 marks)

(a) Find:

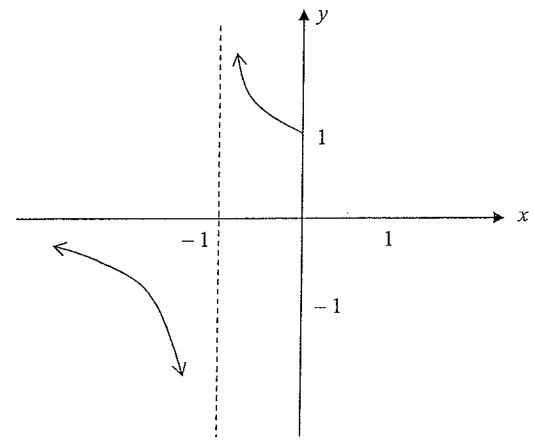
(i) $\int \frac{dx}{\sqrt{16-x^2}}$ **2**

(ii) $\int x(x-2)^5\,dx$ using the substitution $u = x-2$ **2**

(b) Evaluate:

$$\int_{-1/4}^{1/2} \frac{dx}{\sqrt{1-4x^2}}$$ **2**

(c) The graph of $y = f(x)$ is drawn below: **2**



On the answer sheet provided sketch $y = f^{-1}(x)$

Question 1 continued on the next page

Marks

Question 1 (continued)

(d) The half life of a radioactive substance is the time it takes a given amount of the substance to lose one half of its mass. It is given that the half-life of plutonium-239 is 24 000 years. **2**

Assume that plutonium-239 decays according to the law:

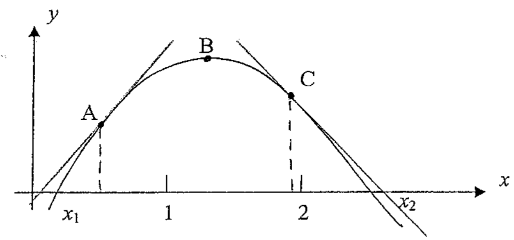
$M = M_0 e^{-kt}$ where M = mass

t = time in years

M_0 and k are constants

Find how long it would take for an amount of plutonium-239 to lose 80% of its mass. (Answer to nearest year)

(e) Consider the graph of $y = f(x)$



Siobhan wants to calculate the value of x_1. It is the root of the equation $f(x) = 0$ between 0 and 1. She uses Newton's Method.

(i) Explain, graphically, why the x – value of A is a better choice than the x – value of C as a first approximation of x_1. **1**

(ii) Explain what would happen if Siobhan used the x – value of B. **1**

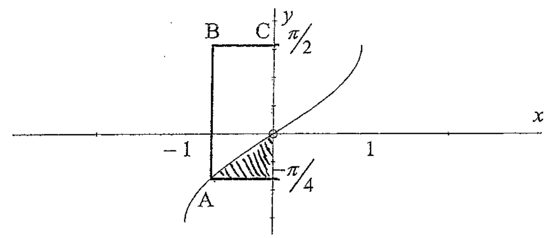
Start A New Page

Marks

Question 2: (13 marks)

(a) $x^4 - 10x + 7 = 0$ has a root between 0.6 and 0.9. Use halving the interval method twice to show the root lies between 0.675 and 0.75. **2**

(b) The diagram shows a sketch of the function $y = \sin^{-1} x$



(i) What are the coordinates of B? **1**

(ii) Show the area of the shaded region is $\frac{2-\sqrt{2}}{2}$ units^2. **3**

(iii) Hence calculate the area bounded by $y = \sin^{-1} x$, the y – axis and the intervals AB and BC **1**

(c) An ice cube tray is filled with water at a temperature of $18^0 C$ and placed in a freezer that has a constant temperature of $-19^0 C$. The cooling rate of the water is proportional to the difference between the temperature of the freezer and the temperature of the water T.

