Student Number: ..................................................

# ACADEMIC RESOURCES CENTRE

## 1999
## TRIAL EXAMINATION

# MATHEMATICS

## 3 Unit (Additional)
## and
## 3/4 Unit (Common)

Time Allowed - Two (2) hours
(plus 5 minutes reading time)

## Directions to Candidates

- Attempt ALL questions
- Show all necessary working, marks may be deducted for careless or untidy work
- Standard integrals are printed on the last page
- Board-approved calculators may be used
- Additional Answer Booklets are available

## Directions to School or College

## STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a}e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{(a^2 - x^2)}} \, dx = \sin^{-1}\frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{(x^2 - a^2)}} \, dx = \ln\left\{x + \sqrt{(x^2 - a^2)}\right\}, \; |x| > |a|$$

$$\int \frac{1}{\sqrt{(x^2 + a^2)}} \, dx = \ln\left\{x + \sqrt{(x^2 + a^2)}\right\}$$

NOTE : $\ln x = \log_e x; \quad x > 0$

## Question One

(a) For the function $f(x) = e^{x+1}$ find the inverse function $f^{-1}(x)$ and hence show that $f[f^{-1}(x)] = f^{-1}[f(x)] = x$ *(3 marks)*

(b) Solve the inequality $\frac{1}{x+2} \geq \frac{2}{x-3}$ and represent the solution on a number line. *(3 marks)*

(c) If $\sum_{r=1}^{n} \frac{3}{2}(2)^{r-1} = 766\frac{1}{2}$, find $n$ *(3 marks)*

(d) The word **EQUATION** contains all five vowels. How many 3-letter 'words' consisting of at least 1 vowel and 1 consonant can be made from the letters of **EQUATION**? *(3 marks)*

## Question Two

(a) Prove that $\frac{2\cos A}{\operatorname{cosec} A - 2\sin A} = \tan 2A$ *(3 marks)*

(b) Show that $\tan^{-1} x = \sin^{-1} \frac{x}{\sqrt{1+x^2}}$ *(3 marks)*

(c) Write down the expansion for $\sin(\alpha - \beta)$ and hence prove that $\sin(-\theta) = -\sin\theta$ *(3 marks)*

(d) Find $\frac{d}{dx}\left[\frac{\ln x}{x}\right]$ and hence find the primitive function of $\frac{2 - \ln x}{x^2}$ *(4 marks)*

## Question Three

(a) The sides of a square sheet of cardboard are each 12m long. At each corner a square of $x^2 \text{m}^2$ is cut away. The sides of the sheet are then turned up to form a box. Calculate:

(i) the values of $x$ so that the box has a volume of $108\text{m}^3$

(ii) the value of $x$ so that the box has a maximum volume *(5 marks)*

(b) Use mathematical induction to show that for all positive integers $n$, $\sum_{r=1}^{n} a^{-r} = \frac{a^n - 1}{(a-1)a^n}$ *(4 marks)*

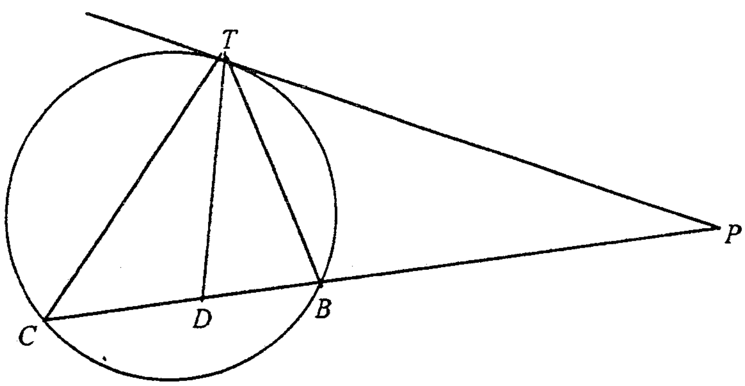
(c) A polynomial $P(x) = ax^3 + bx^2 + cx + d$ has zeroes at -2, 2 and $\frac{3}{2}$.
It leaves a remainder of 12 when divided by $x$ - 1.
Find the values of $a$, $b$, $c$ and $d$. *(3 marks)*

