SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2013

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- Show all necessary working in Questions 11–14

Total Marks - 70 Marks

Section I **10 Marks**

- Attempt Questions 1–10
- Allow about 15 minutes for this section.

Section II **60 Marks**

- Attempt Questions 11–14
- Allow about 1 hour 45 minutes for this section.

Examiner: *External Examiner*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

TABLE OF STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1};\quad n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

Note: $\ln x = \log_e x,\quad x > 0$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

1 What is the expression $\dfrac{2\tan A}{1+\tan^2 A}$ equal to?

(A) $\cos 2A$

(B) $\sin 2A$

(C) $\tan 2A$

(D) $\cot 2A$

2 The polynomial, $p(x)$ is defined by $p(x) = x^3 - x^2 + x + 3$.
What is the remainder when $p(x)$ is divided by $(x-1)$?

(A) 0

(B) 2

(C) 3

(D) 4

3 What is the domain and range of $y = 2\sin^{-1}\left(\dfrac{x}{3}\right)$?

(A) $|x| \le 3, |y| \le \pi$.

(B) $|x| \le 1, |y| \le 3$.

(C) $|x| \le 1, |y| \le \pi$.

(D) $|x| \le 3, |y| \le 2$.

4 What ratio does the point $P(10, 11)$ divide the interval AB, where $A(-2,3)$ and $B(7,9)$?

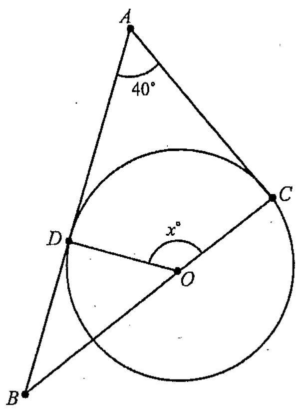
(A) $1:4$

(B) $4:-1$

(C) $1:-4$

(D) $4:1$

5 In the figure below, a circle with centre O is tangent to AB at point D and tangent to AC at point C.



If $\angle A = 40°$, what is the value of x?

(A) 140

(B) 145

(C) 150

(D) 155

6 The function $f(x) = \sin x - \dfrac{2}{3}x$ has a real root close to $x = 1{\cdot}5$
Let $x = 1{\cdot}5$ be a first approximation to the root.
What is the second approximation to the root using Newton's method?

(A) 1·495
(B) 1·496
(C) 1·503
(D) 1·504

7 A test is administered with 15 questions. Students are allowed to answer any ten.
How many choices of ten questions are there?

(A) 150

(B) 250

(C) 3003

(D) 3000

8 The graph of $f(x) = 0{\cdot}6\cos^{-1}(x-1)$, defines a curve that, when rotated about the y-axis, will produce a solid that is to be the shape and size of a new biscuit.
Which integral expression will give the volume of the biscuit?

(A) $\pi\int_0^{0\cdot6}\left[\cos\left(\frac{3}{5}y\right)+1\right]^2 dy$

(B) $\pi\int_0^{0\cdot6}\left[\cos\left(\frac{5}{3}y\right)+1\right]^2 dy$

(C) $\pi\int_0^{0\cdot6\pi}\left[\cos\left(\frac{3}{5}y\right)+1\right]^2 dy$

(D) $\pi\int_0^{0\cdot6\pi}\left[\cos\left(\frac{5}{3}y\right)+1\right]^2 dy$

9 What is the value of $\lim_{n\to\infty}\left(n\sin\frac{\pi}{n}\right)$?

(A) $-\infty$

(B) 0

(C) π

(D) ∞

10 What is the x-intercept of the normal to the parabola $x^2 = 4ay$ at the point $(2ap, ap^2)$ on the parabola?

(A) $ap(p^2 + 1)$

(B) $ap(p^2 + 2)$

(C) ap^2

(D) $-ap^2$

BLANK PAGE

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a NEW writing booklet. Extra pages are available

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Start a NEW Writing Booklet

(a) Evaluate $\int_0^1 \frac{dx}{\sqrt{2-x^2}}$ **2**

(b) Find the acute angle between the lines $y=\sqrt{3}x-2$ and $y=-\sqrt{3}x+1$. **2**

(c) The point $(-6t, 9t^2)$, where t is a variable, lies on a curve. **2**
Find the Cartesian equation of the curve.

(d) Use the substitution $u=x^4+2$ to evaluate $\int_0^1 \frac{x^7}{(x^4+2)^2}\,dx$, **3**
leaving your answer in the form $p\ln q+r$.

