SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2013

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 180 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- Show all necessary working in Questions 11–16

Total Marks - 100 Marks

Section I **10 Marks**

- Attempt Questions 1–10
- Allow about 15 minutes for this section.

Section II **90 Marks**

- Attempt Questions 11–16
- Allow about 2 hour 45 minutes for this section.

Examiner: *External Examiner*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\quad x > 0$

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1) Given $z = r(\cos\theta + i\sin\theta)$, then $\left|\frac{\bar{z}}{z^2}\right|$ equals

(A) r

(B) r^2

(C) $\frac{1}{r}$

(D) $-r$

2) If $1+i$ is a root of the polynomial $x^3 - 4x^2 + 6x - 4 = 0$. The other roots are:

(A) $1-i$ and -2

(B) $1+i$ and -2

(C) $1-i$ and 2

(D) $1+i$ and 2

3) If the polynomial equation $P(x) = 0$, has roots α, β, γ then the roots of the polynomial equation $P(3x+2) = 0$ are

(A) $\frac{\alpha}{3} - 2,\ \frac{\beta}{3} - 2,\ \frac{\gamma}{3} - 2$

(B) $\frac{\alpha-2}{3},\ \frac{\beta-2}{3},\ \frac{\gamma-2}{3}$

(C) $3\alpha + 2,\ 3\beta + 2,\ 3\gamma + 2$

(D) $\alpha + \frac{2}{3},\ \beta + \frac{2}{3},\ \gamma + \frac{2}{3}$

4) The gradient of the tangent to the curve $2x^3 - y^2 = 7$ at the point $(2, -3)$ is:

(A) -4

(B) -2

(C) 2

(D) 4

5) The area bounded by the parabola $x^2 = 4ay$ and the line $y = a$ is rotated about the line $y = a$. To find the volume of the resulting solid, the slicing technique is used.

The area of a typical slice is given by

(A) $\pi(a-y)^2$

(B) $\pi(a^2+y^2)$

(C) $\pi(a-x)^2$

(D) $\pi(a^2+x^2)$

6) The equation of the conic whose distance from the point $(1,0)$ is half its distance from the line $x = 4$ is given by:

(A) $\frac{x^2}{4} + \frac{y^2}{3} = 1$

(B) $\frac{x^2}{4} + \frac{y^2}{4} = 1$

(C) $\frac{x^2}{4} - \frac{y^2}{3} = 1$

(D) $\frac{x^2}{3} - \frac{y^2}{4} = 1$

7) The number of different arrangements of the letters of the word SERVICES which begin and end with letter S is:

(A) $\frac{8!}{2!}$

(B) $\frac{6!}{2!}$

(C) $\frac{6!}{(2!)^2}$

(D) $\frac{8!}{(2!)^2}$

8) Given the curve $y = f(x)$, then the curve $y = f(|x|)$ is represented by

(A) A reflection of $y = f(x)$ in the y-axis

(B) A reflection of $y = f(x)$ in the x-axis

(C) A reflection of $y = f(x)$ for $x \geq 0$ in the y-axis

(D) A reflection of $y = f(x)$ for $y \geq 0$ in the x-axis

9) Using the substitution $x = \pi - y$, the definite integral

$$\int_0^{\pi} x \sin x \,.\, dx$$

will simplify to:

(A) 0

(B) $\frac{\pi^2}{4}$

(C) $\frac{\pi}{2}\int_0^{\pi} \sin x \,.\, dx$

(D) $\int_0^{\pi} \sin x \,.\, dx$

10) Which of the following statements is false?

(A) $\int_{-3}^{3} x^3 e^{-x^2} .\, dx = 0$

(B) $\int_{-4}^{4} \frac{x^2}{x^2+4} .\, dx = 2\int_0^{4} \frac{x^2}{x^2+4} .\, dx$

(C) $\int_0^{\pi} \sin^4 \theta \,.\, d\theta > \int_0^{\pi} \sin 4\theta \,.\, d\theta$

(D) $\int_0^{1} x^4 .\, dx < \int_0^{1} x^5 .\, dx$

End of Section I

Section II **Free response questions**
Total marks – 90
Attempt Questions 11 – 16

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Find **2**

$$\int x \tan^{-1} x \,.\, dx$$

(b) Use completion of squares to evaluate **2**

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{15 - 4x - 4x^2}}$$

(c) Use the substitution $t = \tan\frac{\theta}{2}$ to evaluate **2**

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{d\theta}{2\cot\frac{\theta}{2} - \sin\theta}$$

(d)

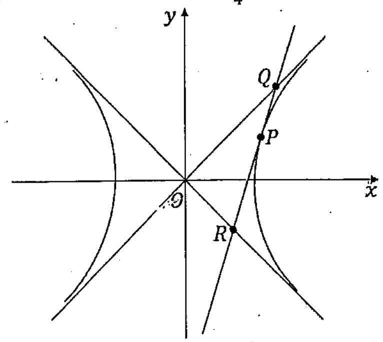
(i) Find the real number A, B and C such that **2**

$$\frac{3x^2 - x + 8}{(1-x)(x^2+1)} = \frac{A}{1-x} + \frac{Bx + C}{x^2+1}$$

(ii) Hence find **2**

$$\int \frac{3x^2 - x + 8}{(1-x)(x^2+1)} .\, dx$$

(e) $P(2\sec\theta, \tan\theta)$ is a point on the hyperbola $H: \frac{x^2}{4} - y^2 = 1$



If the tangent at P cuts the asymptotes at Q and R as shown in the figure above, find the coordinates of Q and R in terms of θ. Show that P is the mid-point of QR. **3**

(f) Find a and b where a and b are real numbers if $(a + ib)^2 = 21 - 20i$. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Given that α, β and γ are the roots of the equation

$$2x^3 + 3x^2 - 5x + 8 = 0$$ **2**

find the polynomial equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$.

(b) Let $z = 3(\cos\theta + i\sin\theta)$

(i) Find $\overline{1 - z}$ **1**

(ii) Express the imaginary part of $\frac{1}{1-z}$ in terms of θ. **1**

(c) Sketch the region for z in the Argand plane defined by:

$$|z - 1 + i| < 2 \quad \text{and} \quad -\frac{\pi}{4} \le \arg(z - 1 + i) \le \frac{5\pi}{4}$$ **2**

(d) If $z_1 = 2i$ and $z_2 = 1 + 3i$ are two complex numbers, describe the loci of z such that:

$z = z_1 + k(z_2 - z_1)$, when

(i) $k = 1$ **1**

(ii) $0 < k < 1$ **1**

(iii) k is any real number. **1**

(e) The polynomial $P(x) = x^3 + ax + b$ has zeroes α, β and $2(\alpha - \beta)$.

