ASCHAM SCHOOL

MATHEMATICS EXTENSION 2

TRIAL EXAMINATION

2005

Time : 3 hours + 5 minutes reading tir

Instructions:

Attempt all questions

All questions are of equal value

All necessary working should be shown for every question.

Full marks may not be awarded for careless or badly arranged work

A Table of Standard Integrals is provided

Approved calculators may be used

Each question should be answered in a separate booklet

Question 1

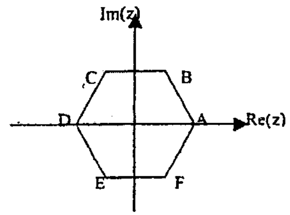
(a) If $z_1 = 3+4i$ and $z_2 = 2-i$ find $\frac{z_1}{z_2}$ in the form $a+ib$ where a and b are real. [2]

(b) (i) Find all pairs of integers x and y that satisfy $5-12i = (x+iy)^2$. [2]

(ii) Hence or otherwise solve $z^2 - (1-4i)z - (5-i) = 0$. [2]

(c) Given that $1+2i$ is a root of the polynomial equation $z^3 + z^2 - z + 15 = 0$ find all the roots over the set of complex numbers. [3]

(d)



ABCDEF is a regular hexagon drawn on an Argand diagram with vertex A at the point (4,0). O (0,0) is the centre of the hexagon.

(i) State the complex number represented by E in modulus argument form. [2]

(ii) The hexagon is rotated anticlockwise about the origin through an angle of $\frac{\pi}{4}$. Express in the form r(cosθ + i sinθ) the new position of E. [2]

(iii) On a copy of the given diagram show the region within the hexagon where both the inequalities $|z| \geq 2$ and $-\frac{\pi}{3} < \arg z < \frac{2\pi}{3}$ hold. [2]

Question 2 (Start a new booklet)

(a) Find $\int \frac{4}{\sqrt{2x^2+3}}\,dx$ [2]

(b) Evaluate $\int_0^{\frac{\pi}{3}} \cos 3x \sin 2x \, dx$ [2]

(c) Find $\int x e^{2x}\,dx$ [2]

(d) Find $\int \frac{x-4}{\sqrt{8-x^2-2x}}\,dx$ [3]

(e) (i) If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$ show that $I_n = \frac{n-1}{n} I_{n-2}$ [3]

(ii) Hence find $\int_0^1 \frac{x^3}{\sqrt{1-x^2}}\,dx$ by using the substitution $x = \cos\theta$. [3]

Question 3 (Start a new booklet)

(a) The vertices of a conic are (2,0) and (-2,0) and the foci are (3,0) and (-3,0).

(i) Determine the equations of the directrices. **[2]**

(ii) Find the equation of the conic. **[2]**

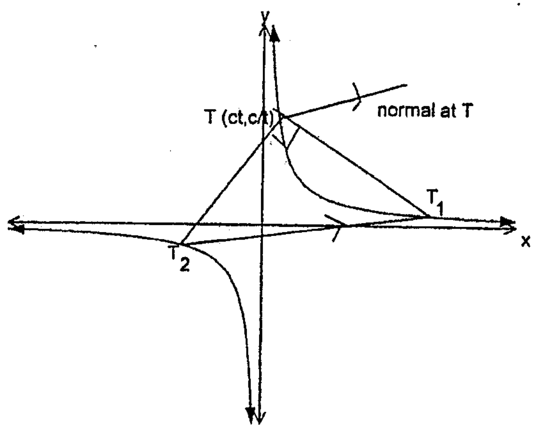
(b) (i) Show the equation of the normal to the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ at the point P(5cosθ,3sinθ) is $\frac{5x}{\cos\theta}-\frac{3y}{\sin\theta}=16$ **[3]**

(ii) This normal intersects the major and minor axes at A and B respectively. Show that as P moves on the ellipse, the midpoint of AB describes another ellipse and find its eccentricity. **[3]**

(c) As shown in the diagram below, T_1 and T_2 are two points on the rectangular hyperbola $xy=c^2$ with parameters t_1 and t_2 and T is a third point on it with parameter t such that $\angle T_1TT_2$ is a right angle.

