Student Number:......................................

2002
HIGHER SCHOOL CERTIFICATE
Sample Examination Paper

MATHEMATICS Extension 2

General Instructions

Reading time – 5 minutes
Working time _ 3 hours

Attempt ALL questions
All questions are of equal value.

Write your answer to each question in SEPARATE answer booklets provided.

All necessary work must be shown.

Only Board approved calculators may be used.

Question 1 (15 marks)

a) Find

i) $\displaystyle\int \frac{du}{u(\ln u)^6}$ 2

ii) $\displaystyle\int x^2 e^x dx$ 3

iii) $\displaystyle\int \frac{3x^2+1}{x(x+1)^2} dx$ 4

b) i) If $C_n = \displaystyle\int_0^{\frac{\pi}{2}} \cos^n x dx$ then show that $C_n = \dfrac{n-1}{n} C_{n-2}$. 4

ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \cos^5 dx$ 2

Question 2 (15 marks)

a) Given the complex number $z = 2\sqrt{3} - 5i$, find

i) $|z|$ 2

ii) $\arg z$ 2

iii) $\arg \overline{z}$ 2

iv) $|z^2|$ 2

b) Sketch the region in the Argand Plane consisting of those points z for which

$|\arg z| \geq \frac{\pi}{3}$ intersecting with $|z| \leq 3$ 3

c) Solve the equation

$2z^4 + 3z^3 + 5z^2 + 3z + 2 = 0$

(Hint: divide by z^2) 4

Question 3 (15 marks)

Sketch the following curves on separate axes for each part showing all intercepts and turning points.

a) $y = x^3 - 4x$ and hence $y = |x^3 - 4x|$ (in the domain: $-3 \le x \le 3$) 4

b) i) $y = 1 - 2\sin x$ (in the domain: $0 \le x \le 2\pi$) 2

ii) hence $y = |1 - 2\sin x|$ (in the domain: $0 \le x \le 2\pi$) 2

iii) hence $y = \ln|1 - 2\sin x|$ (in the domain: $0 \le x \le 2\pi$) 2

c) $y = \sqrt{4 - x^2} + 2^x$ (in the domain: $-2 \le x \le 2$) 3

(Hint: use the addition of ordinates)

d) $|y| = 1 - \dfrac{1}{x}$ (in the domain: $-3 \le x \le 3$) 2

Question 4 (15 marks)

a) i) For the ellipse $\frac{y^2}{a^2}+\frac{x^2}{b^2}=1$, state the equation of the tangents at $P\left(a\cos\theta, b\sin\theta\right)$ and at the ends of the major axes. 1

ii) Find the coordinates of the points Q and R where the tangent at P meets the two tangents at the extremities of the major axis 2

iii) Hence prove that the interval QR subtends a right angle at either focus 4

b) $P\left(4p, \frac{4}{p}\right)$ and $Q\left(4q, \frac{4}{q}\right)$ are points on the rectangular hyperbola $xy=16$.

i) Derive the equations of the tangents at P and Q.. 2

ii) The tangents at P and Q intersect at the point R.

Derive the coordinates of the point R 2

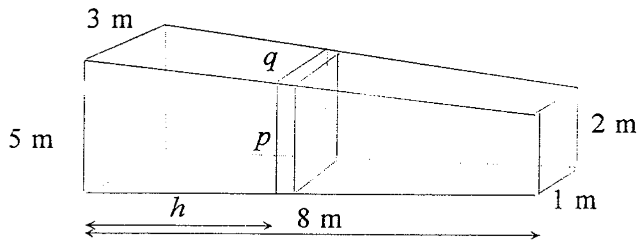
iii) If the chord PQ passes through the point $(4,0)$, derive the locus of R. 4

Question 5 (15 marks)

a) A doughnut is formed by rotating the area of the circle $(x-3)^2 + y^2 = 4$ about the y-axis.

Calculate the volume of this doughnut using cylindrical shells. 5

b) A wooden beam of length 6 metres has plane sides with cross-sections parallel to the ends being rectangular with dimensions as shown.



i) Express p and q in terms of h. 2

ii) Calculate the area of the cross-section pq. 1

iii) Hence calculate the volume of the beam.. 3

c) Six letters are chosen from the letters of the word PYTHAGORAS. These six letters are then placed alongside each other to form a six-letter arrangement. Find the number of distinct six-letter arrangements which are possible, considering all the choices. 4

Question 6 (15 marks)

a) The equation $x^3 - mx - n = 0$ has roots α, β, γ. Find

i) $\alpha^2 + \beta^2 + \gamma^2$ 2

ii) $\alpha^3 + \beta^3 + \gamma^3$ 2

b) Find the zeros of $P(x) = x^4 - 5x^3 + 7x^2 + 3x - 10$ over $\mathbb{C}$ if $2 - i$ is a zero.