(i) Show that $a = -13\alpha^2$ and $b = 12\alpha^3$. **2**

(ii) Deduce that the zeroes of $P(x)$ are $-\frac{13b}{12a}, -\frac{13b}{4a}$ and $\frac{13b}{3a}$. **2**

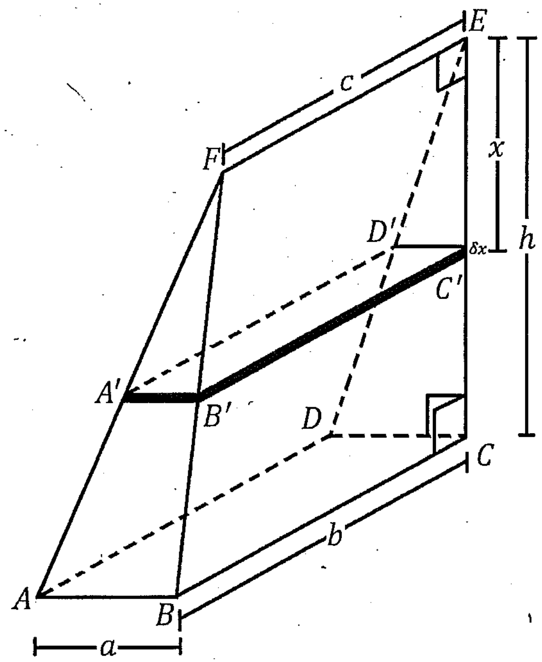
(f) Given $1, \omega$ and ω^2 are the cube roots of unity and each are represented by the points A_1, A_2 and A_3 respectively on an Argand diagram.

Find the value of $A_1A_2 \times A_1A_3$, where 'A_1A_2' represents the length A_1A_2. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Consider solid $ABCDEF$ whose height is h, and whose base is a rectangle $ABCD$, where $AB = a,\ BC = b$ and the top edge $EF = c$.



Consider a rectangular slice $A'B'C'D'$ (parallel to the base $ABCD$) which is x units from the top edge with width δx

(Note: $B'C' || BC$ and $A'B' || AB$)

(i) Show that the volume δV of the slice is given by

$$\delta V = \left(\frac{a}{h}x\right)\left(c + \frac{b-c}{h}x\right)\delta x$$ **3**

(ii) Hence show that the volume of the solid $ABCDEF$ is

$$\frac{ha}{6}(2b + c)$$ **2**

Question 13 continues on the next page

(b) The acceleration of a motor car on a straight road is $a - bv^2$ where v is the velocity; a and b are positive constants. Let x be the displacement of the motor car from its starting point at time t. Initially, $x = 0, v = 0$.

(i) Show that at time t, the velocity is given by **3**

$$v = \sqrt{\frac{a}{b}}(1 - e^{-2bx})^{1/2}$$

(ii) Show that the velocity of the motor car has a limiting value of V where V is $\sqrt{\frac{a}{b}}$. **1**

(iii) The velocity p is attained in a displacement l after starting and the velocity q is attained after a further displacement of l where p and q are positive constants and $0 < q < \sqrt{2}p$. Show that **3**

$$V = \frac{p^2}{\sqrt{2p^2 - q^2}}$$

(c) The region bounded by $y = 0, y = e^x, x = 0$ and $x = 2$ is revolved about the line $y = 0$. Find the volume of the resulting solid by using the ***cylindrical shell method.*** **3**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Given that $y = \cos^n x \sin nx$

(i) Show that **3**

$$\frac{dy}{dx} = 2n\cos^n x \cos nx - n\cos^{n-1} x \cos(n-1)x$$

Hence show that

$$2n\int \cos^n x \cos nx\, dx - n\int \cos^{n-1} x \cos(n-1)x\, dx = \cos^n x \sin nx + C$$

(ii) Using the result of (i), show that **2**

$$\int_0^{\frac{\pi}{2}} \cos^n x \cos nx\,.dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos(n-1)x\,.dx$$

(iii) Let **2**

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x \cos nx\,.dx$$

Find I_1, hence find I_8.

(iv) Hence find **2**

$$\int_0^{\frac{\pi}{2}} \sin^8 x \cos 8x\,.dx$$

(b) There are eleven men waiting for their turn in a barber shop. Three particular men are A, B and C. There is a row of 11 seats for the customers. Find the number of ways of arranging them so that no two of A, B and C are adjacent. **3**

(c) The curve C has equation $y = \frac{(x-1)^2}{x+2}$

(i) Obtain the equations of the asymptotes of the curve C. **1**

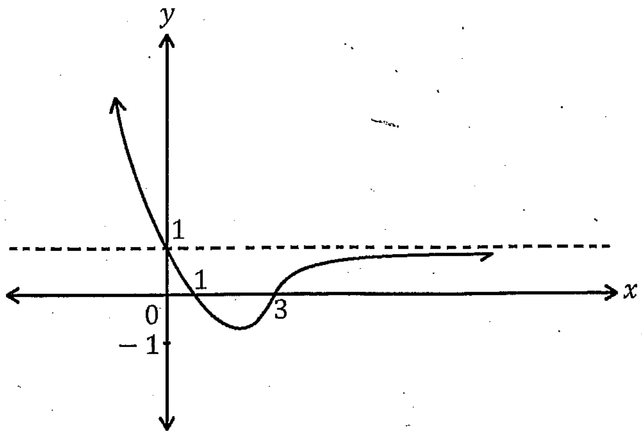
(ii) On the same diagram, draw a sketch of C and of the curve with equation $y = -\frac{1}{x}$. **2**

Deduce the number of real roots of the equation $x^3 - 2x^2 + 2x + 2 = 0$.

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) The graph of $y = f(x)$ is shown below. On separate axes, neatly sketch:



(i) $y = [f(x)]^2$ **1**

(ii) $y = f(1-x)$ **1**

(iii) $y = \ln[f(x)]$ **1**

(b)

(i) Prove **2**

$$\frac{\cos\theta + i\sin\theta - 1}{\cos\theta + i\sin\theta + 1} = i\tan\frac{\theta}{2}$$

(ii) Find the five roots of the equation $\omega^5 = 1$ and express your answers in the form of $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$. **2**

(iii) Hence show that the roots of the equation **2**

$$\left(\frac{2+z}{2-z}\right)^5 = 1 \qquad (*)$$

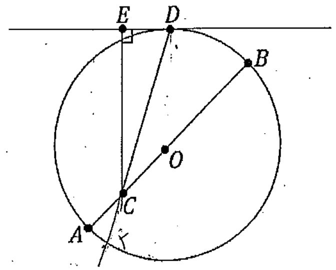
are $2i\tan\left(\frac{k\pi}{5}\right)$, where $k = 0, \pm1, \pm2$.