(i) Show that the gradient of T_1T is $-\frac{1}{t_1t}$ and deduce that since $\angle T_1TT_2$ is a right angle then $t^2=-\frac{1}{t_1t_2}$. **[3]**

(ii) Hence prove that T_1T_2 is parallel to the normal at T. **[2]**



Question 4 (Start a new booklet)

(a) Find the primitive of $\frac{1}{1+\sin x}$ **[3]**

(b) (i) State the condition for m to be a zero of multiplicity 2 of the polynomial P(x). **[1]**

(ii) The polynomial ax^8+bx^7+2 is divisible by $(x+1)^2$. Find a and b. **[2]**

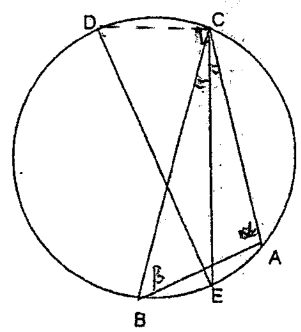
(c) The roots of $x^3-3x^2+2x-1=0$ are α, β and γ

(i) Find the equation whose roots are α^2, β^2 and γ^2. **[2]**

(ii) Hence or otherwise find the value of $\alpha^2+\beta^2+\gamma^2$ **[1]**

(iii) Find $\sum\alpha^3$ **[2]**

(d)



ABC is a triangle inscribed in a circle and E is the midpoint of arc AB. ED is the diameter through E. Let ∠CAB = α° and ∠CBA = β° where α > β.

(i) Mark relevant information on a copy of the diagram provided.

(ii) Find an expression for ∠DCA in terms of α° and β°. **[2]**

(iii) Show that ∠DEC = $\frac{1}{2}(\alpha-\beta)°$. **[2]**

Question 5 (Start a new booklet)

(a) If $\omega = 1 - i$

(i) Express ω in mod arg form. **[1]**

(iii) Hence or otherwise determine the points on an Argand diagram representing the complex numbers ω^4 and $\frac{1}{\omega^2}$. **[2]**

(iii) Sketch the locus of the points z such that $\left|z-\omega^4\right| = \left|z - \frac{1}{\omega^2}\right|$ and find the Cartesian equation of this locus. **[3]**

(b) The area enclosed by the curve $y=(x-3)^2$ and the line y = 9 is rotated about the y-axis. Use the method of cylindrical shells to find the exact volume of the solid formed. **[4]**

(c) The base of a solid is the region bounded by the curves $y = 6 - x^2$ and $y = 2x^2$. Cross sections by planes perpendicular to the x-axis are semi-ellipses with their major axes in the base of the solid. The minor axes are one third the length of the major axes. Find the volume of the solid to the nearest whole cubic unit. (Assume that the area of an ellipse is πab) **[5]**

Question 6 (Start a new booklet)

(a) A circle of radius r is drawn with its centre on the circumference of another circle of radius r. Show that the area common to both circles is $\frac{1}{6}r^2\left(4\pi - 3\sqrt{3}\right)$ square units. **[3]**

(b) A ball of mass 3kg is thrown vertically upward from the origin with an initial speed of 6m/s. The ball is subject to a downward gravitational force of 18 newtons and air resistance of $\frac{v^2}{2}$ newtons in the opposite direction to the velocity, v metres per second.

(i) Show that until the ball reaches its highest point, the equation of motion is $\ddot{y} = -\frac{1}{6}\left(v^2 + 36\right)$ where y metres is its height. **[2]**

(ii) Show that while the ball is rising $v^2 = 36\left(2e^{-\frac{y}{3}} - 1\right)$. **[4]**

(iii) Show that the ball's maximum height is $3\ln 2$ metres. **[2]**

(iv) Show that the ball's speed on its return to the origin is $3\sqrt{2}$ m/s. **[4]**

Question 7 (Start a new booklet)

(a) (i) Show that i is a root of $z^6 + 1 = 0$. **[1]**

(ii) Hence or otherwise show the 6 sixth roots of −1 on an Argand Diagram. **[3]**

(iii) Show that $\left(z^2+1\right)\left(z^4 - z^2 + 1\right) = z^6 + 1$ **[1]**

(iv) Hence find the 4 roots of $z^4 - z^2 + 1 = 0$ **[2]**

(b) If $f(x) = \frac{(x+2)}{x}$, sketch the following on separate graphs showing important features such as intercepts on the axes and asymptotes.