**Question Four**

(a) The tangent at the point $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$ cuts the $y$-axis at $T$. The line through the focus $S$ parallel to this tangent cuts the directrix at $V$. $M$ is the mid-point of $TV$. Find the locus of $M$ as $P$ moves on the parabola. *(5 marks)*

(b) If $3^x = 2^y = 6^z$, prove that $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$ *(3 marks)*

(c)




$PT$ is a tangent to the circle and $PBC$ is a secant. $D$ is a point on $PBC$ such that $TD = TB$.
Prove that $\angle CTD = \angle P$. *(4 marks)*

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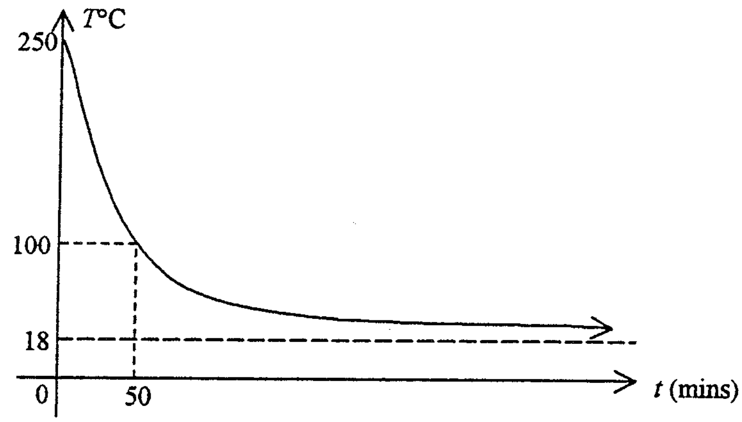
## Question Five

(a) The graph shown below shows the cooling curve for a container of paraffin oil which as been heated to a temperature of 250° then allowed to cool in air whose temperature is 18°C.




It is known that the rate at which the temperature $T$ of the oil is changing is given by $\frac{dT}{dt} = k(T - M)$ where $M$ is the temperature of the surrounding air and t is the time elapsed after cooling begins, in minutes.

(i) Show that $T = M + Ae^{kt}$ is a solution to the given equation.

(ii) Use the graph to write down the values of $M$ and $A$.

(iii) Find the value of $k$ to one significant figure if the temperature of the oil drops to 103.3°C in 50 minutes of cooling.

*(6 marks)*

(b) Write the equation $2\tan\theta - 3\sec\theta = -2\sqrt{3}$ in the form $a\sin\theta + b\cos\theta = c$, and then by expressing the left hand side as sine of a compound angle, solve the equation for $0 \leq \theta \leq 360°$

*(6 marks)*

## Question Six

(a) A particle moving in simple harmonic motion, passes through the centre of oscillation $O$ with a velocity of 5cm/s. If it has a period of $\pi$ seconds, find

(i) the value of $n$

(ii) the amplitude of the motion

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(iii) the time taken for the particle to first reach $x = 1.5$cm *(5 marks)*

(b) Express $\sec(\sin^{-1} x)$ in terms of $x$ and hence write down $\int \sec(\sin^{-1} x)dx$ $(-1<x<1)$ *(2 marks)*

(c) The acceleration of a body moving in a straight line is given by

$$\frac{d^2x}{dt^2} = -\frac{24}{x^2}$$

where $x$ is the displacement from the origin after $t$ seconds. When $t = 0$ the body is 3 metres to the right of the origin with a velocity of 4m/s.

(i) Show that the velocity $v$ of the body in terms of $x$, is

$$v = \frac{4\sqrt{3}}{\sqrt{x}}$$

(ii) Find an expression for $t$ in terms of $x$

(iii) How long does it take for the body to reach a point 10m to the right of the origin? *(5 marks)*

**Question Seven**

(a) (i) Show that the range of flight of a projectile fired at an angle of $\alpha$ to the horizontal and at a velocity $v$ is

$$\frac{v^2 \sin 2\alpha}{g}$$

where $g$ is the acceleration due to gravity

(ii) A cannon fires a shell at an angle of 45° to the horizontal and strikes a point 50m beyond its target. When fired with the same velocity at an angle of 30° it hits a point 20m in front of the target. Calculate