Hence factorise completely over $\mathbb{C}$. 3

c) Factorise completely if $3x^4 + 8x^3 + 6x^2 - 1 = 0$ if it has a root of multiplicity 3. 3

d) i) Derive the five roots of the equation $z^5 - 1 = 0$ 2

ii) Hence find the exact value of $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5}$ 3

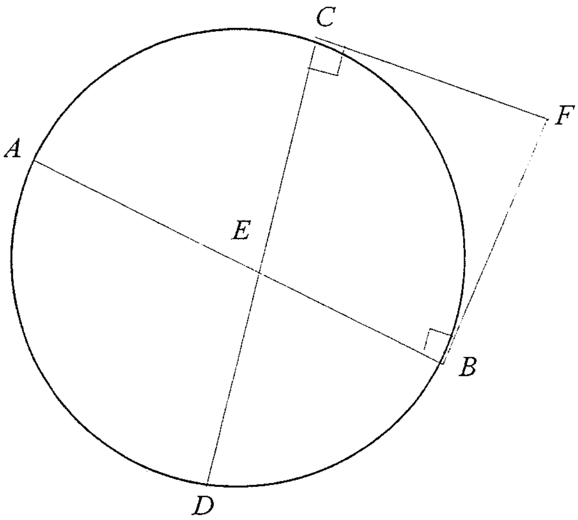
Question 7 (15 marks)

a) Find the general solution to $\sin 4x + \sin 6x = \sin 10x$ 5

b) If A, B, C are the angles of a triangle, prove that

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$ 5

c) In the figure AB and CD are two chords of the circle. AB and CD intersect at E. F is a point such that $\angle ABF$ and $\angle DCF$ are right angles.



Prove that FE produced is perpendicular to AD. 5

Question 8 (15 marks)

a) Given that $p > 0, q > 0, r > 0$, prove that $(p+2q)(2q+3r)(3r+p) > 48pqr$ 3

b) A sequence T_n is such that $T_1 = 4$ and $T_2 = 8$ and $T_{n+2} = 6T_{n+1} - 5T_n$

Prove by mathematical induction that $T_n = 5^{n-1} + 3$ 6

c) A car takes a banked curve of a racing track at speed p m/s, the lateral gradient angle ϕ being designed to reduce the tendency to side-slip to zero for a lower speed q.

Show that the coefficient of friction necessary to prevent side-slip for the greater speed p must be at least

$$\frac{(p^2 - q^2)\sin\phi\cos\phi}{p^2\sin^2\phi + q^2\cos^2\phi}$$ 6

End of Paper.

Table of Standard Integrals.

$\int x^n dx = \frac{x^{n+1}}{n+1},$	$n \neq -1; x \neq 0$ if $n < 0$
$\int \frac{1}{x} dx = \ln x,$	$x > 0$
$\int e^{ax} dx = \frac{1}{a} e^{ax},$	$a \neq 0$
$\int \cos ax dx = \frac{1}{a} \sin ax,$	$a \neq 0$
$\int \sin ax dx = -\frac{1}{a} \cos ax,$	$a \neq 0$
$\int \sec^2 ax dx = \frac{1}{a} \tan ax,$	$a \neq 0$
$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax,$	$a \neq 0$
$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a},$	$a \neq 0$
$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}$	$a > 0, -a < x < a$
$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left\{x + \sqrt{x^2 - a^2}\right\},$	$\lvert x \rvert > \lvert a \rvert$
$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left\{x + \sqrt{x^2 + a^2}\right\}$	

Note : $\ln x \equiv \log_e x$

2002 Extension 2 Trial Mathematics Examination Mapping Grid

For each item in the examination, the grid shows the marks allocated the syllabus content and syllabus outcomes to which it relates, and the bands on the performance scale it is targeting. The range on the bands shows the performance candidates may be expected to demonstrate in their responses. That is, if an item is shown as targeting Bands E3 – E4, candidates who demonstrate performance equivalent to Band E3 descriptions should be able to score some of the marks on the item, while those who perform at Band E4 or above could be expected to gain high marks. In the case of one-mark items, candidates who demonstrate performance at or above the bands shown generally could be expected to answer the item correctly.