(e) Find $\frac{d}{dx}(x^2\tan^{-1}x)$. **2**

(f) (i) Find a general solution of the equation **3**

$$\cos\left(3x-\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$

(ii) Hence, find the smallest solution of this equation which is greater than 5π. **1**

Question 12 (15 Marks) Start a NEW Writing Booklet

(a) The cubic equation $x^3+px^2+qx+r=0$, where p, q and r are real, has roots α, β and γ.

(i) Given that $\alpha+\beta+\gamma=4$ and $\alpha^2+\beta^2+\gamma^2=20$, find the values of p and q. **2**

(ii) Given further that one root is 4, find the value of r. **1**

(b) (i) Show that $(2\sin x+\cos x)^2$ can be written in the form **2**

$$\frac{5}{2}+2\sin 2x-\frac{3}{2}\cos 2x$$

(ii) Find the exact volume when the graph of $y=2\sin x+\cos x$ for $0\le x\le\frac{\pi}{2}$ is rotated about the x-axis. **2**

(c) A particle moves in simple harmonic motion about a fixed origin O with a period of $\frac{2\pi}{5}$ seconds. Initially, $x=1$ and $\dot{x}=-5\sqrt{3}$.

(i) Find x as a function of t. **3**

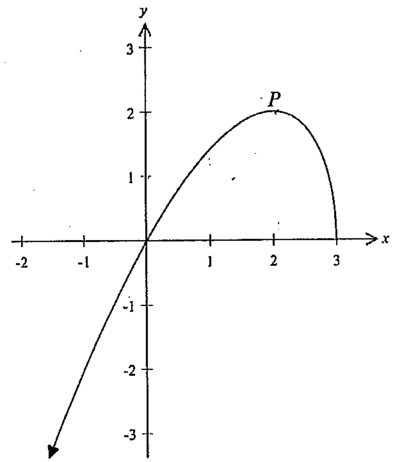
(ii) Find the first time that the particle passes back through $x=1$. **2**

(d) Solve the inequality $\frac{x}{(x+1)(x-2)}\le-\frac{1}{2}$ **3**

End of Question 12

Question 13 (15 Marks) Start a NEW Writing Booklet

(a) The diagram below shows the graph of the curve $y = x\sqrt{3-x}$.
The point P has coordinates (2, 2) and is a stationary point.
The function $f(x)$ is defined by $f(x) = x\sqrt{3-x}$, $x \leq 2$.



(i) On the same diagram, sketch the region satisfied by **3**

$$y \leq f(x) \text{ and } y \geq f^{-1}(x)$$

(ii) Explain why the area A of the shaded region in (i) is given by **2**

$$A = 2\int_0^2 \left(x\sqrt{3-x} - x\right)dx$$

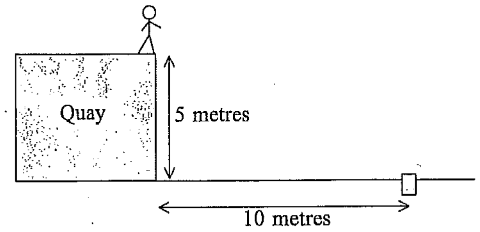
Do NOT attempt to evaluate this integral.

(b) Nine different pies are to be divided between three people so that each person gets an odd number of pies. **2**
Find the number of ways this can be done.

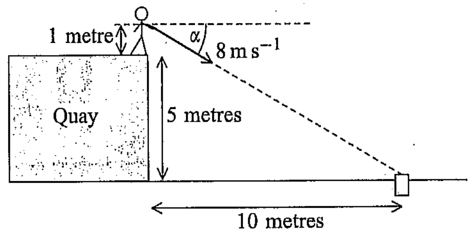
(c) Show that $\lim_{x \to 5} \dfrac{x^3 - x^2 - 100}{x - 5} = 65$ **2**

Question 13 continues on page 9

(d) A girl stands at the edge of a quay and sees a tin can floating in the water. The water level is 5 metres below the top of the quay and the can is at a horizontal distance of 10 metres from the quay, as shown in the diagram.



The girl decides to throw a stone at the can. She throws the stone from a height of 1 metre above the top of the quay. The initial velocity of the stone is 8 ms^{-1} at an angle α below the horizontal, so that the initial velocity of the stone is directed at the can, as shown in the diagram below.



Assume that the stone is a particle and that it experiences no air resistance as it moves. The equations of motion of the stone are

$x = 8t\cos\alpha$ and $y = 6 - 8t\sin\alpha - 4\cdot 9t^2$. (Do NOT prove this.)

(i) Find α. **2**
Leave your answer correct to the nearest degree.

(ii) Find the time that it takes for the stone to reach the level of the water. **2**
Leave your answer correct to 2 significant figures.