T satisfies the equation $\frac{dT}{dt} = k(T + 19)$

(i) Show that $T = -19 + Ae^{kt}$ satisfies the equation for $\frac{dT}{dt}$ and find the value of A. **2**

(ii) After 5 minutes in the freezer the temperature of the water $3^0 C$. Find the time for the water to reach $-18.9^0 C$. **3**

(iii) Sketch a graph of Temperature versus Time labelling all important features. **1**

Start A New Page

Marks

Question 3: (13 marks)

(a) (i) Show that the equation $e^x = x+2$ has a solution in the interval $1<x<2$. 1

(ii) Letting $x_1 = 1.5$ use one application of Newton's Method to approximate the solution to 3 decimal places. 3

(b) By using the substitution $x = \tan\theta$ evaluate $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dx}{\left(1+x^2\right)^{3/2}}$ 3

(c) Consider the function $y = \cos^{-1}(x-1)$.

(i) Find the domain of the function. 1

(ii) Sketch the graph of the curve $y = f(x)$ showing clearly the coordinates of the endpoints. 2

(iii) The region in the first quadrant bounded by the curve $y = f(x)$ and the coordinate axes is rotated about the y – axis. Find the exact value of the volume of the solid or revolution. 3

Start A New Page

Marks

Question 4: (12 marks)

(a) Find the exact value of $\sin\left[\tan^{-1}\left(\frac{-1}{2}\right)+\cos^{-1}\left(\frac{2}{3}\right)\right]$ 3

(b) (i) Show that $\frac{u}{u+1} = 1 - \frac{1}{u+1}$ 1

(ii) Hence find $\int \frac{dx}{1+\sqrt{x}}$ using the substitution $x = u^2$ 2

(c) $y = \sinh x$ is an example of a hyperbolic function. It is defined as:

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

(i) Find $\frac{dy}{dx}$. 1

(ii) Explain why $y = \sinh x$ has an inverse function 1

(iii) Show that $f^{-1}(x) = \ln\left(x+\sqrt{x^2+1}\right)$ for all x. 4

END OF PAPER

Mathematics Extension 1 - Assessment Task 2 HSC 2007 - Solutions

/2 a) i) $\int \frac{dx}{\sqrt{16-x^2}} = \sin^{-1}\frac{x}{4} + c$ Calc - 2

/2 ii) $\int x(x-2)^5 dx$ let $u = x-2$, $du = 1.dx$

$= \int (u+2)u^5 du$ ✓

$= \int u^6 + 2u^5 du$

$= \frac{u^7}{7} + \frac{u^6}{3} + c$

$= \frac{(x-2)^7}{7} + \frac{(x-2)^6}{3} + c$ ✓ Calc - 2

Remember to change back to x!

/2 b) $\int_{-1/4}^{1/2} \frac{dx}{\sqrt{1-4x^2}}$

$= \frac{1}{2}\int_{-1/4}^{1/2} \frac{dx}{\sqrt{\frac{1}{4}-x^2}}$

$= \frac{1}{2}\left[\sin^{-1}2x\right]_{-\frac{1}{4}}^{1/2}$ ✓

$= \frac{1}{2}\left(\sin^{-1}1 - \sin^{-1}\left(-\frac{1}{2}\right)\right)$

$= \frac{1}{2}\left(\frac{\pi}{2} - -\frac{\pi}{6}\right)$

$= \frac{\pi}{3}$ ✓ Calc - 2

/2 c) Refer to answer sheet ✓✓

Communication - 2

Intersection must be on the line $y = x$.

/2 d) find k if $M = \frac{M_0}{2}$ when $t = 24\,000$

$\therefore \frac{M_0}{2} = M_0 e^{k \times 24000}$

$\frac{1}{2} = e^{k \times 24000}$

$\ln\frac{1}{2} = 24000 \times k$

$k = \frac{\ln(\frac{1}{2})}{24000}$ ✓

$= -2.89 \times 10^{-5}$ (3 sig fig)

find t when $M = 0.2M_0$

$\therefore 0.2M_0 = M_0 e^{kt}$

$0.2 = e^{kt}$ Reasoning - 2

$\ln(0.2) = kt$

$t = \frac{\ln(0.2)}{k}$

$= \frac{\ln(0.2)}{\ln(\frac{1}{2})} \times 24000$

$= 55\,726$ (to nearest whole no.) ✓

$\therefore$ It will take 55 726 years to lose 80% of mass

note: $M \neq 0.8M_0$!!