(iv) By expressing the equation in part (iii) $(*)$ in the form of $z^5 + mz^3 + nz = 0$, show that **2**

$$\tan\frac{\pi}{5}\tan\frac{2\pi}{5} = \sqrt{5}$$

Question 15 continues on the next page

(c)



AB is a diameter of the circle and O is the centre. C is a point on AB and D is a point on the circle. DE is the tangent of the circle at D and $CE \perp DE$. Copy the diagram in your booklet.

Extend DC to intersect the circle at F such that DF is a chord of the circle. Similarly, join DO and extend DO to intersect the circle at G. DG now is another diameter for the circle $BDFG$.

(i) Prove ΔCED is similar to ΔDFG. **2**

(ii) Hence or otherwise, prove that $AB \times CE = AC \times CB + CD^2$. **2**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) It is given that

$$\frac{r+1}{r-1} = 1 + \frac{2}{r} + \frac{2}{r^2} + \cdots + \frac{2}{r^n} + \frac{2}{r^n(r-1)}$$

(i) Show that **2**

$$\sum_{r=2}^{n} [\ln(r+1) - \ln(r-1)] = \ln\frac{n(n+1)}{2}$$

(ii) Hence prove by mathematical induction that **3**

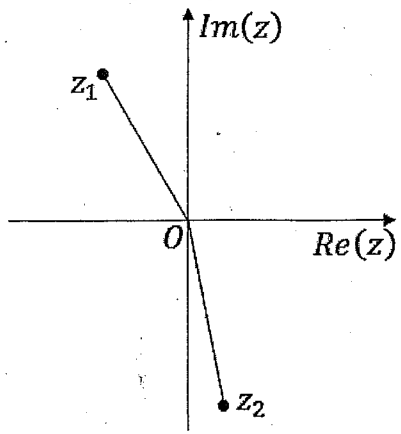
$$\sum_{r=2}^{n} \ln\left[1 + \frac{2}{r} + \frac{2}{r^2} + \cdots + \frac{2}{r^n} + \frac{2}{r^n(r-1)}\right] = \ln\frac{n(n+1)}{2} \quad \text{for } n = 2, 3, 4, \ldots$$

Question 16 continues on the next page

(b) Given that z_1, z_2 and z_3 are three complex numbers which satisfy

$$z_1 + z_2 + z_3 = 0$$

(i) z_1 and z_2 are indicated in the Argand diagram as shown in the figure below.



Copy the diagram in your booklet and sketch on the same diagram a possible location for z_3. Explain your decision. **1**

(ii) Given that the arguments of z_1, z_2 and z_3 are α, β and γ respectively, and their moduli are $1, k$ and $2-k$ respectively, where $0 < k < 2$. **2**
Express z_1, z_2 and z_3 in mod-arg form.

(iii) Prove that **1**

$$\begin{cases} \cos\alpha + k\cos\beta + (2-k)\cos\gamma = 0 \\ \sin\alpha + k\sin\beta + (2-k)\sin\gamma = 0 \end{cases}$$

(iv) From (iii), by eliminating α or otherwise, prove that **3**

$$k^2 + (2-k)^2 + 2k(2-k)\cos(\beta - \gamma) = 1$$

(v) By considering $|\cos(\beta - \gamma)| \le 1$, find the range of values of k. **3**

End of paper

Question (11)

(a) $\int x \tan^{-1}x \, dx$

$= \int \tan^{-1}x \, \frac{d}{dx}\left(\frac{1}{2}x^2\right) dx$

$= \frac{1}{2}x^2 \tan^{-1}x - \frac{1}{2}\int \frac{x^2 \, dx}{1+x^2}$

$= \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2}\int \frac{(1+x^2)-1}{1+x^2} \, dx.$

$= \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2}\left[x - \tan^{-1}x\right]$

$= \frac{1}{2}\left(x^2 \tan^{-1}x + \tan^{-1}x - x\right) + c$

(b) $\int_{-1/2}^{1/2} \frac{dx}{\sqrt{15-4x-4x^2}}$

$= \int_{-1/2}^{1/2} \frac{dx}{\sqrt{-4\left(x^2+x-\frac{15}{4}\right)}}$

$= \int_{-1/2}^{1/2} \frac{dx}{\sqrt{-4\left(x+\frac{1}{2}\right)^2+4^2}}$

$= \frac{1}{2}\int_{-1/2}^{1/2} \frac{dx}{\sqrt{4-\left(x+\frac{1}{2}\right)^2}}$

(b) $= \frac{1}{2}\left[\sin^{-1}\left(\frac{x+\frac{1}{2}}{2}\right)\right]_{-1/2}^{1/2}$

$= \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{12}.$

(c) $\int_{\pi/2}^{2\pi/3} \frac{d\theta}{2\cot\frac{\theta}{2} - \sin\theta}$

$\boxed{\text{let } t = \tan\frac{\theta}{2} \quad d\theta = \frac{2\,dt}{1+t^2}, \quad -\pi < \theta < \pi}$

$= \int_1^{\sqrt{3}} \left(\frac{2\,dt}{1+t^2}\right) \times \frac{1}{\left(\frac{2}{t} - \frac{2t}{1+t^2}\right)}$

$= \cancel{2}\int_1^{\sqrt{3}} \frac{dt}{\cancel{1+t^2}} \times \frac{t\cancel{(1+t^2)}}{\cancel{2}+2\cancel{t^2}-2\cancel{t^2}} = \int_1^{\sqrt{3}} t\,dt$

$= \left[\frac{1}{2}t^2\right]_1^{\sqrt{3}} = 1$

(d) $3x^2 - x + 8 = A(x^2+1) + (Bx+C)(1-x)$

When $x = 1$, $2A = 10 \Rightarrow A = 5$

$A - B = 3 \therefore B = 2, \quad A + C = 8 \Rightarrow C = 3$

$\therefore \int \frac{(3x^2-x+8)\,dx}{(1-x)(x^2+1)} = 5\int \frac{dx}{1-x} + \int \frac{(2x+3)\,dx}{x^2+1}$

$= -5\ln(1-x) + \ln(x^2+1) + 3\tan^{-1}x + C.$

YEAR 12 - EXTENSION 2 - MATHEMATICS - 2013 TRIAL

ANSWER SHEET

INSTRUCTIONS:

- Cross the box that indicates the correct answer

1	A	B	X	D
2	A	B	X	D
3	A	X	C	D
4	X	B	C	D
5	X	B	C	D
6	X	B	C	D
7	A	X	C	D
8	A	B	X	D
9	A	B	X	D
10	A	B	C	X

Question (11)

(f) $(a+ib)^2 = 21 - 20i$

$a^2 + i2ab - b^2 = 21 - 20i$

Equate real and imaginary pts

$$\begin{cases} a^2 - b^2 = 21 \\ 2ab = -20 \end{cases}$$

Now, $(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2$

$= 21^2 + 20^2$

$= 841$

$\therefore\ a^2 + b^2 = 29$

$a^2 - b^2 = 21$

$\therefore\ 2a^2 = 50,\ a^2 = 25$

$\therefore\ a = \pm 5.$

If $a = 5,\ b = -2$

$a = -5,\ b = 2.$

Question (11)