(iii) Find the distance between the stone and the can, when the stone hits the water. **2**
Leave your answer correct to 2 significant figures.

End of Question 13

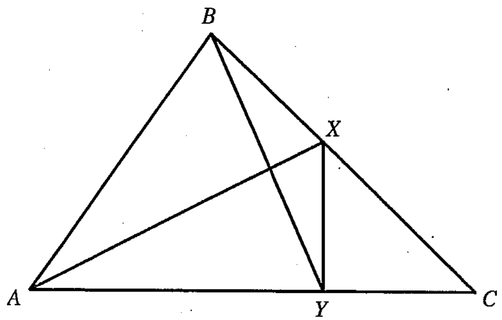
Question 14 (15 Marks) Start a NEW Writing Booklet

(a) (i) Show that $\dfrac{1}{(k+2)!}-\dfrac{k+1}{(k+3)!}=\dfrac{2}{(k+3)!}$, where k is an integer. **1**

(ii) Prove by induction that, for all positive integers n, **3**

$$\sum_{r=1}^{n}\frac{r\times 2^{r}}{(r+2)!}=1-\frac{2^{n+1}}{(n+2)!}$$

(b) X and Y are points on the sides BC and AC of a triangle ABC respectively such that $\angle AXC=\angle BYC$ and $BX=XY$.



Copy or trace the diagram into your answer booklet.

(i) Show that $\angle XAC=\angle YBC$. **2**

(ii) Hence, explain why $ABXY$ is a cyclic quadrilateral. **1**

(iii) Prove that AX bisects the angle $\angle BAC$. **2**

Question 14 continues on page 11

(c) If $\dfrac{dx}{dt}=-2(x-6)^{\frac{1}{2}}$, and $x=70$ when $t=0$, find x as a function of t. **3**

(d) Liquid fuel is stored in a tank. At time t minutes, the depth of fuel in the tank is x cm. Initially there is a depth of 70 cm of fuel in the tank.
There is a tap 6 cm above the bottom of the tank.
The flow of fuel out of the tank is modeled by the differential equation

$$\frac{dx}{dt}=-2(x-6)^{\frac{1}{2}}$$

(i) Explain what happens when $x=6$. **1**

(ii) Find how long it will take for the depth of fuel to fall from 70 cm to 22 cm. **2**

End of paper

SYDNEY BOYS HIGH SCHOOL

MOORE PARK, SURRY HILLS

2013

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Sample Solutions

Section I

1 What is the expression $\dfrac{2\tan A}{1+\tan^2 A}$ equal to?

(A) $\cos 2A$

(B) $\sin 2A$

(C) $\tan 2A$

(D) $\cot 2A$

If $t=\tan\dfrac{\theta}{2}$ then $\sin\theta=\dfrac{2t}{1+t^2}$;

2 The polynomial, $p(x)$ is defined by $p(x)=x^3-x^2+x+3$.

What is the remainder when $p(x)$ is divided by $(x-1)$?

(A) 0

(B) 2

(C) 3

(D) 4

Remainder $=p(1)=1-1+1+3=4$

3 What is the domain and range of $y=2\sin^{-1}\left(\dfrac{x}{3}\right)$?

(A) $|x|\le 3,\ |y|\le \pi.$

(B) $|x|\le 1,\ |y|\le 3.$

(C) $|x|\le 1,\ |y|\le \pi.$

(D) $|x|\le 3,\ |y|\le 2.$

$-\dfrac{\pi}{2}\le\dfrac{y}{2}=\sin^{-1}\left(\dfrac{x}{3}\right)\le\dfrac{\pi}{2}\Rightarrow -\pi\le y\le\pi$

$-1\le\dfrac{x}{3}\le 1\Rightarrow -3\le x\le 3$

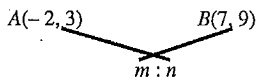
4 What ratio does the point $P(10, 11)$ divide the interval AB, where $A(-2,3)$ and $B(7,9)$?

(A) $1:4$

(B) $4:-1$

(C) $1:-4$

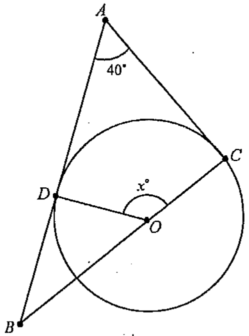
(D) $4:1$



$10=\dfrac{-2n+7m}{m+n}\Rightarrow 10m+10n=-2n+7m$

$\therefore 12n=-3m\Rightarrow\dfrac{m}{n}=-4$

5 In the figure below, a circle with centre O is tangent to AB at point D and tangent to AC at point C.



If $\angle A = 40°$, what is the value of x?