/1 e) i) the tangent at A cuts the x-axis closer to x_1 than the tangent at C. ✓

Communication - 1

/1 ii) the gradient of the tangent is zero

$\therefore f'(x) = 0$ $\therefore \frac{f(x)}{f'(x)}$ is undefined and

Newton's Method does not work. ✓

or

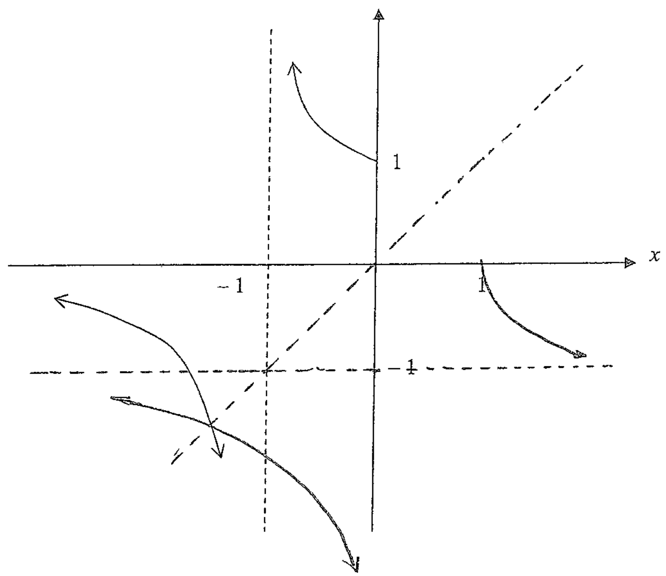
The tangent does not cut the x-axis.

Communication - 1

draw a diagram must mention the tangent and that the new approximation is where the tangent crosses the x axis.

Start A New Page

Answer Sheet for Question 1 (c)



Q2 a) $f(x) = x^4 - 10x + 7$

$\frac{}{2}$ $f(0.6) = 1.1296$

$f(0.9) = -1.3439$

$\frac{0.6+0.9}{2} = 0.75$

$f(0.75) = -0.18359...$ ✓

$\therefore$ root lies between 0.6 and 0.75

$\frac{0.6+0.75}{2} = 0.675$

$f(0.675) = 0.45759....$ ✓

$\therefore$ root lies between 0.675 and 0.75

To draw conclusions you must state the value of $f(0.6)$ & $f(0.9)$

because it is a 'show' question you must state the value of $f(0.75)$ & $f(0.675)$

$\frac{}{1}$ b) i) $B\left(-\frac{1}{\sqrt{2}}, \frac{\pi}{2}\right)$ ✓

done well

$\frac{}{3}$ ii) $y = \sin^{-1} x$

$x = \sin y$

$$\text{Area} = \left|\int_{-\frac{\pi}{4}}^{0} \sin y \, dy\right| \checkmark$$

$$= \left|\left[-\cos y\right]_{-\frac{\pi}{4}}^{0}\right|$$

$$= \left|(-\cos 0) - \left(-\cos\left(-\tfrac{\pi}{4}\right)\right)\right| \checkmark$$

$$= \left|-1 - \left(-\tfrac{1}{\sqrt{2}}\right)\right|$$

$$= \left|\tfrac{1}{\sqrt{2}} - 1\right|$$

$$= \left|\frac{1-\sqrt{2}}{\sqrt{2}}\right|$$

Reasoning − 3

$$= \left|\frac{\sqrt{2}-2}{2}\right| \checkmark$$

$$= \frac{2-\sqrt{2}}{2} \text{ units}^2$$

some interesting statements here!

because it is a show question you must be particular about how you find this area

eg's why do these work:

· $\int_0^{\pi/4} \sin y \, dy$

· $-\int_0^{\pi/4} \sin y \, dy$.

etc

$\frac{}{1}$ iii) $\text{Area} = \frac{1}{\sqrt{2}} \times \frac{3\pi}{4} - \left(\frac{2-\sqrt{2}}{2}\right)$ ✓ Reasoning − 1