(e) $\dfrac{x^2}{4} - y^2 = 1,\quad P(2\sec\theta, \tan\theta).$

$a = 2,\ b = 1.$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sec^2\theta}{2\sec\theta\tan\theta}$$

$\therefore$ Equation of tgt. at P'

$$y - \tan\theta = \frac{\sec\theta}{2\tan\theta}(x - 2\sec\theta)$$

i.e

$(\sec\theta)x - (2\tan\theta)y = 2(\sec^2\theta - \tan^2\theta).$

$$\boxed{\left(\frac{\sec\theta}{2}\right)x - (\tan\theta)y = 1}$$

The asymptotos are $\begin{cases} y_1 = \frac{1}{2}x_1 \\ y_2 = -\frac{1}{2}x_2 \end{cases}$

(i) When $y_1 = \frac{x_1}{2}.$

$(\sec\theta)x_1 - (\tan\theta)x_1 = 2$

$\therefore\ x_1(\sec\theta - \tan\theta) = 2.$

$\therefore\ x_1 = \dfrac{2}{\sec\theta - \tan\theta},\ y_1 = \dfrac{1}{\sec\theta - \tan\theta}$

(ii) When $y_2 = \dfrac{-x_2}{2}.$

$(\sec\theta)x_2 + (\tan\theta)x_2 = 2$

$\therefore\ x_2 = \dfrac{2}{\sec\theta + \tan\theta}$

$y_2 = \dfrac{1}{\sec\theta + \tan\theta}$

Now, $\dfrac{x_1 + x_2}{2} = \dfrac{1}{\sec\theta - \tan\theta} + \dfrac{1}{\sec\theta + \tan\theta}$

$= \dfrac{2\sec\theta}{\sec^2\theta - \tan^2\theta}$

$= 2\sec\theta$

$$\frac{y_1 + y_2}{2} = \frac{(\sec\theta + \tan\theta) - (\sec\theta - \tan\theta)}{2(\sec^2\theta - \tan^2\theta)}$$

$= \tan\theta.$

$\therefore$ The mid-pt of QR is $(2\sec\theta, \tan\theta)$

i.e the mid-pt is P

Question (12)

(e) (i) $\sum \alpha_i = 0 \Rightarrow \alpha + \beta + 2\alpha - 2\beta = 0$

$\therefore 3\alpha = \beta$

$\sum_{i \neq j} \alpha_i \alpha_j = a$, $\therefore \alpha\beta + 2\alpha(\alpha-\beta) + 2\beta(\alpha-\beta) = a$.

Expanding,

$\alpha\beta + 2\alpha^2 - 2\alpha\beta + 2\alpha\beta - 2\beta^2 = a$.

but $3\alpha = \beta$ from (i),

$\therefore 3\alpha^2 + 2\alpha^2 - 18\alpha^2 = a$

$\Longrightarrow a = -13\alpha^2$

Product of roots $= -b$

$\therefore 2\alpha\beta(\alpha - \beta) = -b$.

but $3\alpha = \beta$ $\therefore 6\alpha^2(-2\alpha) = -b$

$\Longrightarrow b = 12\alpha^3$

$\therefore \frac{b}{a} = \frac{-12\alpha}{13} \Rightarrow \alpha = \frac{-13b}{12a}$.

but $3\alpha = \beta$ $\therefore \beta = 3\left(\frac{-13b}{12a}\right)$

$= \frac{-13b}{4a}$.

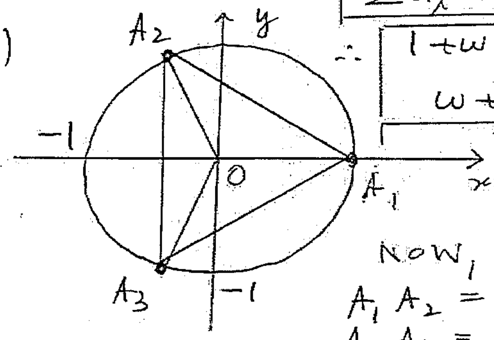
(ii) (cont.) Now, $2(\alpha - \beta)$

$= 2\left(\frac{-13b}{12a} + \frac{13b}{4a}\right)$

$= \frac{13b}{3a}$

$\boxed{\therefore \text{roots of } P(x) \text{ are: } \frac{-13b}{a}, \frac{-13b}{4a}, \frac{13b}{3a}}$

(f)



$\boxed{\begin{array}{l} z^3 - 1 = 0 \\ \sum \alpha_i = 0 \end{array}}$

$\therefore \boxed{\begin{array}{l} 1 + \omega + \omega_2 = 0 \\ \omega + \omega^2 = -1 \end{array}}$

Now,

$A_1A_2 = |\omega - 1|$

$A_1A_3 = |\omega^2 - 1|$

$\therefore A_1A_2 \times A_1A_3 = |\omega - 1||\omega^2 - 1|$

$= |\omega^3 - \omega - \omega^2 + 1|$

$= |2 - (\omega + \omega^2)|$ $\boxed{\because \omega + \omega^2 = -1}$

$= |2 - (-1)|$

$= |3| = 3$.

Question (12)

(a) $2x^3 + 3x^2 - 5x + 8 = 0$

Let $y = \frac{1}{x}$, $x = \frac{1}{y}$.

$\therefore \frac{2}{y^3} + \frac{3}{y^2} - \frac{5}{y} + 8 = 0$

$\Longrightarrow 8y^3 - 5y^2 + 3y + 2 = 0$

(b) $z = 3\cos\theta + i\,3\sin\theta$.

$1 - z = (1 - 3\cos\theta) - i\,3\sin\theta$.

$\overline{1 - z} = (1 - 3\cos\theta) + i\,3\sin\theta$.

$\frac{1}{1-z} = \frac{1}{(1 - 3\cos\theta) - i\,3\sin\theta} \times \frac{(1 - 3\cos\theta) + i\,3\sin\theta}{(1 - 3\cos\theta) + i\,3\sin\theta}$

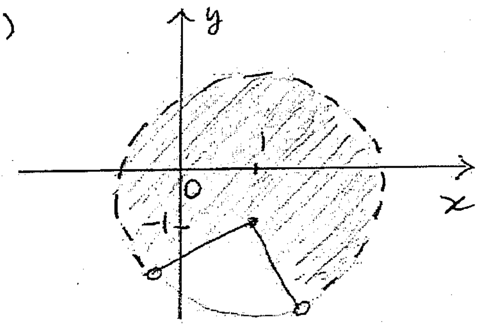
$= \frac{1 - 3\cos\theta + i\,3\sin\theta}{(1 - 3\cos\theta)^2 + 9\sin^2\theta}$

$= \frac{(1 - 3\cos\theta) + i\,3\sin\theta}{1 - 6\cos\theta + 9\cos^2\theta + 9\sin^2\theta}$

$= \frac{1 - 3\cos\theta + i\,3\sin\theta}{10 - 6\cos\theta}$.