(I) the distance of the target from the cannon

(II) the correct angles required to hit the target *(7 marks)*

(b) By considering the value of $(1+x)^{2n}$ when $x = 1$, prove that

$$\sum_{k=0}^{n} \binom{2n}{k} = 2^{2n-1} + \frac{(2n)!}{2(n!)^2}$$ *(5 marks)*

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(1)(a) $f^{-1}(x) = \ln x - 1$

(b) $x \leq -7$ or $-2 < x < 3$

(c) $n = 9$

(d) 270

(2) (a) , (b) Proofs

(c) $\sin\alpha\cos\beta - \cos\alpha\sin\beta$; Proof

(d) $\frac{\ln x}{x} - \frac{1}{x} + c$

(3)(a)(i) $x = 3$ and $\frac{9 - 3\sqrt{5}}{2} \approx 1.15$m

(ii) $x = 2$m

(b) Proof

(c) $a = 8, b = -12, c = -32, d = 48$

(4) (a) $y = -\frac{a}{2}\left\{\left(\frac{x}{a}\right)^2 - 1\right\}$ $= -\frac{a}{2}\left\{\left(\frac{a}{x}\right)^2 - 1\right\}$

(b), (c) Proofs

(5) (a) (i) Proof

(ii) $M = 18^0C$; $A = 232^0C$

(iii) $k \approx -0.002$(to 1 s.f.)

(b) $\theta = 71^0 25'$ or $348^0 35'$

(6) (a) (i) $n = 2$

(ii) $a = 2.5$cm

(iii) $t = 0.32$ sec

(b) $\frac{1}{\sqrt{1 - x^2}}$; $\sin^{-1}x + c$

(c) (i) Proof

(ii) $t = \frac{x^2}{6\sqrt{3}} - \frac{1}{2}$

(iii) $t \approx 2.5$ sec

(7)(a)(i) Proof

(ii) (I) $T = 472.5$m

(II) $\alpha = 32^0 22'$ or $57^0 38'$

(b) Proof

A.R.C SUGGESTED SOLUTIONS for 3 UNIT MATHS - 1999

## Question One

(a)

$f(x) = y = e^{x+1}$
$x = e^{y+1}$

$y + 1 = \ln x$
$y = \ln x - 1$ the inverse function $f^{-1}(x)$

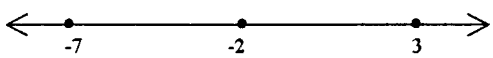
$f[f^{-1}(x)] = e^{\ln x - 1 + 1}$
$= e^{\ln x}$
$= x$

$f^{-1}[f(x)] = \ln[e^{x+1}] - 1$
$= (x+1)\ln e - 1$
$= x + 1 - 1$
$= x$

(b)

$\frac{1}{x+2} \ge \frac{2}{x-3}$
$(x-3)^2(x+2) \ge 2(x-3)(x+2)^2$
$(x-3)^2(x+2) - 2(x-3)(x+2)^2 \ge 0$
$(x-3)(x+2)[x-3-2(x+2)] \ge 0$
$(x-3)(x+2)[x-3-2x-4] \ge 0$
$(x-3)(x+2)(-x-7) \ge 0$ ..........(1)

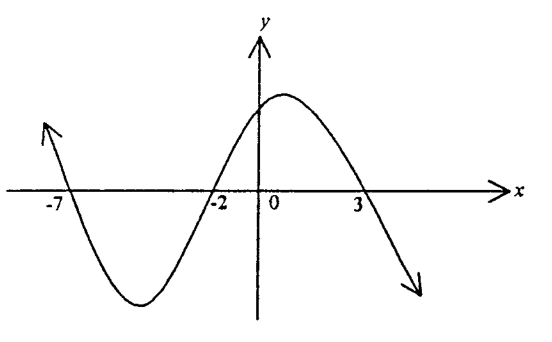
Consider $(x-3)(x+2)(-x-7) = 0$




Test $x$ = -10, inequality (1) holds true
Test $x$ = -5, inequality (1) does not hold true
Test $x$ = 0, inequality (1) holds true
Test $x$ = 4, inequality (1) does not hold true