Question	Marks	Content	Syllabus outcomes	Targeted performance bands
1(a)(i)	2	Integration	E8, HE6	E2-E3
1(a)(ii)	3	Integration	E8	E2-E3
1(a)(iii)	4	Integration	E8	E3-E4
1(b)(i)	4	Integration	E8	E3-E4
1(b)(ii)	2	Integration	E8	E3-E4
2(a)(i)	2	Complex Numbers	E3	E2
2(a)(ii)	2	Complex Numbers	E3	E2-E3
2(a)(iii)	2	Complex Numbers	E3	E2-E3
2(a)(iv)	2	Complex Numbers	E3	E3-E4
2(b)	3	Complex Numbers	E3	E3-E4
2(c)	4	Complex Numbers	E3	E3-E4
3(a)	4	Curve Sketching	E6	E2-E3
3(b)(i)	2	Curve Sketching	E6	E2-E3
3(b)(ii)	2	Curve Sketching	E6	E2-E3
3(b)(iii)	2	Curve Sketching	E6	E3-E4
3(c)	3	Curve Sketching	E6	E3-E4
3(d)	2	Curve Sketching	E6	E3-E4
4(a)(i)	1	Conics	E4	E2
4(a)(ii)	2	Conics	E4	E2-E3
4(a)(iii)	4	Conics	E4	E3-E4
4(b)(i)	2	Conics	E4	E2-E3
4(b)(ii)	2	Conics	E4	E3-E4
4(b)(iii)	4	Conics	E4	E3-E4
5(a)	5	Volumes	E7	E3-E4
5(b)(i)	2	Volumes	E7	E3-E4
5(b)(ii)	1	Volumes	E7	E3-E4
5(b)(iii)	3	Volumes	E7	E3-E4
5(c)	4	Harder Extension 1	E9, PE3	E3-E4
6(a)(i)	2	Polynomials	E4	E2-E3
6(a)(ii)	2	Polynomials	E4	E3-E4
b)	3	olynomials	E4	E3-E4
c)	3	olynomials	E4	E2-E3
d)(i)	2	olynomials, Complex Numbers	E3,E4	E2-E3
d)(ii)	3	olynomials, Complex Numbers	E3, E4	E3-E4
a)	5	arder Extension 1	E9	E3-E4
b)	5	arder Extension 1	E9, E1	E3-E4
c)	5	arder Extension 1	E9, PE3	E3-E4
a)	3	Harder Extension 1	E9, PE3	E3-E4
b)	6	arder Extension 1	E9, E1	E3-E4
c)	6	echanics	E5	E3-E4

Question 1 (15 marks)

a) i)

$$\int \frac{du}{u(\ln u)^6} = \int (\ln u)^{-6} \frac{1}{u} du \quad \mathbf{1}$$

$$= \frac{(\ln u)^{-5}}{-5} + c$$

$$= -\frac{1}{5(\ln u)^5} + c \quad \mathbf{1}$$

Alternatively,

Put $x = \ln u$

$$dx = \frac{1}{u} du$$

$$I = \int x^{-6} dx$$

$$= \frac{x^{-5}}{-5} + c$$

$$= -\frac{1}{5(\ln u)^5} + c$$

ii)

$$\int x^2 e^x dx \qquad v' = e^x \Rightarrow v = e^x \quad \mathbf{1}$$

$$u = x^2 \Rightarrow u' = 2x$$

$$= x^2 e^x - \int 2xe^x dx \quad \mathbf{1}$$

$$= x^2 e^x - 2xe^x + \int 2e^x dx$$

$$= x^2 e^x - 2xe^x + 2e^x + c \quad \mathbf{1}$$

iii)

$$\int \frac{3x^2+1}{x(x+1)^2} dx = \int \left(\frac{a}{x} + \frac{b}{x+1} + \frac{c}{(x+1)^2} \right) dx \quad \mathbf{1}$$

$$3x^2 + 1 = a(x+1)^2 + bx(x+1) + cx$$

Put $x = 0$ $\qquad a = 1$

Put $x = -1$ $\qquad 3 + 1 = -c$ $\qquad \therefore c = -4$ $\quad$ **1**

Put $x = 1$ $\qquad 3 + 1 = 4 + 2b - 4$ $\qquad \therefore b = 2$

$$I = \int \left(\frac{1}{x} + \frac{2}{x+1} - \frac{4}{(x+1)^2} \right) dx \quad \mathbf{1}$$

$$= \ln x + 2\ln(x+1) + \frac{4}{x+1} + c \quad \mathbf{1}$$

b) i)

$$C_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos x\,dx \qquad \text{Let } u = \cos^{n-1} x, \quad v' = \cos x \quad \therefore v = \sin x$$

$$= \left[\sin x \cos^{n-1} x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x (n-1)\cos^{n-2} x(-\sin x)\,dx$$ 1