$\therefore \mathrm{Im}\left(\frac{1}{1-z}\right) = \frac{3\sin\theta}{2(5 - 3\cos\theta)}$

(c)



(d) $z_1 = 2i$, $z_2 = 1 + 3i$

(i) $k = 1$, $z = z_1 + (z_2 - z_1)$

$= z_2$.

$\therefore$ locus of z is a point $(1, 3)$.

(ii) $0 < k < 1$, locus of z is the line interval between $(0,2)$ and $(1,3)$

(iii) $k \in \mathbb{R}$, locus of z is the line through $(0,2)$ and $(1,3)$.

i.e. $y = x + 2$

Question (13)

(b) $v\frac{dv}{dx} = a - bv^2$.
separating variables, we have

$$\int \frac{v}{a-bv^2}\,dv = \int dx.$$

i.e $$\int \frac{1}{a-bv^2}\, d\left(\frac{a-bv^2}{-2b}\right) = \int dx.$$

$\therefore$ $$-\frac{1}{2b}\ln(a-bv^2) = x + c$$

When $t=0$, $x=0$ and $v=0$,
we have $c = -\frac{1}{2b}\ln a$.

$\therefore$ $$-\frac{1}{2b}\ln(a-bv^2) = x - \frac{1}{2b}\ln a$$

ie $$\boxed{v = \sqrt{\frac{a}{b}}\,(1-e^{-2bx})^{\frac{1}{2}}} \quad (\because v>0).$$

(ii) As $x \to \infty$ $v \to \sqrt{\frac{a}{b}}$.

Since $e^{-2bx} \to 0$
The limiting value V of the motor car is $\sqrt{\frac{a}{b}}$.

(iii) When $V = \sqrt{\frac{a}{b}}$

$$v = V\sqrt{1-e^{-2bx}}$$

When $v=p$ and $x=l$
$\therefore$ $p = V\sqrt{1-e^{2bl}}$

i.e $$e^{-2bl} = \frac{V^2-p^2}{V^2}$$

When $v=q$ and $x = l+l = 2l$,
we have $q = V\sqrt{1-e^{4bl}}$

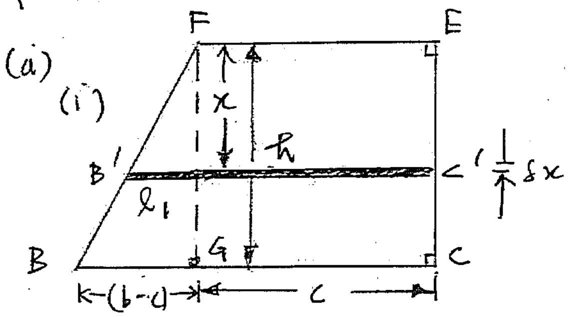
i.e $$e^{-4bl} = \frac{v^2-q^2}{v^2}$$

Eliminating displacement l,
we have $$\left(\frac{V^2-p^2}{V^2}\right)^2 = \frac{V^2-q^2}{V^2}$$

i.e $$V = \frac{p^2}{\sqrt{2p^2-q^2}} \quad (\because V>0)$$

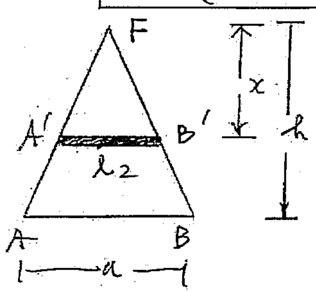
Question (13)

(a)
(i)



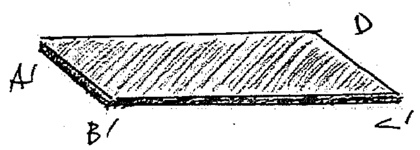
In $\triangle FBG$ $$\frac{l_1}{b-c} = \frac{x}{h}$$

$\therefore$ $$\boxed{l_1 = \left(\frac{b-c}{h}\right)x} \quad \text{———} (1)$$



$$\frac{l_2}{a} = \frac{x}{h}$$

$\therefore$ $$\boxed{l_2 = \left(\frac{a}{h}\right)x} \quad \text{———} (2)$$



$$\delta v = (A'B')(B'C')\,\delta x \quad \text{———} (3)$$

$$A'B' = l_2 = \left(\frac{a}{h}x\right)$$

$$B'C' = c + \left(\frac{b-c}{h}\right)x$$

$$\boxed{\therefore\ \delta v = \left(\frac{a}{h}x\right)\left[c + \left(\frac{b-c}{h}\right)x\right]\delta x} \quad (4)$$

(ii) $$V = \lim_{\delta x \to 0} \sum_0^h \left(\frac{a}{h}x\right)\left(c + \left(\frac{b-c}{h}\right)x\right)\delta x$$

$\therefore$ $$V = \int_0^h \left\{\left(\frac{ac}{h}\right)x + \left[\frac{a(b-c)}{h^2}\right]x^2\right\} dx.$$

$$= \left[\left(\frac{ac}{h}\right)\frac{x^2}{2} + \frac{a(b-c)}{h^2}\frac{x^3}{3}\right]_0^h$$

$$= \frac{3ach + 2ah(b-c)}{6}$$

$$= \frac{ha}{6}(2b+c) \quad \text{———} (5)$$

Question (14)

(a) $y = \cos^n x \sin nx$.

(i) $\frac{dy}{dx} = n\cos^n x \cos nx - n\sin nx \cos^{n-1} x \sin x$

$= 2n\cos^n x \cos nx - n\cos^{n-1} x(\sin nx \sin x + \cos x \cos nx)$

$= 2n\cos^n x \cos nx - n\cos^{n-1} x \cos(n-1)x$.

$\int d(\cos^n x \sin nx)$

$= \int 2n\cos^n x \cos nx\,dx - n\int \cos^{n-1} x \cos(n-1)x\,dx$

$= 2n\int \cos^n x \cos nx\,dx - n\int \cos^{n-1} x \cos(n-1)x\,dx$

$= \cos^n x \sin nx + C$.