$\therefore$ Solution is $x \le -7$ or $-2 < x < 3$
$(x \ne -2), (x \ne 3)$

From the graph of $y = (x-3)(x+2)(-x-7)$



Solution is $x \le -7$ or $-2 < x < 3$

(c)

$$\sum_{r=1}^{n} \frac{3}{2}(2)^{r-1} = \frac{3}{2} + \frac{3}{2}(2) + \frac{3}{2}(2)^2 + \ldots + \frac{3}{2}(2)^n$$

$$a = \frac{3}{2}; r = 2$$

$$S_n = \frac{a(r^n - 1)}{r-1}$$

$$\therefore \quad \frac{\frac{3}{2}(2^n - 1)}{2-1} = 766\tfrac{1}{2}$$

$$\tfrac{3}{2}(2^n - 1) = \frac{1533}{2}$$

$$3(2^n - 1) = 1533$$

$$2^n - 1 = 511$$

$$2^n = 512$$

$$n = 9$$

(d)

There are 5 vowels and 3 consonants. For words with 2 vowels and 1 consonant the number of selections is ${}^5C_2 \times {}^3C_1 = 30$. For words with 1 vowel and 2 consonants the number of selections is ${}^5C_1 \times {}^3C_2 = 15$.

$\therefore$ The total number of selections = 45

But each selection may be arranged 3! ways.

$\therefore$ The total number of words $= 45 \times 3!$

$= 270$

**Question Two**

(a)

$$\text{LHS} = \frac{2\cos A}{\operatorname{cosec} A - 2\sin A}$$

$$= \frac{2\cos A}{\frac{1}{\sin A} - 2\sin A}$$

$$= \frac{2\cos A}{\frac{1 - 2\sin^2 A}{\sin A}}$$

$$= \frac{2\sin A\cos A}{1 - 2\sin^2 A}$$

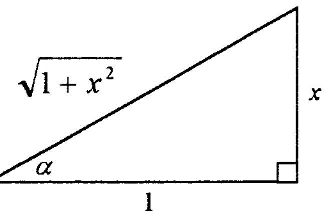
$$= \frac{\sin 2A}{\cos 2A}$$

$$= \tan 2A$$

$$= \text{RHS}$$

(b) Let $\tan^{-1} x = \alpha$

$\therefore \tan\alpha = x$




From the triangle $\sin\alpha = \frac{x}{\sqrt{1+x^2}}$

$$\therefore \alpha = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$

$$\therefore \tan^{-1} x = \frac{\sin^{-1} x}{\sqrt{1+x^2}}$$

(c) $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Put $\alpha = 0$ and $\beta = \theta$

$\therefore$

$$\sin(-\theta) = \sin 0\cos\theta - \cos 0\sin\theta$$
$$= 0 - (1)\sin\theta$$
$$= -\sin\theta$$

(d)

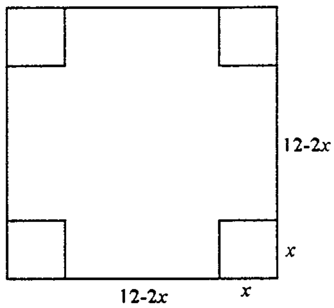
$$\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{x\cdot\frac{1}{x} - \ln x(1)}{x^2}$$
$$= \frac{1-\ln x}{x^2}$$

$\therefore$

$$\int\left(\frac{2-\ln x}{x^2}\right)dx = \int\left(\frac{1+1-\ln x}{x^2}\right)dx$$
$$= \int\left(\frac{1}{x^2} + \frac{1-\ln x}{x^2}\right)dx$$
$$= \int x^{-2}dx + \int\left(\frac{1-\ln x}{x^2}\right)dx$$
$$= \frac{x^{-1}}{-1} + \frac{\ln x}{x} + c$$
$$= \frac{\ln x}{x} - \frac{1}{x} + c$$

**Question Three**

(a) (i)




$$V = x(12-2x)^2$$
$$= 144x - 48x^2 + 4x^3$$
$$\therefore 4x^3 - 48x^2 + 144x = 108$$
$$x^3 - 12x^2 + 36x - 27 = 0$$

Let $f(x) = x^3 - 12x^2 + 36x - 27$

$$f(3) = 27 - 108 + 108 - 27$$
$$= 0$$

$\therefore x - 3$ is a factor

$$\begin{array}{r l}
 & x^2 - 9x + 9 \\
x - 3 & \overline{)\, x^3 - 12x^2 + 36x - 27} \\
 & \underline{x^3 - 3x^2} \\
 & -9x^2 + 36x \\
 & \underline{-9x^2 + 27x} \\
 & 9x - 27 \\
 & \underline{9x - 27} \\
 & 0
\end{array}$$