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x\,dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x\left(1-\cos^2 x\right)dx$$ 1

$$= \sin x\cos^{n-1} x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$ 1

$$= \sin x\cos^{n-1} x + (n-1)C_{n-2} - (n-1)C_n$$

$$nC_n = (n-1)C_{n-2}$$

$$C_n = \frac{n-1}{n}C_{n-2}$$ 1

ii)

$$C_5 = \frac{4}{5}C_3$$ 1

$$= \frac{4}{5}\left(\frac{2}{3}C_1\right)$$

$$= \frac{8}{15}\int_0^{\frac{\pi}{2}} \cos x\,dx$$

$$= \frac{8}{15}\left[\sin x\right]_0^{\frac{\pi}{2}}$$ 1

$$= \frac{8}{15}$$

Question 2 (15 marks)

a) $z = 2\sqrt{3} - 5i$

i) $|z| = \sqrt{12+25}$ 1

$= \sqrt{37}$ 1

ii) $\arg z = \tan^{-1}\frac{-5}{2\sqrt{3}}$ 1

$= -55^0 17'$ 1

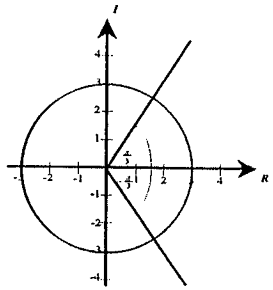
iii) $\arg \overline{z} = \tan^{-1}\frac{5}{2\sqrt{3}}$ 1

$= 55^0 17'$ 1

iv) $z^2 = 37 - 20i$ 1

$|z^2| = \sqrt{1369+400} = \sqrt{1769}$ 1

b)



3

c)

$2z^2 + 3z + 5 + 3z^{-1} + 2z^{-2} = 0$ 1

$2\left(z^2 + z^{-2}\right) + 3\left(z + z^{-1}\right) + 5 = 0$

Let $u = z + z^{-1}$

$u^2 = \left(z + z^{-1}\right)^2$

$= z^2 + z^{-2} + 2$

$2\left(u^2 - 2\right) + 3u + 5 = 0$ 1

$2u^2 + 3u + 1 = 0$

$(2u+1)(u+1) = 0$

$2z + 2z^{-1} + 1 = 0$ or $z + z^{-1} + 1 = 0$ 1

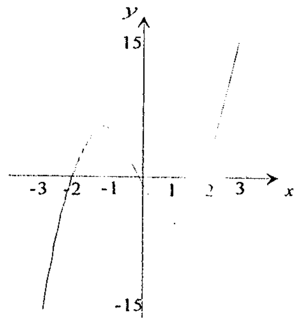
$2z^2 + z + 2 = 0$ or $z^2 + z + 1 = 0$

$z = \dfrac{-1 \pm \sqrt{1-16}}{4}$ or $z = \dfrac{-1 \pm \sqrt{1-4}}{2}$ 1

$z = \dfrac{-1 \pm \sqrt{15}i}{4}$ or $z = \dfrac{-1 \pm \sqrt{3}i}{2}$

Question 3 (15 marks)

a)



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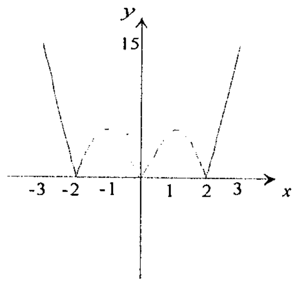

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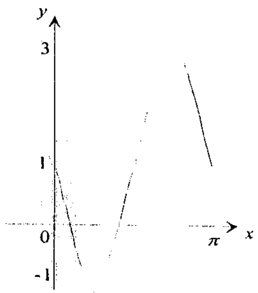
b) i)



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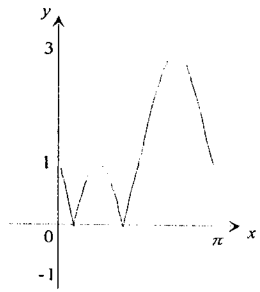
ii)