(ii) $\left[\cos^n x \sin nx\right]_0^{\pi/2}$

$= 2n\int_0^{\pi/2} \cos^n x \cos nx\,dx - n\int_0^{\pi/2} \cos^{n-1} x \cos(n-1)x\,dx$

$\therefore 0 = 2n\int_0^{\pi/2} \cos^n x \cos nx\,dx - n\int_0^{\pi/2} \cos^{n-1} x \cos(n-1)x\,dx$

$\therefore \int_0^{\pi/2} \cos^n x \cos nx\,dx = \frac{1}{2}\int_0^{\pi/2} \cos^{n-1} x \cos(n-1)x\,dx$

(iii) $I_n = \int_0^{\pi/2} \cos^n x \cos nx\,dx$

$\therefore I_1 = \int_0^{\pi/2} \cos^2 x\,dx = \int_0^{\pi/2} \left(\frac{1+\cos 2x}{2}\right) dx$

$\therefore I_1 = \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi/2}$

$= \frac{\pi}{4}$.

From (ii) $I_8 = \frac{1}{2} I_7$

$= \frac{1}{2^2} I_6$

$= \vdots$

$= \frac{1}{2^7} I_1$

$= \frac{\pi}{512}$.

(iv) Let $y = \frac{\pi}{2} - x$, $dy = -dx$

and $x = \frac{\pi}{2} - y$. When $x = 0$, $y = \frac{\pi}{2}$;

When $x = \frac{\pi}{2}$, $y = 0$

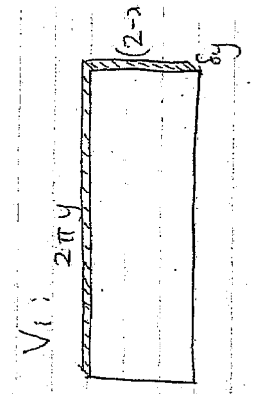
$\therefore \int_0^{\pi/2} \sin^8 x \cos 8x\,dx$

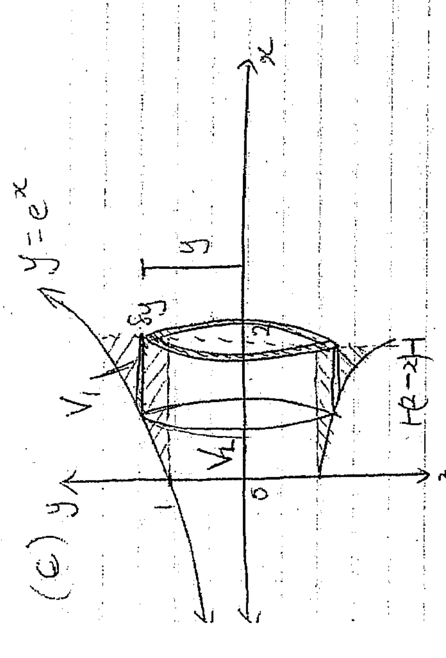
$= \int_{\pi/2}^{0} \sin^8\left(\frac{\pi}{2} - y\right) \cos\left[8\left(\frac{\pi}{2} - y\right)\right](-dy)$

$= \int_0^{\pi/2} \cos^8 y \cos 8y\,dy = \frac{\pi}{512}$.

4

(c)





$V_1 = \int_1^{e^2} 2\pi y(2-x)\,dy$

$= 2\pi\int_1^{e^2} y(2 - \ln y)\,dy$

$= 2\pi\int_1^{e^2} 2y - y\ln y\,dy$

$= 2\pi\left\{\left[y^2\right]_1^{e^2} - \int_1^{e^2} y\ln y\,dy\right\}$

$= 2\pi\left\{e^4 - 1 - \left(\left[\frac{y^2\ln y}{2}\right]_1^{e^2} - \int_1^{e^2} \frac{y^2}{2}\cdot\frac{1}{y}\,dy\right)\right\}$ 1

$= 2\pi\left\{e^4 - 1 - \left(e^4 - \left[\frac{y^2}{4}\right]_1^{e^2}\right)\right\}$

$= 2\pi\left\{e^4 - 1 - \left(e^4 - \frac{e^4}{4} + \frac{1}{4}\right)\right\}$

$= 2\pi\left(\frac{e^4}{4} - \frac{5}{4}\right)$ 1

$= \frac{\pi}{2}(e^4 - 5)$

$V_2 = \pi(1)^2(2)$

$V_2 = 2\pi$

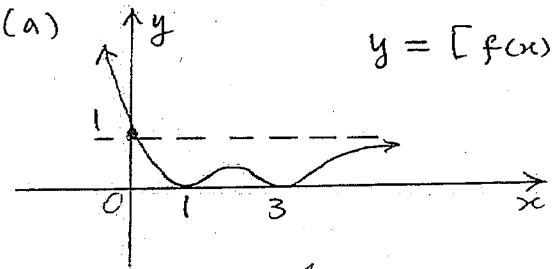
$V_{Total} = V_1 + V_2$

$= \frac{\pi}{2}(e^4 - 5) + 2\pi$

$= \boxed{\frac{\pi}{2}(e^4 - 1)}$ 1

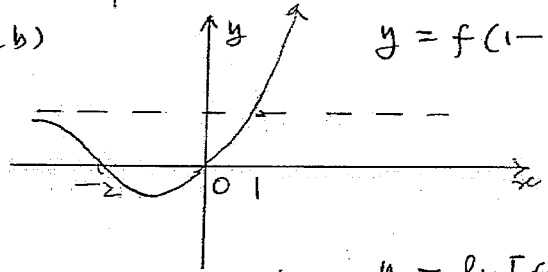
Question (15)

(a)



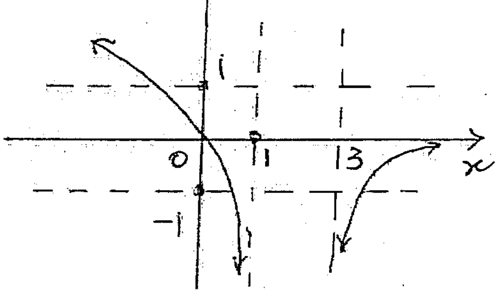
$y = [f(x)]^2$

(b)



$y = f(1-x)$

(c)



$y = \ln[f(x)]$

(b) $\dfrac{\cos\theta + i\sin\theta - 1}{\cos\theta + i\sin\theta + 1}$

$= \dfrac{\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) + i\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) - \left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right)}{\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) + i\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) + \left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right)}$

$= \dfrac{-2\sin^2\frac{\theta}{2} + i\, 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} + i\, 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$= \dfrac{i\, 2\sin\frac{\theta}{2}\left(\cancel{i\sin\frac{\theta}{2} + \cos\frac{\theta}{2}}\right)}{2\cos\frac{\theta}{2}\left(\cancel{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}\right)}$

$= i\tan\frac{\theta}{2}$.