(A) 140
(B) 145
(C) 150
(D) 155

$\angle ADO = \angle ACO = 90°$ (radius and tangent)
$\therefore x + 2\times 90 + 40 = 360$ (angle sum $ADOC$)
$\therefore x = 140$

6 The function $f(x) = \sin x - \frac{2}{3}x$ has a real root close to $x = 1\cdot5$

Let $x = 1\cdot5$ be a first approximation to the root.
What is the second approximation to the root using Newton's method?

(A) 1·495
(B) 1·496
(C) 1·503
(D) 1·504

$f'(x) = \cos x - \frac{2}{3}$

$x_1 = 1\cdot5 - \frac{f(1\cdot5)}{f'(1\cdot5)}$

$= 1\cdot5 - \frac{\sin 1\cdot5 - \frac{2}{3}\times\frac{3}{2}}{\cos 1\cdot5 - \frac{2}{3}}$

$= 1\cdot496$

7 A test is administered with 15 questions. Students are allowed to answer any ten.
How many choices of ten questions are there?

(A) 150
(B) 250
(C) 3003
(D) 3000

${}^{15}C_{10} = 3003$

8 The graph of $f(x) = 0\cdot6\cos^{-1}(x-1)$, defines a curve that, when rotated about the y-axis, will produce a solid that is to be the shape and size of a new biscuit.
Which integral expression will give the volume of the biscuit?

(A) $\pi\int_0^{0\cdot6}\left[\cos\left(\frac{3}{5}y\right)+1\right]^2 dy$

(B) $\pi\int_0^{0\cdot6}\left[\cos\left(\frac{5}{3}y\right)+1\right]^2 dy$

(C) $\pi\int_0^{0\cdot6\pi}\left[\cos\left(\frac{3}{5}y\right)+1\right]^2 dy$

(D) $\pi\int_0^{0\cdot6\pi}\left[\cos\left(\frac{5}{3}y\right)+1\right]^2 dy$

$y = \frac{3}{5}\cos^{-1}(x-1)$

$\therefore x = \cos\left(\frac{5}{3}y\right)+1$

$0 \le \frac{5}{3}y = \cos^{-1}(x-1) \le \pi$

$\therefore 0 \le y \le \frac{3}{5}\pi$

9 What is the value of $\lim_{n\to\infty}\left(n\sin\frac{\pi}{n}\right)$?

(A) $-\infty$
(B) 0
(C) π
(D) ∞

$\lim_{n\to\infty}\left(n\sin\frac{\pi}{n}\right) = \lim_{n\to\infty}\left(\frac{\sin\frac{\pi}{n}}{\frac{1}{n}}\right)$

$= \lim_{u\to0}\left(\frac{\sin\pi u}{u}\right) \quad \left[u = \frac{1}{n}\right]$

$= \pi$

10 What is the x-intercept of the normal to the parabola $x^2 = 4ay$ at the point $(2ap, ap^2)$ on the parabola?

(A) $ap(p^2 + 1)$
(B) $ap(p^2 + 2)$
(C) ap^2
(D) $-ap^2$

$x + py = 2ap + ap^3$
$\therefore y = 0,\ x = 2ap + ap^3$

Section II

Question 11

(a) Evaluate $\int_0^1 \frac{dx}{\sqrt{2-x^2}}$ **2**

$$\int_0^1 \frac{dx}{\sqrt{2-x^2}} = \left[\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^1$$
$$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$
$$= \frac{\pi}{4}$$

(b) Find the acute angle between the lines $y=\sqrt{3}x-2$ and $y=-\sqrt{3}x+1$. **2**

$m_1=\sqrt{3},\ m_2=-\sqrt{3}$

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$$
$$= \left|\frac{2\sqrt{3}}{1-3}\right|$$
$$= \sqrt{3}$$
$$\theta = 60°$$

Accept an answer of $\theta=\frac{\pi}{3}$.

(c) The point $(-6t, 9t^2)$, where t is a variable, lies on a curve. **2**

Find the Cartesian equation of the curve.

$x=-6t \Rightarrow t=-\tfrac{1}{6}x$

$\therefore y=9t^2=9\left(-\tfrac{1}{6}x\right)^2$

$\therefore x^2=4y$

(d) Use the substitution $u=x^4+2$ to evaluate $\int_0^1 \frac{x^7}{(x^4+2)^2}\,dx$, **3**

leaving your answer in the form $p\ln q+r$.