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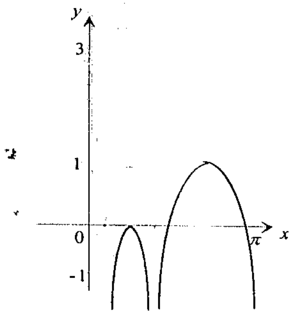
iii)



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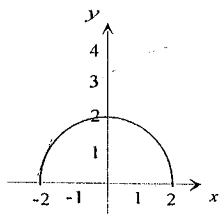
c)



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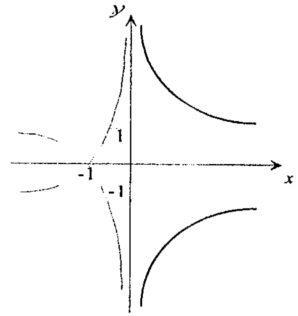
d)



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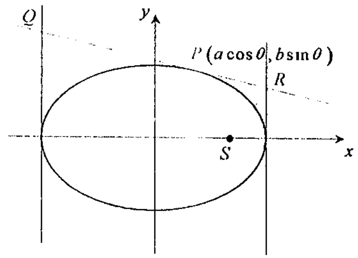
Question 4 (15 marks)

a)



i)

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$ 1

ii)

$Q:\ x=-a \Rightarrow -\cos\theta+\dfrac{y\sin\theta}{b}=1$

$y=\dfrac{b(1+\cos\theta)}{\sin\theta}$ 1

$Q\left(-a,\dfrac{b(1+\cos\theta)}{\sin\theta}\right)$

$R:\ x=a \Rightarrow \cos\theta+\dfrac{y\sin\theta}{b}=1$

$y=\dfrac{b(1-\cos\theta)}{\sin\theta}$ 1

$R\left(a,\dfrac{b(1-\cos\theta)}{\sin\theta}\right)$

iii) $S(ae,0)$

$$m_{SQ}=\frac{\frac{b(1+\cos\theta)}{\sin\theta}}{-a-ae}=\frac{b(1+\cos\theta)}{-a(1+e)\sin\theta} \qquad m_{SR}=\frac{\frac{b(1-\cos\theta)}{\sin\theta}}{a-ae}=\frac{b(1-\cos\theta)}{a(1-e)\sin\theta}$$ 1 1

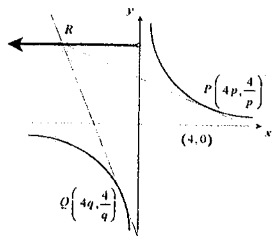
$$m_{SQ}.m_{SR}=\frac{b(1+\cos\theta)}{-a(1+e)\sin\theta}\times\frac{b(1-\cos\theta)}{a(1-e)\sin\theta}$$

$$=\frac{b^2(1-\cos^2\theta)}{a^2(1-e^2)\sin^2\theta}=-1$$ 1

$\therefore SQ\perp SR$ 1

b)

b)



i)

$$y=\frac{16}{x}$$

$$\frac{dy}{dx}=-\frac{16}{x^2}$$ 1

At P $\frac{dy}{dx}=-\frac{16}{16p^2}=-\frac{1}{p^2}$

Tangent at P is $y-\frac{4}{p}=-\frac{1}{p^2}(x-4p)\Rightarrow x+p^2y=8p$ 1

Tangent at P is $x+q^2y=8q$

ii) Solving simultaneously,

$$(p^2-q^2)y=8(p-q)$$

$$y=\frac{8}{p+q}$$ 1

$$x+\frac{8p^2}{p+q}=8p$$

$$x=\frac{8pq}{p+q} \quad\Rightarrow\quad R\left(\frac{8pq}{p+q},\frac{8}{p+q}\right)$$ 1

iii) Chord PQ:

$$m_{PQ}=\frac{\frac{4}{p}-\frac{4}{q}}{4p-4q}=-\frac{1}{pq}$$ 1

$$PQ:\quad y-\frac{4}{p}=-\frac{1}{pq}(x-4p)$$

$$(4,0)\Rightarrow -\frac{4}{p}=-\frac{1}{pq}(4-4p)$$

$4q=4-4p$ 1

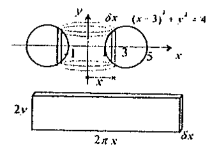
$p+q=1$

$R \quad x=8pq \qquad y=8 \quad (\text{since } p+q=1)$ 1

Hence the locus of R is the ray $y=8,\ x<0$ 1

Question 5 (15 marks)