(ii) Let $\omega = r\,\mathrm{cis}\,\theta$, $\omega^5 = 1$

$(r\,\mathrm{cis}\,\theta)^5 = 1$, $r^5\,\mathrm{cis}\,5\theta = \mathrm{cis}(2k\pi)$

$\therefore r^5 = 1 \Rightarrow r = 1$

Now, $5\theta = 2n\pi$, $\theta = \frac{2k\pi}{5}$ $(k = 0, 1, 2, 3, 4)$

$\therefore$ roots are $\mathrm{cis}\,0$, $\mathrm{cis}\left(-\frac{2\pi}{5}\right)$, $\mathrm{cis}\left(-\frac{4\pi}{5}\right)$, $\mathrm{cis}\left(\frac{2\pi}{5}\right)$, $\mathrm{cis}\left(\frac{4\pi}{5}\right)$

Question (14)

(b) We first arrange the 8 persons (excluding A, B and C) in a row in 8! ways. Fix one of these ways, say

$\_\_ X_1 \_\_ X_2 \_\_ X_3 \_\_ X_4 \_\_ X_5 \_\_ X_6 \_\_ X_7 \_\_ X_8 \_\_$

(1) (2) (3) (4) (5) (6) (7) (8) (9)

We now consider A. There are 9 ways to place A in one of 9 boxes, say box (4):

$\_\_ X_1 \_\_ X_2 \_\_ X_3$ A $X_4 \_\_ X_5 \_\_ X_6 \_\_ X_7 \_\_ X_8 \_\_$

(1) (2) (3) (4) (5) (6) (7) (8) (9)

Next, consider B. Since A and B cannot be adjacent, B can be placed only in one of the remaining 8 boxes. Likewise, C can be placed only in one of the remaining 7 boxes.

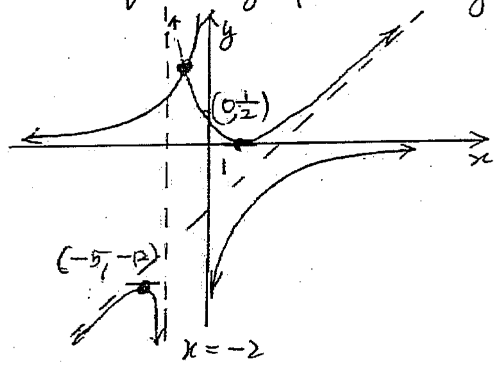
$\therefore (8!)\, 9 \times 8 \times 7 = 20321280$

(c) Since $x \neq -2$, a vertical asymptote is $x = -2$

(i) $y = \dfrac{(x-1)^2}{x+2} = \dfrac{x^2 - 2x + 1}{x+2} = x - 4 + \dfrac{9}{x+2}$

An oblique asymptote is $y = x - 4$

(ii)



Given: $x^3 - 2x^2 + 2x + 2 = 0$

$\Rightarrow x(x^2 - 2x + 1) + x + 2 = 0$

$x(x^2 - 2x + 1) = -(x+2)$

$\therefore \dfrac{x^2 - 2x + 1}{x+2} = -\dfrac{1}{x}$

Since there is one intersection between the graphs of C and $y = -\frac{1}{x}$, there is one real root for $x^3 - 2x^2 + 2x + 2 = 0$

Question 15 (c)

Produce DC to intersect the circle at F.
Similarly, extend DO to intersect the circle at G.

Now $AC \cdot CB = FC \cdot CD$.
(product of intercepts of intersecting chords are equal)

Now
$FC \cdot CD + CD^2$
$= CD\,(FC + CD)$
$= CD \cdot FD$.

$AB \times CE$
$= 2\,(\text{radius})\,CE$
$= DG \cdot CE$.

To prove.
$AB \times CE = AC \times CB + CD^2$.
it is sufficient to prove
$CD \cdot FD = CE \cdot DG$

or $\dfrac{FD}{CE} = \dfrac{DG}{CD}$.

Now In $\triangle CED$ and $\triangle DFG$.
$\angle DFG = 90^\circ$ (Angle in a semi-circle)
$= \angle CED$
$\angle EDC = \angle DGF$ (alternate segment theorem.)
$\therefore \triangle CED \parallel\!\!| \triangle DFG$. (equiangular).
$\therefore \dfrac{FD}{CE} = \dfrac{DG}{CD}$.
(corresponding sides of similar triangles $\triangle CED$ and $\triangle DFG$ are in the same ratios.)

$\therefore AC \cdot CD + CD^2 = AB \cdot CE$.

Question (15)

(iii) Let $\omega = \left(\dfrac{2+z}{2-z}\right)$
$2\omega - \omega z = 2 + z$
$2\omega - 2 = \omega z + z$
$2(\omega - 1) = z(\omega + 1)$
$\therefore z = 2\left(\dfrac{\omega - 1}{\omega + 1}\right)$

Now, from (ii) $\omega = cis\left(\dfrac{2n\pi}{5}\right)$
for $n = \pm 1, \pm 2, 0$.
From (ii) we have
$\omega = \cos\dfrac{2k\pi}{5} + i\sin\dfrac{2k\pi}{5}$,
$k = 0, \pm 1, \pm 2$. When $k=0$, $z = 0$

When $k=1$, $z = 2\left[\dfrac{\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} - 1}{\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + 1}\right]$
$= 2i\tan\frac{2\pi}{6}$,

When $k=-4$, $z = 2\left[\dfrac{\cos(-\frac{4\pi}{5}) + i\sin(-\frac{4\pi}{5}) - 1}{\cos(-\frac{4\pi}{5}) + i\sin(-\frac{4\pi}{5}) + 1}\right]$

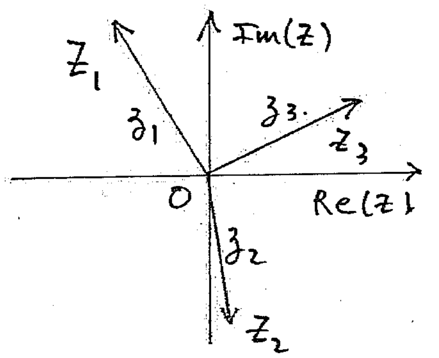
$\therefore z = 2i\tan\left(\frac{n\pi}{5}\right)$ for $n = 0, \pm 1, \pm 2$.

(iv) $\left(\dfrac{2+z}{2-z}\right)^5 = 1$
$(2+z)^5 = (2-z)^5$
$32 + 80z + 80z^2 + 40z^3 + 10z^4 + z^5$
$= 32 - 80z + 80z^2 - 40z^3 + 10z^4 - z^5$
$\therefore 2z^5 - 80z^3 + 160z = 0$
$z^5 - 40z^3 + 80z = 0$
$\therefore z(z^4 - 40z^2 + 80) = 0$
[Note: $M = -40$, $n = 80$].
Product of roots (excluding $z=0$) $= 80$
i.e
$$\boxed{2^4(i^4)\tan\left(\tfrac{2\pi}{5}\right)\tan\left(\tfrac{-2\pi}{5}\right) \times \tan\left(\tfrac{\pi}{5}\right)\tan\left(\tfrac{-\pi}{5}\right) = 80}$$

$\therefore 16\left[-\tan^2\frac{2\pi}{5}\right]\left[-\tan^2\frac{\pi}{5}\right] = 80$
$\therefore \tan^2\left(\frac{2\pi}{5}\right)\tan^2\left(\frac{\pi}{5}\right) = 5$
i.e $\tan\left(\frac{\pi}{5}\right)\tan\left(\frac{2\pi}{5}\right) = \sqrt{5}$.