$\therefore (x-3)(x^2 - 9x + 9) = 0$

$x$=3 or

$$x = \frac{9 \pm \sqrt{81 - 4(1)(9)}}{2}$$
$$= \frac{9 \pm \sqrt{81 - 36}}{2}$$
$$= \frac{9 \pm \sqrt{45}}{2}$$
$$= \frac{9 \pm 3\sqrt{5}}{2}$$

$x = \dfrac{9 + 3\sqrt{5}}{2}$ is rejected since it is >12

$\therefore$ Values of $x$ so that box has a volume of $108\text{m}^3$ are

3m and $\frac{9-3\sqrt{5}}{2}$ m $(\approx 1.15\text{m})$

(ii) $V = 4x^3 - 48x^2 + 144x$

$^{dv}/_{dx} = 12x^2 - 96x + 144 = 0$

for maximum/minimum

$x^2 - 8x + 12 = 0$

$(x-6)(x-2) = 0$

reject $x = 6$ ; $x = 2$

$^{d^2v}/_{dx^2} = 24x - 96$

When $x = 2$, $\quad ^{d^2v}/_{dx^2} = 48 - 96$

$= -48 < 0$

$\therefore$ maximum

$\therefore$ the box has a maximum volume when $x = 2$m

(b) $\sum_{r=1}^{n} a^{-r} = \frac{1}{a} + \frac{1}{a^2} + \ldots + \frac{1}{a^n}$

We are required to prove that

$\frac{1}{a} + \frac{1}{a^2} + \ldots + \frac{1}{a^n} = \frac{a^n - 1}{(a-1)a^n}$ for all positive integers $n$

Test $n = 1$:

$LHS = \frac{1}{a}$

$RHS = \frac{a-1}{(a-1)a}$

$= \frac{1}{a}$

$\therefore$ The result is true for $n = 1$

Assume that it is true for $n = k$

$\frac{1}{a} + \frac{1}{a^2} + \ldots + \frac{1}{a^k} = \frac{a^k - 1}{(a-1)a^k}$

Put $n = k + 1$

$$LHS = \frac{1}{a} + \frac{1}{a^2} + \ldots + \frac{1}{a^k} + \frac{1}{a^{k+1}}$$
$$= \frac{a^k - 1}{(a-1)a^k} + \frac{1}{a^{k+1}}$$
$$= \frac{a(a^k - 1) + (a-1)}{(a-1)a^{k+1}}$$
$$= \frac{a^{k+1} - a + a - 1}{(a-1)a^{k+1}}$$
$$= \frac{a^{k+1} - 1}{(a-1)a^{k+1}}$$

Hence if the result is true for $n = k$, it is true for $n = k + 1$. But it is true for $n = 1$. Therefore it is true for $n = 2$ and so, by mathematical induction, for all positive integers $n$.

(c) Since $P(x)$ has zeros at -2, 2 and $\frac{3}{2}$, $P(x)$ may be expressed in the form

$P(x) = k(x+2)(x-2)(2x-3)$ where $k$ is a real number.