$$= \frac{3\sqrt{2}\pi}{8} - \left(\frac{2-\sqrt{2}}{2}\right)$$

$= \frac{1}{8}\left(3\sqrt{2}\pi + 4\sqrt{2} - 8\right)$ units^2

/2 c) i) $T = -19 + Ae^{kt}$

$\frac{dT}{dt} = kAe^{kt}$

$\therefore$ LHS $= \frac{dT}{dt}$ $\quad$ RHS $= k(T+19)$

$= kAe^{kt}$ $\quad$ $= k\left(-19 + Ae^{kt} + 19\right)$ ✓

$= kAe^{kt}$

$\therefore$ LHS = RHS

$\therefore$ $T = -19 + Ae^{kt}$ satisfies the equation

when $t=0$ $T=18$

$18 = -19 + Ae^{k\times 0}$

$37 = Ae^{0}$

$A = 37$ ✓

done well

/3 ii) find k: $T=3$ $t=5$

$\therefore$ $3 = -19 + 37e^{k\times 5}$

$22 = 37e^{k\times 5}$

$\frac{22}{37} = e^{k\times 5}$

$\ln\left(\frac{22}{37}\right) = k\times 5$

$k = \frac{1}{5}\ln\left(\frac{22}{37}\right)$ ✓ $(-0.1039\ldots)$

$\therefore$ find t when $T = -18.9$

$-18.9 = -19 + 37e^{kt}$ ✓

$0.1 = 37e^{k\times t}$

$\frac{0.1}{37} = e^{kt}$

$\ln\left(\frac{0.1}{37}\right) = kt$ $\quad$ Reasoning - 3

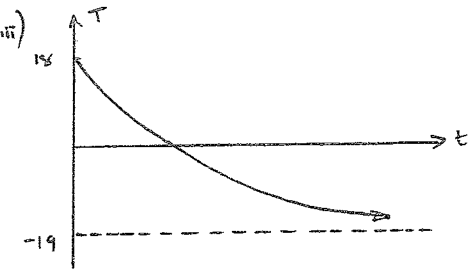
$t = \dfrac{\ln\left(\frac{0.1}{37}\right)}{k}$

$= 56.87\ldots$

$\therefore$ 57 minutes ✓

done well by nearly all candidates. Errors were made by using incorrect values for T & t

/1 iii)



✓ Communication - 1

Q3 a) i) $e^x = x+2 \rightarrow e^x - x - 2 = 0$

/1 Let $f(x) = e^x - x - 2$

$f(1) = e^1 - 1 - 2$ $\quad$ $f(2) = e^2 - 2 - 2$

$= -0.281\ldots < 0$ $\quad$ $= 3.389\ldots > 0$ ✓

since $f(1) < 0$ and $f(2) > 0$ and $f(x)$ is continuous then $f(x)$ has a root between $x=1$ and $x=2$.

Communication - 1

curve must be continuous

/3 ii) $f'(x) = e^x - 1$

$x_2 = 1.5 - \frac{f(1.5)}{f'(1.5)}$ ✓

$= 1.5 - \frac{e^{1.5} - 1.5 - 2}{e^{1.5} - 1}$ ✓

$= 1.218$ (to 3 dec. pl.) ✓

(correct rounding)

/3 b) $\displaystyle\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{(1+x^2)^{3/2}}$ $\quad$ $x = \tan\theta$ $\quad$ $x=\sqrt{3}$ $\theta = \frac{\pi}{3}$

$dx = \sec^2\theta\, d\theta$ $\quad$ $x = \frac{1}{\sqrt{3}}$ $\theta = \frac{\pi}{6}$

$= \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^{3/2}}$ ✓

$= \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^{3/2}}$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{d\theta}{\sec\theta}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos\theta \, d\theta \quad ✓$$

$$= \sin\theta \Big]_{\pi/6}^{\pi/3}$$

$$= \sin\frac{\pi}{3} - \sin\frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2}$$

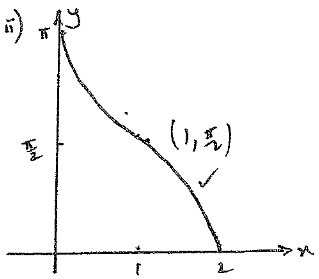
$$= \frac{\sqrt{3}-1}{2} \quad ✓$$

Calc – 3

/1 c) i) $-1 \le x-1 \le 1$

$0 \le x \le 2$ ✓

/2 ii)