a)



$$\delta V\approx 4\pi xy\delta x$$

$$V=\lim_{\delta x\to 0}\sum_{x=1}^{5}4\pi xy\delta x$$ 1

$$=4\pi\int_1^5 x\sqrt{4-(x-3)^2}\,dx$$

Put $x-3=2\sin\theta$

$dx=2\cos\theta d\theta$

$x=1\Rightarrow\theta=-\frac{\pi}{2}$ 1

$x=5\Rightarrow\theta=\frac{\pi}{2}$

$$V=4\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(3+2\sin\theta)2\cos\theta 2\cos\theta d\theta$$ 1

$$=8\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(12\cos^2\theta+8\cos^2\theta\sin\theta)d\theta$$

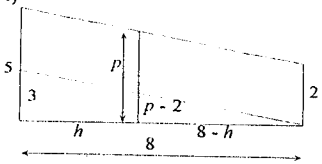
$$=48\pi\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)d\theta-64\pi\int_0^{\frac{\pi}{2}}\cos^2\theta(-\sin\theta)d\theta$$ 1

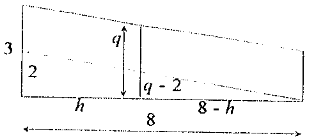
$$=48\pi\left[\theta+\tfrac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}-64\pi\left[\frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{2}}$$

$$=48\pi\left[\tfrac{\pi}{2}+0\right]-64\pi\left[0-\tfrac{1}{3}\right]$$ 1

$$=24\pi^2-\frac{64\pi}{3}$$

b) i)





$\frac{p-2}{3}=\frac{8-h}{8}$ $\qquad$ $\frac{q-1}{2}=\frac{8-h}{8}$

$8p-16=24-3h$ $\qquad$ 1 $\qquad$ $8q-8=16-2h$ $\qquad$ 1

$p=5-\frac{3h}{8}$ $\qquad$ $q=3-\frac{h}{4}$

ii)

$A(h)=pq$

$=\left(5-\frac{3h}{8}\right)\left(3-\frac{h}{4}\right)$ $\qquad$ 1

$=15-\frac{19h}{8}+\frac{3h^2}{32}$

iii)

$\delta V\approx A(h)\delta h$

$V=\lim_{\delta h\to 0}\sum_{x=0}^{8}\left(15-\frac{19h}{8}+\frac{3h^2}{4}\right)\delta h$ $\qquad$ 1

$=\int_0^8\left(15-\frac{19h}{8}+\frac{3h^2}{4}\right)dx$

$=\left[15h-\frac{19h^2}{16}+\frac{h^3}{32}\right]_0^8$ $\qquad$ 1

$=120-76+16$

$=60\text{ m}^3$ $\qquad$ 1

c) c) PYTHAGORAS

Number of selections of 2A and 4 non-A $= {}^2\,{}^8C_4$

Number of selections of 1A and 5 non-A $= {}^8C_5$ $\qquad$ 2

Number of selections of 0A and 6 non-I $= {}^8C_6$

Number of arrangements

$= {}^8C_4\times\frac{6!}{2!}+{}^8C_5\times 6!+{}^8C_6\times 6!$ $\qquad$ 1

$=6!(35+56+28)$

$=85680$ $\qquad$ 1

Question 6 (15 marks)

a) $x^3-mx-n=0$ $\qquad$ Roots are α,β,γ

i) $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ $\qquad$ 1

$=0-2(-m)$ $\qquad$ $\alpha+\beta+\gamma=0$ $\qquad$ 1

$=6$ $\qquad$ $\alpha\beta+\beta\gamma+\gamma\alpha=-m$

ii) $\sum\alpha^3-m\sum\alpha-3n=0$ $\qquad$ 1

$\sum\alpha^3=m\sum\alpha+3n$

$\therefore \alpha^3+\beta^3+\gamma^3=3n$ $\qquad$ 1

b) $P(x)=x^4-5x^3+7x^2+3x-10$

If $2-i$ is a zero, then $2+i$ is also a zero. Hence the factors are

$(x-2+i)(x-2-i)\Rightarrow(x^2-4x+5)$ $\qquad$ 1

$$\begin{array}{r} x^2-x-2 \\ x^2-4x+5\overline{)x^4-5x^3+7x^2+3x-10} \\ \underline{x^4-4x^3+5x^2} \\ -x^3+2x^2 \\ \underline{-x^3+4x^2-5x} \\ -2x^2+8x \\ \underline{-2x^2+8x-10} \\ 0 \end{array}$$ $\qquad$ 1