(i) $y = f(x)$ **[2]**

(ii) $y = \frac{1}{f(x)}$ **[2]**

(iii) $y = \log\left(f(x)\right)$ **[2]**

(iv) $y^2 = f(x)$ **[2]**

Question 8 (Start a new booklet)

(a) (i) Use the substitution $x = \pi - u$ to show that $\int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$. **[3]**

(ii) Hence evaluate $\int_0^\pi x\cos^2 x\,dx$ **[5]**

(b) Given that $a+b+c=1$ and $a+b+c \geq 3\sqrt[3]{abc}$ where a, b and c are positive real numbers :

(i) Prove that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$ **[2]**

(ii) Hence or otherwise find the minimum value of $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)$ **[5]**

End of Examination

2005 Ascham D. Ong.

1.(a). $z_1 = 3+4i$

$z_2 = 2-i$

$\frac{z_1}{z_2} = \frac{3+4i}{2-i} \times \frac{(2+i)}{(2+i)}$

$= \frac{2+11i}{5}.$

(b)(i) $5-12i = (x+iy)^2$

$5-12i = (x^2-y^2+2ixy)$

$x^2-y^2=5 \qquad -12 = 2xy$

$x^2 - \frac{36}{x^2} = 5. \qquad \frac{-6}{x} = y$

$x^4 - 5x^2 - 36 = 0.$

$x^2 = 9, \quad x^2 = -4.$

$\therefore x = \pm 3 \quad , y = \mp 2.$

$\therefore z = \pm(3-2i)$

(ii). $z^2 - (1-4i)z - (5+i) = 0.$

$z = \frac{(1-4i) \pm \sqrt{(1-4i)^2 + 4(5-i)}}{2.}$

$= \frac{(1-4i) \pm \sqrt{5-12i}}{2.}$

$= \frac{(1-4i) \pm (3-2i)}{2.}$

$= 2-3i$ or $-1-i$

(c). $z^3 + z^2 - z + 15 = 0.$

$P(1+2i) = 0. \quad$ if coefficients real, then $(1-2i)$ is another root.

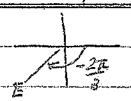
$\therefore z^2 - 2z + 5$ is a factor.

$\therefore$ By Inspection other factor $(z+3).$

$\therefore$ roots $(1+2i), (1-2i), (-3)$

(d)(i) $E: |z| = 4, \quad \arg(z) = \frac{-2\pi}{3}$

$E = 4\,\text{cis}(\frac{-2\pi}{3})$



(ii). $E' = 4\,\text{cis}(\frac{-2\pi}{3})\,\text{cis}(\frac{\pi}{4}) \qquad = 4\,\text{cis}(-\frac{2\pi}{3}+\frac{\pi}{4})$

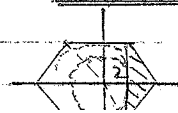
$= (2+i2\sqrt{3})(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}) \qquad = 4\,\text{cis}(-\frac{5\pi}{12})$

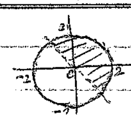
$= \frac{2}{\sqrt{2}} + i\frac{2}{\sqrt{2}} + i\frac{2\sqrt{3}}{\sqrt{2}} - \frac{2\sqrt{3}}{\sqrt{2}}$

$= (\sqrt{2}+\sqrt{6}) + i(\sqrt{2}-\sqrt{6})$

$|z| \geq 2$ is the region outside the circle with centre O + radius 2 units

(iii).