Now $P(1) = k(3)(-1)(-1) = 12$

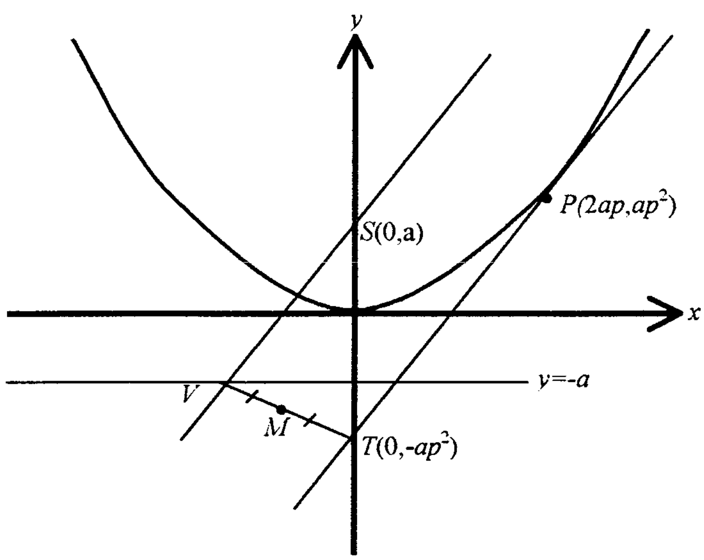
$\therefore k = 4$

$$\therefore P(x) = 4(x+2)(x-2)(2x-3)$$
$$= (x^2 - 4)(8x - 12)$$
$$= 8x^3 - 12x^2 - 32x + 48$$

$\therefore a = 8, b = -12, c = -32, d = 48$

**Question Four**

(a)




Gradient of tangent at $P = p$

Equation of tangent at $P$: $y = px - ap^2$

$\therefore$ It cuts the y-axis at $-ap^2$

$\therefore$ $T(0,-ap^2)$

Equation of $VS$: $\quad y = px + a$

y-coordinate of $V = -a$

$$\therefore \quad -a = px_v + a$$

$$x_v = \frac{-2a}{p}$$

$$\therefore \quad V\left(\frac{-2a}{p}, -a\right)$$

$$\therefore \quad x_m = \frac{\frac{-2a}{p} + 0}{2} \qquad y_m = \frac{-ap^2 - a}{2}$$

$$M\left(\frac{-a}{p}, \frac{-ap^2 - a}{2}\right)$$

$$x = \frac{-a}{p} \qquad \therefore p = \frac{-a}{x}$$

$$y = \frac{-ap^2 - a}{2} = \frac{-a\left(\frac{-a}{x}\right)^2 - a}{2}$$

$\therefore$ $$y = \frac{-a^3}{2x^2} - \frac{a}{2}$$ which is the locus of $M$.

(b)

$$3^x = 6^z$$

$$\therefore x\log 3 = z\log 6$$

Similarly $\quad y\log 2 = z\log 6$

so that

$$x = \frac{z\log 6}{\log 3}$$

$$y = \frac{z\log 6}{\log 2}$$

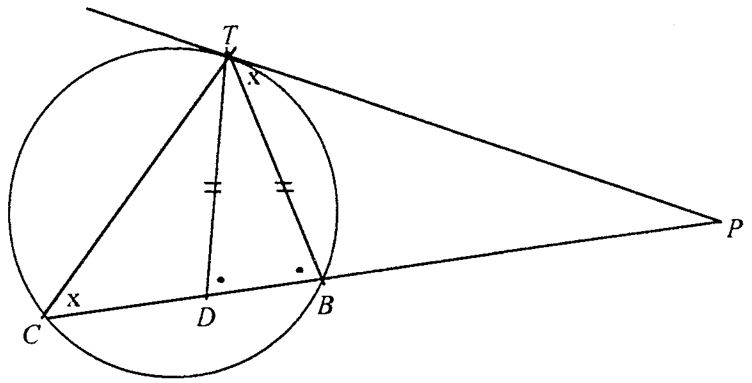
$$\therefore \frac{1}{x} + \frac{1}{y} = \frac{\log 3}{z\log 6} + \frac{\log 2}{z\log 6}$$

$$= \frac{\log 3 + \log 2}{z\log 6}$$

$$= \frac{\log 6}{z\log 6}$$

$$= \frac{1}{z}$$

(c)




$\angle TDB = \angle C + \angle CTD$ (exterior $\angle$ of $\Delta CTD$)
and $\angle TBD = \angle P + \angle PTB$ (exterior $\angle$ of $\Delta PTB$)
But $\angle TDB = \angle TBD$ (base $\angle$s of isosceles $\Delta TBD$)
$\therefore \angle C + \angle CTD = \angle P + \angle PTB$
But $\angle PTB = \angle C$ ($\angle$ between tangent and chord)
$\therefore \angle CTD = \angle P$