Hence $P(x)=(x-2+i)(x-2-i)(x-2)(x+1)$ $\qquad$ 1

c)

$P(x)=3x^4+8x^3+6x^2-1$

$P'(x)=12x^3+24x^2+12x$

$P''(x)=36x^2+48x+12$

For root of multiplicity 3, $P''(x)=0$ $\qquad$ 1

$36x^2+48x+12=0$

$3x^2+4x+1=0$

$(3x+1)(x+1)=0$ $\qquad$ 1

$x=-\frac{1}{3}$ or -1

$P(-1)=3-8+6-1=0$

$\therefore x=-1$ is the root of multiplicity 3. $\qquad$ 1

$\therefore P(x)=(x+1)^3(3x-1)$

d) i) $z^5-1=0$

Let $z=\cos\theta+i\sin\theta$

$\therefore z^5=\cos 5\theta+i\sin 5\theta$

$\cos 5\theta=1$ $\qquad$ $\sin\theta=0$

$5\theta=0,2\pi,4\pi,6\pi,8\pi$ $\qquad$ 1

$\theta=\frac{2n\pi}{5}$ $\qquad$ $n=0,1,2,3,4$ $\qquad$ 1 ii)

$z_1 = cis0 = 1$

$z_2 = cis\frac{2\pi}{5}$

$z_3 = cis\frac{4\pi}{5}$ 1

$z_4 = cis\frac{6\pi}{5} = cis\left(-\frac{4\pi}{5}\right)$

$z_5 = cis\frac{8\pi}{5} = cis\left(-\frac{2\pi}{5}\right)$

$\sum \alpha = z_1 + z_2 + z_3 + z_4 + z_5 = 0$

$1 + cis\frac{2\pi}{5} + cis\frac{4\pi}{5} + cis\left(-\frac{4\pi}{5}\right) + cis\left(-\frac{2\pi}{5}\right) = 0$ 1

$1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = 0$

$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$ 1

Question 7 (15 marks)

$\cos 6x + \sin 4x = \sin 10x$

a) $2\sin\frac{6x+4x}{2}\cos\frac{6x-4x}{2} = 2\sin 5x\cos 5x$ 1

$2\sin 5x\cos x = 2\sin 5x\cos 5x$

$2\sin 5x\cos x - 2\sin 5x\cos 5x = 0$

$2\sin 5x(\cos x - \cos 5x) = 0$ 1

$\sin 5x = 0$ or $\cos x = \cos 5x$

$5x = 0, \pi, 2\pi, 3\pi, \ldots$

$x = \frac{n\pi}{5}$ 1

$5x = \pm x, 2\pi \pm x, 4\pi \pm x, \ldots$

$4x = 0, 2\pi, 4\pi, \ldots$

$x = \frac{n\pi}{2}$ 1

$6x = 0, 2\pi, 4\pi, \ldots$

$x = \frac{n\pi}{3}$ 1

$\therefore x = \frac{n\pi}{5}, \frac{n\pi}{3}, \frac{n\pi}{2}$

b)

We need the following identities.

Since $A + B + C = \pi$, then $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$

$\frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2} \Rightarrow \sin\frac{A+B}{2} = \cos\frac{C}{2}, \cos\frac{A+B}{2} = \sin\frac{C}{2}$

Also, $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A + \sin B + \sin C = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2}$ 1

$= 2\cos\frac{C}{2}\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2}$ 1

$= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\right)$

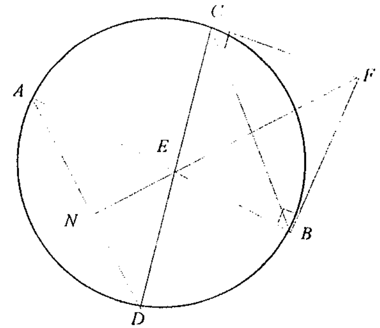
$= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} + 2\cos\frac{A+B}{2}\right)$ 1

$= 2\cos\frac{C}{2}\left(2\cos\frac{A+B+A-B}{4}\cos\frac{A+B-A+B}{4}\right)$ 1

$= 2\cos\frac{C}{2}\left(2\cos\frac{A}{2}\cos\frac{B}{2}\right)$ 1

$= 2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

c)



Construction: Join AD and CB and produce FE to meet AD in N.