$u=x^4+2 \Rightarrow du=4x^3dx$

$x=0,\ u=2$

$x=1,\ u=3$

$$\int_0^1 \frac{x^7}{(x^4+2)^2}\,dx = \frac{1}{4}\int_0^1 \frac{x^4(4x^3dx)}{(x^4+2)^2} = \frac{1}{4}\int_2^3 \frac{u-2}{u^2}\,du = \frac{1}{4}\int_2^3\left(\frac{1}{u}-\frac{2}{u^2}\right)du$$
$$= \frac{1}{4}\left[\ln u+\frac{2}{u}\right]_2^3 = \frac{1}{4}\left[\left(\ln 3+\frac{2}{3}\right)-\left(\ln 2+\frac{2}{2}\right)\right] = \frac{1}{4}\left(\ln\frac{3}{2}-\frac{1}{3}\right)$$
$$= \frac{1}{4}\ln\frac{3}{2}-\frac{1}{12}$$

Question 11 (continued)

(e) Find $\frac{d}{dx}\left(x^2\tan^{-1}x\right)$. **2**

$$\frac{d}{dx}\left(x^2\tan^{-1}x\right) = x^2\times\frac{1}{1+x^2}+2x\tan^{-1}x$$
$$= \frac{x^2}{1+x^2}+2x\tan^{-1}x$$

(f) (i) Find a general solution of the equation **3**

$$\cos\left(3x-\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$

$$3x-\frac{\pi}{6}=2n\pi\pm\cos^{-1}\left(\frac{\sqrt{3}}{2}\right),\ n\in\mathbb{Z}$$
$$\therefore 3x=2n\pi\pm\frac{\pi}{6}+\frac{\pi}{6}$$
$$=2n\pi+\frac{\pi}{3},\ 2n\pi$$
$$\therefore x=\frac{2}{3}n\pi+\frac{\pi}{9},\ \frac{2}{3}n\pi$$

(ii) Hence, find the smallest solution of this equation which is greater than 5π. **1**

$\frac{2}{3}n\pi+\frac{\pi}{9}\ge 5\pi \Rightarrow n\ge\frac{22}{3}$

$\therefore n=8$

Smallest angle $=\frac{2}{3}\times 8\pi=\frac{16}{3}\pi$

Question 12

(a) The cubic equation $x^3 + px^2 + qx + r = 0$, where p, q and r are real, has roots α, β and γ.

(i) Given that $\alpha+\beta+\gamma = 4$ and $\alpha^2+\beta^2+\gamma^2 = 20$, find the values of p and q. **2**

$p = -(\alpha+\beta+\gamma)$

$= -4$

$q = \alpha\beta+\beta\gamma+\gamma\alpha$

$\alpha^2+\beta^2+\gamma^2 = 20$

$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$= 16 - 2q$

$\therefore q = -2$

$\therefore p = -4, q = -2$

(ii) Given further that one root is 4, find the value of r. **1**

$\therefore x^3 - 4x^2 - 2x + r = 0$

Substitute $x = 4$: $4^3 - 4\times16 - 2\times4 + r = 0$

$\therefore r = 8$

(b) (i) Show that $(2\sin x + \cos x)^2$ can be written in the form **2**

$$\frac{5}{2} + 2\sin 2x - \frac{3}{2}\cos 2x$$

$(2\sin x + \cos x)^2 = 2\times2\sin^2 x + 2\times2\sin x\cos x + \cos^2 x$

$= 2(1-\cos 2x) + 2\sin 2x + \frac{1}{2}(1+\cos 2x)$

$= \frac{5}{2} + 2\sin 2x - \frac{3}{2}\cos 2x$

(ii) Find the exact volume when the graph of $y = 2\sin x + \cos x$ for $0 \le x \le \frac{\pi}{2}$ is rotated about the x-axis. **2**

$\text{Volume} = \pi\int_0^{\frac{\pi}{2}} (2\sin x + \cos x)^2\, dx$

$= \pi\int_0^{\frac{\pi}{2}} \left(\frac{5}{2} + 2\sin 2x - \frac{3}{2}\cos 2x\right) dx$

$= \pi\left[\frac{5}{2}x - \cos 2x - \frac{3}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$

$= \pi\left[\left(\frac{5\pi}{4} - \cos\pi - \frac{3}{4}\sin\pi\right) - (-1)\right]$

$= \frac{5\pi^2}{4} + 2\pi$

(c) A particle moves in simple harmonic motion about a fixed origin O with a period of $\frac{2\pi}{5}$ seconds. Initially, $x = 1$ and $\dot{x} = -5\sqrt{3}$.

(i) Find x as a function of t. **3**

$T = \frac{2\pi}{5} = \frac{2\pi}{n} \Rightarrow n = 5$

Let $x = A\cos(5t+\varepsilon)$, where $A, \varepsilon > 0$.