2(a). $\int \frac{4}{\sqrt{2x^2+3}}$

$\frac{4}{\sqrt{2}} \int \frac{dx}{\sqrt{x^2+3/2}}$

$= \frac{4}{\sqrt{2}} \ln(x + \sqrt{x^2+3/2})$

$= 2\sqrt{2}\ln(x+\sqrt{x^2+3/2})$

(b). $\int_0^{\pi/3} \cos 3x \sin 2x \, dx \qquad \Rightarrow$ Quicker to use the result:

$2\int_0^{\pi/3} (4\cos^3 x - 3\cos x)\sin x \cos x \, dx \qquad 2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$

Let $\cos x = u. \qquad$ At $x = \pi/3, u = 1/2$

$du = -dx.\sin x \qquad x = 0, u = 1 \quad \therefore \cos 3x \sin 2x$

$I = -2\int_1^{1/2} (4u^3-3u)u \, du. \qquad = \frac{1}{2}[\sin 5x - \sin x]$

$= 2\int_{1/2}^1 (4u^4 - 3u^2)\,du$

$= 2\left[\frac{4u^5}{5} - \frac{3u^3}{3}\right]_{1/2}^1 \qquad \therefore \int_0^{\pi/3} \frac{1}{2}(\sin 5x - \sin x)\,dx$

$= 2[(4/5 - 1) - (\frac{1}{8} - \frac{1}{8})] \qquad = \frac{1}{2}\left[-\frac{\cos 5x}{5} + \cos x\right]_0^{\pi/3}$

$= -2/5. \qquad = \frac{1}{2}\left[\left(-\frac{\cos\frac{5\pi}{3}}{5} + \cos\frac{\pi}{3}\right) - \left(-\frac{1}{5}+1\right)\right]$

(c). $\int x e^{2x}\,dx \qquad = \frac{1}{2}\left[-\frac{1}{10}+\frac{1}{2}-\frac{4}{5}\right]$

Let $u = x \quad v = \frac{e^{2x}}{2}$

$u' = 1 \quad v' = e^{2x} \qquad = -\frac{1}{5}$

$I = \left[\frac{xe^{2x}}{2}\right] - \int \frac{e^{2x}}{2}\,dx$

$= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4}$

$= \frac{e^{2x}}{2}[x - \frac{1}{2}]$

(d). $\int \frac{x-4}{\sqrt{8-x^2-2x}}\,dx$

$\int \frac{x+1}{\sqrt{-x^2-2x+8}} - \frac{5}{\sqrt{-(x^2+2x-8)}}$

$= \frac{1}{2}\int \frac{2x+2}{\sqrt{-x^2-2x+8}} - \frac{5}{\sqrt{-(x^2+2x+1)-9)}}$

$-\frac{1}{2}\int \frac{-2x-2}{\sqrt{-x^2-2x+8}} - \frac{5}{\sqrt{9-(x+1)^2}}$

$-\frac{1}{2}[(-x^2-2x+8)^{1/2} \qquad - 5\sin^{-1}(\frac{x+1}{3})$

$+ 1/2.$

$= -\sqrt{-x^2-2x+8} \quad - 5\sin^{-1}(\frac{x+1}{3}) \quad + c.$

(e)(i) $I_n = \int_0^{\pi/2} \cos^n x.$

$= \int_0^{\pi/2} \cos^{n-1} x \cos x$

$u = \cos^{n-1} x \qquad v = \sin x$

$u' = (n-1)\cos^{n-2} x(-\sin x) \qquad v' = \cos x$

$I_n = [\sin x \cos^{n-1} x]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos^{n-2} \sin^2 x \, dx$

$= (n-1)\int(\cos^{n-2} - \cos^n x \, dx$

$= (n-1)I_{n-2} - (n-1)I_n$

$I_n = \frac{(n-1)}{n} I_{n-2}$

(e)(ii). $\int_0^1 \frac{x^3}{\sqrt{1-x^2}}$

Let $x = \cos\theta$, At $x = 1$, $\cos\theta = 0$

$dx = -\sin\theta \, d\theta$ $\quad x = 0$, $\cos\theta = \pi/2$

$I = \int_0^{\pi/2} \frac{\cos^3\theta}{\cancel{\sqrt{1-\cos^2\theta}}} \cdot \cancel{\sin\theta} \cdot d\theta$

$= \int_0^{\pi/2} \cos^3\theta \cdot d\theta$

$= I_3$

$I_3 = \frac{2}{3} I_2$

$I_2 = \frac{1}{2} I_1$

$I_1 = \int_0^{\pi/2} \cos x$

$= [\sin x]_0^{\pi/2}$

$= 1$

$\therefore I_2 = \frac{1}{2}$.