**Question Five**

(a) (i)

$$T = M + Ae^{kt}$$
$$\tfrac{dT}{dt} = k \cdot Ae^{kt}$$
$$= k(T - M)$$

$\therefore$ $T = M + Ae^{kt}$ is a solution to the equation

(ii) $M = 18°\text{C}$
$A = 250 - 18 = 232°\text{C}$

(iii) $T = 18 + 232e^{kt}$
When $t = 50$ mins and $T = 103.3°\text{C}$
$103.3 = 18 + 232e^{50k}$
$\therefore 232e^{50k} = 85.3$

$$e^{50k} = \frac{85.3}{232}$$
$$50k = \ln\frac{85.3}{232}$$
$$k = \frac{\ln\frac{85.3}{232}}{50}$$
$$\approx -0.002 \text{ (one significant figure)}$$

(b)

$$2\tan\theta - 3\sec\theta = -2\sqrt{3}$$

$$\frac{2\sin\theta}{\cos\theta} - \frac{3}{\cos\theta} = -2\sqrt{3}$$

$$2\sin\theta - 3 = -2\sqrt{3}\cos\theta$$

ie. $2\sin\theta + 2\sqrt{3}\cos\theta = 3$

Let $2\sin\theta + 2\sqrt{3}\cos\theta = r\sin(\theta + \alpha)$

where $r = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{16} = 4$

and $\tan\alpha = \dfrac{2\sqrt{3}}{2} = \sqrt{3}$

$$\alpha = \frac{\pi}{3}$$

$$\therefore\ 4\sin(\theta + \tfrac{\pi}{3}) = 3$$

$$\sin(\theta + \tfrac{\pi}{3}) = \tfrac{3}{4}$$

$\therefore\ \theta + 60° =$ 48°35', 180° - 48°35', 360° + 48°35'
*reject* (48°35')

$\theta = 71°25'$ or $348°35'$

**Question Six**

(a) (i)

$$T = \frac{2\pi}{n}$$

$$\pi = \frac{2\pi}{n}$$

$$n = 2$$

(ii) When $t = 0$, $x = 0$ and $v = 5$cm/s

$$v^2 = n^2(a^2 - x^2)$$

$$25 = 4(a^2 - 0)$$

$$a^2 = \tfrac{25}{4}$$

$$a = \tfrac{5}{2}\text{cm}$$

(iii) Let $x = a\sin(nt + \alpha)$ describe the SHM

When $t = 0$, $x = 0$ so that $\alpha = 0$

$$\therefore\ x = a\sin nt$$

$$x = \tfrac{5}{2}\sin 2t$$

When $x = 1.5$cm

$$\frac{3}{2} = \frac{5}{2}\sin 2t$$

$$\sin 2t = 0.6$$

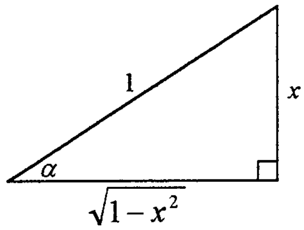
$$2t = 0.64$$

$$t = 0.32\text{s.}$$

(b) Let $\sec(\sin^{-1} x) = y$

Put $\sin^{-1} x = \alpha$

then $\sin \alpha = x$ $(-1 < x < 1)$




$$\therefore y = \sec\alpha$$

$$= \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \int \sec(\sin^{-1} x)dx = \int \frac{1}{1-x^2}dx$$

$$= \sin^{-1} x + c$$

(c) (i)

$$v^2 = 2\int \ddot{x}dx$$

$$= 2\int \tfrac{-24}{x^2}dx$$

$$= -48\int x^{-2}dx$$

$$= 48x^{-1} + c$$

$$v^2 = \tfrac{48}{x} + c$$

When $t = 0$, $v = 4$ and $x = 3$

$$\therefore\ 16 = \tfrac{48}{3} + c$$

$$\therefore\ c = 0$$

$$\therefore\ v^2 = \frac{48}{x}$$

$$v = \sqrt{\frac{48}{x}}$$

$$= \frac{4\sqrt{3}}{\sqrt{x}}$$

(ii) $v = \frac{dx}{dt}$

$= \frac{4\sqrt{3}}{\sqrt{x}}$

$dt/_{dx} = \frac{\sqrt{x}}{4\sqrt{3}}$

$t = \int \frac{\sqrt{x}}{4\sqrt{3}} dx$

$= \frac{1}{4\sqrt{3}} \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \frac{1}{6\sqrt{3}} x^{\frac{3}{2}} + c$