Question (16) (b)

(i)



$z_1 + z_2 + z_3 = 0$

$\overrightarrow{OZ_1} + \overrightarrow{OZ_2} + \overrightarrow{OZ_3} = 0$

(ii) $Z_1 = \cos\alpha + i\sin\alpha$

$Z_2 = k\cos\beta + ik\sin\beta$

$Z_3 = (2-k)\cos\gamma + i(2-k)\sin\gamma$.

where $0 < k < 2$

(iii) $\because z_1 + z_2 + z_3 = 0$

$\therefore \cos\alpha + i\sin\alpha + (k\cos\beta) + (ik\sin\beta) + (2-k)\cos\gamma + i(2-k)\sin\gamma = 0$

$\therefore$ Real (LHS) = Im (LHS) = 0

so

$$\begin{cases} \cos\alpha + k\cos\beta + (2-k)\cos\gamma = 0 \\ \sin\alpha + k\sin\beta + (2-k)\sin\gamma = 0 \end{cases}$$

Make $\cos\alpha$, $\sin\alpha$ the subject

$\therefore \cos\alpha = -[k\cos\beta + (2-k)\cos\gamma]$ —— (1)

$\sin\alpha = -[k\sin\beta + (2-k)\sin\gamma]$ —— (2)

(iv) Square (1) & (2) and find its sum

$\therefore \cos^2\alpha + \sin^2\alpha$

$= k^2\cos^2\beta + (2-k)^2\cos^2\gamma + 2k(2-k)\cos\beta\cos\gamma + k^2\sin^2\beta + (2-k)^2\sin^2\gamma + 2k(2-k)\sin\beta\sin\gamma.$

i.e.

$1 = k^2(\sin^2\beta + \cos^2\beta) + (2-k)^2(\sin^2\gamma + \cos^2\gamma) + 2k(2-k)[\cos\beta\cos\gamma + \sin\beta\sin\gamma]$

$\therefore 1 = k^2 + (2-k)^2 + 2k(2-k)\cos(\beta-\gamma)$ —— (3)

Question (16) (a)

(i) $\sum_{r=2}^{n} [\ln(r+1) - \ln(r-1)]$

$= (\ln 3 - \ln 1) + (\ln 4 - \ln 2) + (\ln 5 - \ln 3) + (\ln 6 - \ln 4) + \cdots + [\ln(n-1) - \ln(n-3)] + [\ln(n) - \ln(n-2)] + [\ln(n+1) - \ln(n-1)]$

$= -\cancel{\ln 1} - \ln 2 + \ln(n) + \ln(n+1)$

$= \ln\dfrac{n(n+1)}{2}$

(ii) $\because$

$\dfrac{r+1}{r-1} = 1 + \dfrac{2}{r} + \dots + \dfrac{2}{r^n} + \dfrac{2}{r^n(r-1)}$

(given)

$\therefore \sum_{r=2}^{n} \ln\left[1 + \dfrac{2}{r} + \dots + \dfrac{2}{r^n} + \dfrac{2}{r^n(r-1)}\right]$

$= \sum_{r=2}^{n} \ln\left(\dfrac{r+1}{r-1}\right)$

Let $P(n)$ be the proposition that $\sum_{r=2}^{n} \ln\left(\dfrac{r+1}{r-1}\right) = \ln\dfrac{n(n+1)}{2}$

When $n = 2$, L.H.S $= \ln 3$ = R.H.S.

$\therefore P(2)$ is true

Assume $P(k)$ is true

i.e $\sum_{r=2}^{k} \ln\left(\dfrac{r+1}{r-1}\right) = \ln\dfrac{k(k+1)}{2}$.

Prove true for $n = k+1$.

Now when $n = k+1$, we have

$\sum_{r=2}^{k+1} \ln\left(\dfrac{r+1}{r-1}\right) = \sum_{r=2}^{k} \ln\left(\dfrac{r+1}{r-1}\right) + \ln\left(\dfrac{k+2}{k}\right)$

$= \ln\left[\dfrac{k(k+1)}{2}\right] + \ln\left(\dfrac{k+2}{k}\right)$

$= \ln\left[\dfrac{\cancel{k}(k+1)(k+2)}{2\cancel{k}}\right]$

$= \dfrac{\ln\{(k+1)[(k+1)+1]\}}{2}$

If the propn. is true for $n=k$, then it is true for $n=k+1$. By the principal of M.I. it is true $\forall\, n \geq 2$.

Expand (3) we have

$$(k^2) + (4 - 4k + k^2) + (4k - 2k^2)\cos(\beta - \gamma) = 1$$

$$\therefore (2k^2 - 4k)\left[1 - \cos(\beta - \gamma)\right] = -3.$$

$$\therefore 2k^2 - 4k = \frac{-3}{\left[1 - \cos(\beta - \gamma)\right]} \quad (4)$$

Let $\boxed{\delta = \dfrac{-3}{1 - \cos(\beta - \gamma)}.} \quad (5)$

Now $|\cos(\beta - \gamma)| \le 1$

i.e.

$$-1 \le \cos(\beta - \gamma) \le 1 \quad (6)$$

Substitute the extreme values of (6) $\Rightarrow$ the first approximation of (5) becomes

$$-\infty < \delta \le -\frac{3}{2} \quad (7)$$

Now from (4) we can form a quadratic equation

$$2k^2 - 4k - \delta = 0$$

where δ is given by (5).

$$\therefore k = \frac{4 \pm \sqrt{16 + 8\delta}}{4}$$

i.e. $k = \dfrac{4 \pm \sqrt{1 + \frac{\delta}{2}}}{4} \quad (8)$

$k = 1 \pm \sqrt{1 + \frac{\delta}{2}}$ Solving for $k \in \mathbb{R}$.

$$1 + \frac{\delta}{2} \ge 0$$

$$\delta \ge -2 \quad (9)$$

$\therefore$ A refinement for δ is: $-2 \le \delta \le -\frac{3}{2}$.

So extreme values of δ are $-2, -3/2$.

$$\therefore 1 - \sqrt{\tfrac{1}{4}} \le k \le 1 + \sqrt{\tfrac{1}{4}}$$

i.e. $\boxed{\dfrac{1}{2} \le k \le \dfrac{3}{2}} \quad (10)$