Proof:

$\angle DAB = \angle DCB$ (angles in the same segment, circle $ACBD$) 1

$\angle ECF + \angle EBF = 180^0$ (given)

$\therefore ECBF$ is a cyclic quadrilateral. 1

$\angle EFB = \angle ECB$ (angles in the same segment, circle $ECFB$) 1

$\therefore \angle DCB = \angle EFB$ (both equal to $\angle ECB$)

$\therefore AFBN$ is a cyclic quadrilateral. 1

$\angle FBA = \angle FNA$ (angles in the same segment, circle $AFBN$)

$\therefore \angle FNA = 90^0$ $(\angle FBA = 90^0)$ 1

$\therefore FN \perp AD$

Question 8 (15 marks)

a)

$\left.\begin{array}{l}\frac{p+2q}{2} \ge \sqrt{2pq} \\ \frac{2q+3r}{2} \ge \sqrt{6qr} \\ \frac{p+3r}{2} \ge \sqrt{3pr}\end{array}\right\}$ (AM ≥ GM) 1

$\left(\frac{p+2q}{2}\right)\left(\frac{2q+3r}{2}\right)\left(\frac{p+3r}{2}\right) \ge \sqrt{2pq}.\sqrt{6qr}.\sqrt{3pr}$ 1

$(p+2q)(2q+3r)(p+3r) \ge 8\sqrt{36p^2q^2r^2}$ 1

$(p+2q)(2q+3r)(p+3r) \ge 48pqr$

b)

$n=1 \Rightarrow T_1 = 5^0+3=4$, true

$n=2 \Rightarrow T_2 = 5^1+3=8$, true 1

Take $n=k$ and $n=k-1$

$T_k = 5^{k-1}+3$ and $T_{k-1}=5^{k-2}+3$ 1

Assume true and use this to prove true for $n=k+1$, ie. $T_{K+1}=5^k+3$

$$\begin{aligned} T_{k+1} &= 6T_k - 5T_{k-1} \\ &= 6\left(5^{k-1}+3\right)-5\left(5^{k-2}+3\right) \\ &= 6.5^{k-1}+18-5.5^{k-2}-15 \\ &= 5^{k-2}(30-5)+3 \\ &= 5^{k-2}.5^2+3 \\ &= 5^k+3 \end{aligned}$$

1 1 1 1

Hence

If true for $n=k-1$ and $n=k$

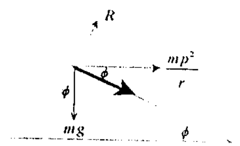
then true for $n=k+1$

However, it was true for $n=1, n=2$ 1

$\therefore$ true for $n=3, n=4$, etc.

$\therefore$ true for all positive integers n.

c)



$$\frac{mp^2}{r}\cos\theta = F + mg\sin\theta$$

$$\frac{mp^2}{r}\sin\theta = R - mg\cos\theta$$ 1

$$F = \frac{mp^2}{r}\cos\theta - mg\sin\theta$$

$$R = \frac{mp^2}{r}\sin\theta + mg\cos\theta$$ 1

At $p=q$, $F=0$

$$0 = \frac{mq^2}{r}\cos\theta - mg\sin\theta$$ 1

$$g\sin\theta = \frac{q^2}{r}\cos\theta \quad \Rightarrow \quad rg\sin\theta = q^2\cos\theta$$

$$F = \frac{mp^2}{r}\cos\theta - \frac{mq^2}{r}\cos\theta$$ 1

$$= \frac{m}{r}\cos\theta\left(p^2-q^2\right)$$

$\mu R \ge F$ for no side-slip.

$$\mu m\left(\frac{p^2}{r}\sin\theta + g\cos\theta\right) \ge \frac{m\cos\theta}{r}\left(p^2-q^2\right)$$ 1

$$\mu\left(p^2\sin\theta + rg\cos\theta\right) \ge \cos\theta\left(p^2-q^2\right)$$

$$\mu\left(p^2\sin^2\theta + rg\sin\theta\cos\theta\right) \ge \sin\theta\cos\theta\left(p^2-q^2\right)$$

$$\mu\left(p^2\sin^2\theta + q^2\cos^2\theta\right) \ge \left(p^2-q^2\right)\sin\theta\cos\theta \qquad \left(\text{since } rg\sin\theta = q^2\cos\theta\right)$$

$$\mu \ge \frac{\left(p^2-q^2\right)\sin\theta\cos\theta}{\left(p^2\sin^2\theta+q^2\cos^2\theta\right)}$$ 1