$\therefore I_3 = 2/3 \times \frac{1}{2}$

$= 1/3$.

3.(a). Vertex $(2,0)$ & $(-2,0)$ Hyperbola. (i). Directrix: $x = \frac{a}{e}$

$= 4/3$.

$\therefore a = 2$.

Foci $(3,0)$, $(-3,0)$

$\therefore 3 = \pm ae \implies e = 3/2$.

$a > b$.

$b^2 = a^2(e^2 - 1)$

$= 4(9/4 - 1)$

$= 9 - 4$.

$b^2 = 5$.

$\therefore$ Eqn: $\frac{x^2}{4} - \frac{y^2}{5} = 1$.

(b)(i). $\frac{x^2}{25} + \frac{y^2}{9} = 1$

$f'(x) = \frac{2x}{25} + \frac{2y \cdot dy}{9\,dx} = 0$.

$\frac{dy}{dx} = \frac{-\frac{2x}{25}}{\frac{2y}{9}}$

$= \frac{-9x}{25y}$

$m_2 = \frac{25y}{9x}$

At $P(5\cos\theta, 3\sin\theta)$

$m_2 = \frac{\cancel{3}5\sin\theta}{3\cos\theta}$

$y - 3\sin\theta = \frac{5\sin\theta}{3\cos\theta}(x - 5\cos\theta)$

$3y\cos\theta - 9\cos\theta\sin\theta = 5\sin\theta x - 25\sin\theta\cos\theta$

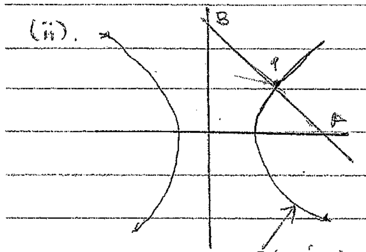
$25\sin\theta\cos\theta - 9\cos\theta\sin\theta = 5\sin\theta x - 3y\cos\theta$

Divide by $\sin\theta\cos\theta$

$25 - 9 = \frac{5x}{\cos\theta} - \frac{3y}{\sin\theta}$

$\frac{5x}{\cos\theta} - \frac{3y}{\sin\theta} = 16$

(ii).



This is a hyperbola

but $E: \frac{x^2}{25} + \frac{y^2}{9} = 1$

Cuts x axis, i.e. $y = 0$.

$5x = 16\cos\theta$

$x = \frac{16\cos\theta}{5}$

$\therefore A = (\frac{16\cos\theta}{5}, 0)$

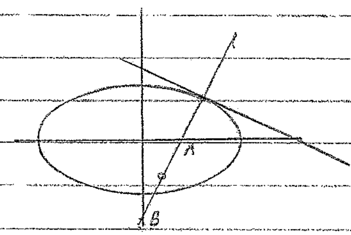
Cuts y axis, i.e. $x = 0$.

$y = -\frac{16\sin\theta}{3}$

$B(0, -\frac{16\sin\theta}{3})$

Midpoint. $x = \frac{16\cos\theta}{10}$ $\quad y = -\frac{16\sin\theta}{6}$.

$\frac{10x}{16} = \cos\theta$ $\quad \frac{-6y}{16} = \sin\theta$



Square All

$\frac{100x^2}{256} = \cos^2\theta$ $\quad \frac{36y^2}{256} = \sin^2\theta$

Add together.

$\frac{25x^2}{64} + \frac{9y^2}{64} = 1$

$\therefore a^2 = \frac{64}{25}$, $b^2 = \frac{64}{9}$.