$t = 0 \Rightarrow 1 = A\cos\varepsilon$ —(1)

$\dot{x} = -5A\sin(5t+\varepsilon)$

$\therefore -5\sqrt{3} = -5A\sin\varepsilon$

$t = 0 \Rightarrow A\sin\varepsilon = \sqrt{3}$ —(2)

Solving (1) and (2) gives $A = 2, \varepsilon = \frac{\pi}{3}$

$\therefore x = 2\cos\left(5t + \frac{\pi}{3}\right)$

(ii) Find the first time that the particle passes back through $x = 1$. **2**

Let the first time be T, where $T > 0$

$\therefore x(T) = x(0)$

$\therefore \cos\left(5T + \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$

$\therefore 5T + \frac{\pi}{3} = \frac{\pi}{3},\ 2\pi - \frac{\pi}{3}, \ldots$

$\therefore 5T + \frac{\pi}{3} = \frac{5\pi}{3}$ $(T > 0)$

$\therefore 5T = \frac{4\pi}{3}$

$\therefore T = \frac{4\pi}{15}$

Question 12 (continued)

(d) Solve the inequality $\frac{x}{(x+1)(x-2)} \le -\frac{1}{2}$ **3**

$x \ne -1, 2$

$2(x+1)^2(x-2)^2 \times \frac{x}{(x+1)(x-2)} \le -\frac{1}{2} \times 2(x+1)^2(x-2)^2$

$\therefore 2x(x+1)(x-2) \le -(x+1)^2(x-2)^2$

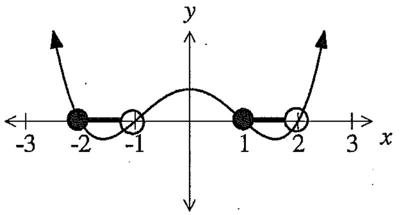
$\therefore (x+1)^2(x-2)^2 + 2x(x+1)(x-2) \le 0$

$\therefore (x+1)(x-2)[(x+1)(x-2)+2x] \le 0$

$\therefore (x+1)(x-2)(x^2+x-2) \le 0$

$\therefore (x+1)(x-2)(x-1)(x+2) \le 0$

$\therefore -2 \le x < -1, 1 \le x < 2$

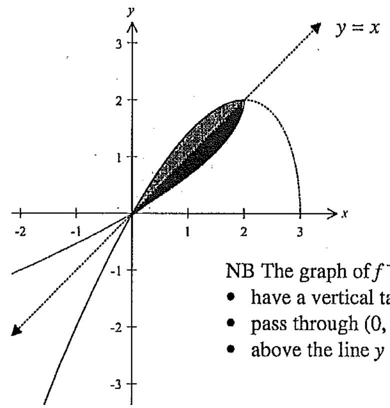


Question 13

(a) The diagram below shows the graph of the curve $y = x\sqrt{3-x}$.
The point P has coordinates (2, 2) and is a stationary point.
The function $f(x)$ is defined by $f(x) = x\sqrt{3-x}$, $x \le 2$.

(i) On the same diagram, sketch the region satisfied by **3**

$$y \le f(x) \text{ and } y \ge f^{-1}(x)$$



NB The graph of $f^{-1}(x)$, should
- have a vertical tangent at (2, 2).
- pass through (0, 0).
- above the line $y = x$ for $x < 0$

(ii) Explain why the area A of the shaded region in (i) is given by **2**

$$A = 2\int_0^2 \left(x\sqrt{3-x} - x\right) dx$$

Do NOT attempt to evaluate this integral.

$y = f^{-1}(x)$ is a reflection of $y = f(x)$ in the line $y = x$.
So the area between $y = f(x)$ and $y = x$ for $0 \le x \le 2$ is the same as the area between $y = f^{-1}(x)$ and $y = x$ for $0 \le x \le 2$.

The area between $y = f(x)$ and $y = x$ for $0 \le x \le 2 = \int_0^2 \left(x\sqrt{3-x} - x\right) dx$

$\therefore$ Shaded area $= 2\int_0^2 \left(x\sqrt{3-x} - x\right) dx$

(b) Nine different pies are to be divided between three people so that each person gets an odd number of pies. **2**
Find the number of ways this can be done.