When $t = 0$, $x = 3\text{m}$

$\therefore 0 = \frac{3\sqrt{3}}{6\sqrt{3}} + c$

$\therefore\ c = -\frac{1}{2}$

$\therefore\ t = \frac{x^{\frac{3}{2}}}{6\sqrt{3}} - \frac{1}{2}$

(iii) When $x = 10$

$t = \frac{10^{\frac{3}{2}}}{6\sqrt{3}} - \frac{1}{2}$

$\approx 2.5\,\text{sec}$

**Question Seven**

(a) (i) Starting with

$y = vt\sin\alpha - \frac{1}{2}gt^2$

$x = vt\cos\alpha$

When $y = 0$,

$vt\sin\alpha - \frac{1}{2}gt^2 = 0$

$t(v\sin\alpha - \frac{1}{2}gt) = 0$

$t = 0 \text{ or } t = \frac{2v\sin\alpha}{g}$

$$\therefore \quad x = \frac{v(2v\sin\alpha)\cos\alpha}{g}$$

$$= \frac{2v^2\sin\alpha\cos\alpha}{g}$$

$$I.e. \quad Range = \frac{v^2\sin 2\alpha}{g}$$

(ii) (I) Let $T$ = distance from cannon to target:

When $\alpha = 45°$, $T + 50 = \dfrac{v^2 \sin 90°}{g}$

$\therefore \; T + 50 = \dfrac{v^2}{g}$ ..............(1)

When $\alpha = 30°$, $T - 20 = \dfrac{v^2 \sin 60°}{g}$

$T - 20 = \dfrac{v^2\sqrt{3}}{2g}$ ...........(2)

(1) ÷ (2):

$$\frac{T+50}{T-20} = \frac{2}{\sqrt{3}}$$

$$\sqrt{3}(T+50) = 2(T-20)$$

$$\sqrt{3}T + 50\sqrt{3} = 2T - 40$$

$$T(2-\sqrt{3}) = 40 + 50\sqrt{3}$$

$$T = \frac{40+50\sqrt{3}}{2-\sqrt{3}}$$

$$= 472.5\text{m}$$

(II) $T + 50 = \dfrac{v^2}{g}$ .........(1)

$T = \dfrac{v^2 \sin 2\alpha}{g}$ .........(2)

Substitute $T = 472.5$,

$$522.5 = \frac{v^2}{g}$$

$$472.5 = \frac{v^2\sin 2\alpha}{g}$$

$$\frac{472.5}{522.5} = \sin 2\alpha$$

$\therefore \; 2\alpha = 64°44'$ or $115°16'$

$\therefore \; \alpha = 32°22'$ or $57°38'$

(b)

$$(1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + \dots + \binom{2n}{n}x^n + \dots + \binom{2n}{2n-1}x^{2n-1} + \binom{2n}{2n}x^{2n}$$

Putting $x = 1$,

$$(2)^{2n} = \binom{2n}{0} + \binom{2n}{1} + \dots + \binom{2n}{n} + \dots + \binom{2n}{2n-1} + \binom{2n}{2n}$$

$$(2)^{2n} = \binom{2n}{0} + \binom{2n}{2n} + \binom{2n}{1} + \binom{2n}{2n-1} \dots + \binom{2n}{n} + \binom{2n}{n} - \binom{2n}{n}$$

$$= 2\left[\binom{2n}{0} + \binom{2n}{1} + \dots + \binom{2n}{n}\right] - \binom{2n}{n}$$

$$= 2\sum_{k=0}^{n}\binom{2n}{k} - \binom{2n}{n}$$

$$\therefore \frac{2^{2n}}{2} = \sum_{k=0}^{n}\binom{2n}{k} - \tfrac{1}{2}\binom{2n}{n}$$

$$2^{2n-1} = \sum_{k=0}^{n}\binom{2n}{k} - \frac{1}{2}\frac{(2n)!}{n!(2n-n)!}$$

$$\therefore \quad 2^{2n-1} = \sum_{k=0}^{n}\binom{2n}{k} - \frac{(2n)!}{2(n!)^2}$$

ie. $$\sum_{k=0}^{n}\binom{2n}{k} = 2^{2n-1} + \frac{(2n)!}{2(n!)^2}$$