1 – for graph

1 – for labelling

Communication – 2

/3 iii) $y = \cos^{-1}(x-1)$

$\cos y = x - 1$

$x = \cos y + 1$

$$V = \pi \int_0^{\pi} (\cos y + 1)^2 \, dy \quad ✓$$

$$= \pi \int_0^{\pi} \cos^2 y + 2\cos y + 1 \, dy$$

now $\cos 2y = 2\cos^2 y - 1$

$\cos^2 y = \frac{1}{2}(1 + \cos 2y)$

$$= \pi \int_0^{\pi} \frac{1}{2}\cos 2y + 2\cos y + \frac{3}{2} \, dy$$

$$= \pi \left[\frac{1}{4}\sin 2y + 2\sin y + \frac{3y}{2} \right]_0^{\pi} \quad ✓$$

$$= \pi \left[\left(\frac{1}{4}\sin 2\pi + 2\sin\pi + \frac{3}{2}\pi \right) - (0+0+0) \right]$$

$$= \pi \times \frac{3\pi}{2}$$

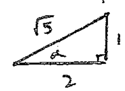
$$= \frac{3\pi^2}{2} \text{ units}^3 \quad ✓$$

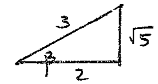
Reasoning – 3

Q4 a) /3 let $\alpha = \tan^{-1}\left(-\frac{1}{2}\right)$ $\qquad \beta = \cos^{-1}\left(\frac{2}{3}\right)$

$\tan\alpha = -\frac{1}{2}$ ✓ $\qquad \cos\beta = \frac{2}{3}$

$\therefore \alpha$ in 4th quad.





$\therefore \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ ✓

$$= \frac{-1}{\sqrt{5}} \times \frac{2}{3} + \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{3}$$

$$= \frac{-2}{3\sqrt{5}} + \frac{2}{3}$$

$$= \frac{-2\sqrt{5}}{15} + \frac{2}{3}$$

$$= \frac{10 - 2\sqrt{5}}{15} \quad ✓$$

Reasoning – 3

Done well and most students recognised α was in 4th quad.

/1 b) i) RHS $= 1 - \frac{1}{u+1}$

$$= \frac{u+1-1}{u+1} \quad ✓$$

$$= \frac{u}{u+1}$$

/2 ii) $\displaystyle\int \frac{dx}{1+\sqrt{x}}$ $\qquad x = u^2$, $dx = 2u\,du$

$$= \int \frac{2u \, du}{1+u} \quad ✓$$

$$= 2\int \frac{u \, du}{1+u}$$

$$= 2\int 1 - \frac{1}{u+1} \, du$$

Calc – 2

$$= 2\left(u - \ln|u+1|\right) + c$$

You must remember to replace u with $\sqrt{x}$ after integrating.

$= 2\sqrt{x} - 2\ln(1+\sqrt{x}) + c$ ✓

1/1 c) i) $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}(e^x - e^{-x})\right)$

$= \frac{1}{2}(e^x + e^{-x})$ ✓

Done well

1/1 ii) since $e^x > 0$ and $e^{-x} > 0$ for all x

$\frac{dy}{dx} > 0$ for all x

$\therefore y = \sinh x$ is a monotonic increasing function

$\therefore y = \sinh x$ has an inverse fn. ✓

Communication-1

Very poor reasoning here. You must link $\frac{dy}{dx} > 0$ with monotonic increasing

4/4 iii) $y = \frac{1}{2}(e^x - e^{-x})$

interchange x and y

$x = \frac{1}{2}(e^y - e^{-y})$ ✓

$2x = e^y - \frac{1}{e^y}$

$2xe^y = e^{2y} - 1$

$e^{2y} - 2xe^y - 1 = 0$ ✓

let $m = e^y$

$m^2 - 2xm - 1 = 0$

$m = \frac{2x \pm \sqrt{(2x)^2 - 4.1.-1}}{2}$

$= \frac{2x \pm \sqrt{4x^2+4}}{2}$

$= \frac{2x \pm 2\sqrt{x^2+1}}{2}$

$= x \pm \sqrt{x^2+1}$ ✓

$\therefore e^y = x \pm \sqrt{x^2+1}$

Reasoning-4

since $\sqrt{x^2+1} > x$ ✓

$e^y = x + \sqrt{x^2+1}$

$y = \ln\left(x + \sqrt{x^2+1}\right)$

Very few candidates know how to progress past $x = \frac{1}{2}(e^y - e^{-y})$