There are 3 cases

1. (1, 3, 5) with 3! different arrangements amongst the 3 people. i.e. the first person gets 1 pie, the second person gets 3 pies and the third gets 5 pies. This could be re-arranged in 3! ways.
$\therefore {}^9C_1 \times {}^8C_3 \times 3!$

2. (1, 1, 7) with 3 different arrangements amongst the 3 people.
$\therefore {}^9C_1 \times {}^8C_1 \times 3$

3. (3, 3, 3) with 1 possibilities
$\therefore {}^9C_3 \times {}^6C_3$

$\therefore {}^9C_1 \times {}^8C_3 \times 3! + {}^9C_1 \times {}^8C_1 \times 3 + {}^9C_3 \times {}^6C_3 = 4920$

(c) Show that $\lim\limits_{x \to 5} \dfrac{x^3 - x^2 - 100}{x - 5} = 65$ **2**

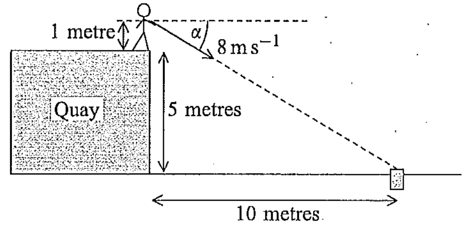
Let $f(x) = x^3 - x^2$.
$f'(x) = 3x^2 - 2x$
$f(5) = 125 - 25 = 100$

$$\begin{aligned}\therefore \lim_{x \to 5} \frac{x^3 - x^2 - 100}{x - 5} &= \lim_{x \to 5} \frac{f(x) - f(5)}{x - 5} \\ &= f'(5) \\ &= 3 \times 25 - 2 \times 5 \\ &= 65\end{aligned}$$

Alternatively:

$$\begin{aligned}\therefore \lim_{x \to 5} \frac{x^3 - x^2 - 100}{x - 5} &= \lim_{x \to 5} \frac{x^3 - 125 - x^2 + 25}{x - 5} \\ &= \lim_{x \to 5} \frac{(x^3 - 125) - (x^2 - 25)}{x - 5} \\ &= \lim_{x \to 5} \left[(x^2 + 5x + 25) - (x + 5)\right] \\ &= 3 \times 25 - 10 \\ &= 65\end{aligned}$$

(d)



$x = 8t\cos\alpha$ and $y = 6 - 8t\sin\alpha - 4 \cdot 9t^2$. (Do NOT prove this.)

(i) Find α. **2**

Given that the initial velocity of the stone is directed at the can then
$\tan\alpha = \dfrac{6}{10} = \dfrac{3}{5}$
$\therefore \alpha = 31°$

(ii) Find the time that it takes for the stone to reach the level of the water. **2**

The stone will reach the water when $y = 0$
$\therefore 6 - 8t\sin\alpha - 4 \cdot 9t^2 = 0$
$\therefore 4 \cdot 9t^2 + 8t\sin\alpha - 6 = 0$
$\therefore 4 \cdot 9t^2 + 8t \times \dfrac{3}{\sqrt{34}} - 6 = 0 \quad \left[\tan\alpha = \dfrac{3}{5} \Rightarrow \sin\alpha = \dfrac{3}{\sqrt{34}}\right]$
$\therefore t = 0 \cdot 76359, -1 \cdot 60$
$\therefore t = 0 \cdot 76$
The stone will reach water level after $0 \cdot 76$ seconds.

(iii) Find the distance between the stone and the can, when the stone hits the water. **2**

The horizontal difference, d m, where $d = 10 - 8t\cos\alpha$
$\therefore d = 10 - 8 \times 0 \cdot 76 \times \dfrac{5}{\sqrt{34}} = 4 \cdot 8$
So the distance between the stone and the can is $4 \cdot 8$ m

Question 14

(a) (i) Show that $\dfrac{1}{(k+2)!}-\dfrac{k+1}{(k+3)!}=\dfrac{2}{(k+3)!}$, where k is an integer. **1**

$$\text{LHS}=\frac{1}{(k+2)!}-\frac{k+1}{(k+3)!}$$
$$=\frac{k+3-(k+1)}{(k+3)!}$$
$$=\frac{2}{(k+3)!}=\text{RHS}$$

(ii) Prove by induction that, for all positive integers n, **3**

$$\sum_{r=1}^{n}\frac{r\times 2^r}{(r+2)!}=1-\frac{2^{n+1}}{(n+2)!}$$

Test $n = 1$:

$$\text{LHS}=\sum_{r=1}^{1}\frac{r\times 2^r}{(r+2)!}=\frac{1\times 2^1}{(1+2)!}=\frac{2}{6}=\frac{1}{3}$$
$$\text{RHS}=1-\frac{2^2}{(1+2)!}=1-\frac{4}{6}=\frac{1}{3}$$

True for $n = 1$.

Assume true for $n = k$ i.e. $\displaystyle\sum_{r=1}^{k}\frac{r\times 2^r}{(r+2)!}=1-\frac{2^{k+1}}{(k+2)!}$

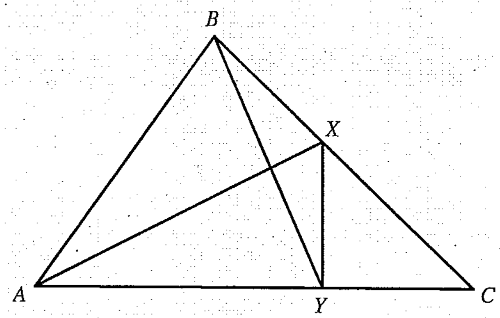
Need to prove true for $n = k + 1$ i.e. $\displaystyle\sum_{r=1}^{k+1}\frac{r\times 2^r}{(r+2)!}=1-\frac{2^{k+2}}{(k+3)!}$

$$\text{LHS}=\sum_{r=1}^{k+1}\frac{r\times 2^r}{(r+2)!}$$
$$=\sum_{r=1}^{k}\frac{r\times 2^r}{(r+2)!}+\frac{(k+1)\times 2^{k+1}}{(k+3)!}$$
$$=1-\frac{2^{k+1}}{(k+2)!}+\frac{(k+1)\times 2^{k+1}}{(k+3)!}\quad\text{[By assumption]}$$
$$=1-2^{k+1}\left[\frac{1}{(k+2)!}-\frac{k+1}{(k+3)!}\right]$$
$$=1-2^{k+1}\times\frac{2}{(k+3)!}\quad\text{[From (a) (i)]}$$
$$=1-\frac{2^{k+2}}{(k+3)!}$$
$$=\text{RHS}$$

So the case $n = k + 1$ is true if the case $n = k$ is true.
So by the principle of mathematical induction, the formula is true for all positive integers.

Question 14 (continued)

(b) X and Y are points on the sides BC and AC of a triangle ABC respectively such that $\angle AXC=\angle BYC$ and $BX=XY$.



(i) Show that $\angle XAC=\angle YBC$. **2**

$\angle XAC+\angle AXC+\angle XCA=180°$ [angle sum ΔAXC]

$\therefore \angle XAC=180°-(\angle AXC+\angle XCA)$

$=180°-\angle BYC-\angle BCY$ [data: $\angle AXC=\angle BYC$]

$=\angle YBC$ [angle sum ΔYBC]

(ii) Hence, explain why $ABXY$ is a cyclic quadrilateral. **1**

As $\angle XAC=\angle YBC$ then $\angle XAY=\angle YBX$

So $ABXY$ is a cyclic quadrilateral (converse of angles in the same segment)

(iii) Prove that AX bisects the angle $\angle BAC$. **2**

$\angle BYX=\angle BAX$ (angles in same segment)

Similarly, $\angle XBY=\angle XAY$.

Now $\angle XBY=\angle BYX$ (equal angles opp. equal sides)

$\therefore \angle XAY=\angle BAX$

i.e. AX bisects $\angle BAC$

(c) If $\dfrac{dx}{dt}=-2(x-6)^{\frac{1}{2}}$, and $x = 70$ when $t = 0$, find x as a function of t. **3**

$$\frac{dt}{dx}=-\frac{1}{2}(x-6)^{-\frac{1}{2}}$$
$$\therefore t=-(x-6)^{\frac{1}{2}}+C$$

Substitute $x = 70$, $t = 0$

$$\therefore 0=-(70-6)^{\frac{1}{2}}+C$$
$$\therefore C=8$$
$$\therefore t=-(x-6)^{\frac{1}{2}}+8$$
$$\therefore (x-6)^{\frac{1}{2}}=8-t$$
$$\therefore x=6+(8-t)^2$$

Question 14 (continued)

(d) Liquid fuel is stored in a tank. At time t minutes, the depth of fuel in the tank is x cm. Initially there is a depth of 70 cm of fuel in the tank. There is a tap 6 cm above the bottom of the tank. The flow of fuel out of the tank is modeled by the differential equation

$$\frac{dx}{dt} = -2(x-6)^{\frac{1}{2}}$$

(i) Explain what happens when $x = 6$. **1**

The fuel ceases to flow out the tap.

(ii) Find how long it will take for the depth of fuel to fall from 70 cm to 22 cm. **2**

From (c), $x = 6 + (8-t)^2$

Find t, when $x = 22$.

$22 = 6 + (8-t)^2$

$\therefore (t-8)^2 = 16$

$\therefore t = 8 \pm 4$

$\therefore t = 4 \qquad (0 \le t \le 8)$

It takes 4 seconds to fall to 22 cm.

End of solutions