$b > a$.

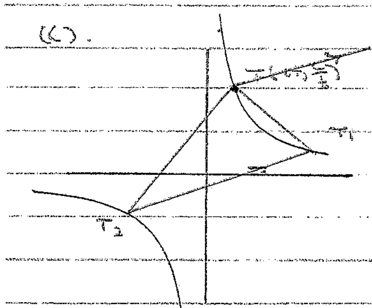
$1 - \frac{a^2}{b^2} = e^2$

$1 - \frac{9}{25} = e^2$

$e^2 = \frac{16}{25}$

$e = \frac{4}{5}$.

(c).



$T_1: x = ct_1, y = \frac{c}{t_1}$

$m_{T_1T} = \frac{\frac{c}{t_1} - \frac{c}{t}}{ct_1 - ct}$

$= \frac{\cancel{c}(\frac{1}{t_1} - \frac{1}{t})}{\cancel{c}(t_1 - t)}$

$= \frac{\frac{t - t_1}{tt_1}}{t_1 - t}$

$= \frac{-(t_1 - t)}{tt_1(t_1 - t)} = -\frac{1}{tt_1}$

since $T_1\hat{T}T_2$

$\therefore m_{TT_1} \times m_{TT_2} = -1$ NB: Similarly $m_{TT_2} = \frac{-1}{t_2 t}$

$\therefore \frac{-1}{t_1 t} \times \frac{-1}{t_2 t} = -1$

$\frac{1}{t_1 t_2 t^2} = -1$

$\underline{\underline{\frac{-1}{t_1 t_2} = t^2}}$

(ii). Normal at T

$xy = c^2$

$y = \frac{c^2}{x}$

$= c^2 x^{-1}$

$\frac{dy}{dx} = -c^2 x^{-2}$

At $(ct, c/t)$

$m = \frac{-c^2}{c^2 t^2}$

$m_2 = t^2$

$\therefore$ ~~Eqn: $y - c/t = t^2(x - ct)$~~

$m_{T_1T_2} = \frac{\frac{c}{t_1} - \frac{c}{t_2}}{ct_1 - ct_2}$ ~~t^2~~

$= \frac{-1}{t_2 t_1}$ (similar as (a))

since $t^2 = \frac{-1}{t_2 t_1}$

$\therefore m_{T_1T_2} = m_2$

$\therefore$ $\underline{\underline{T_1T_2 \parallel \text{Normal at } T}}$

4.(a). $\int \frac{1}{1+\sin x}\,dx$

Let $t = \tan x/2$ $\quad \frac{dt}{dx} = \frac{1}{2}(1+t^2)$

$\sin x = \frac{2t}{1+t^2}$ $\quad \frac{2dt}{1+t^2} = dx$

$I = \int \frac{1}{\frac{1+t^2+2t}{1+t^2}} \cdot \frac{2dt}{1+t^2}$

$= 2\int \frac{dt}{(1+t)^2} \Longrightarrow 2\int (1+t)^{-2}\,dt$

$= \left[-2(1+t)^{-1}\right] + c$

$= \underline{\underline{\frac{-2}{1+\tan x/2}}} + c.$

(b).(i) ~~$m=0$~~ $m = P'(x) = 0$

(ii). $P(x) = ax^8 + bx^7 + 2$

$P(-1) = 0$.

$a - b + 2 = 0$. ——(1)

$P'(x) = 8ax^7 + 7bx^6$

$P'(-1) = -8a + 7b = 0$. ——(2)

$b = a+2$.

$-8a + 7(a+2) = 0$. $\quad b = 14+2$

$-8a + 7a + 14 = 0$ $\quad \underline{\underline{b = 16.}}$

$\underline{\underline{a = 14.}}$

(c). $x^3 - 3x^2 + 2x - 1 = 0$.

~~at~~ For α^2, $y = x^2$

$x = \sqrt{y}$

$y\sqrt{y} - 3y + 2\sqrt{y} - 1 = 0$.

$\sqrt{y}(y+2) = 3y+1$

$y(y+2)^2 = (3y+1)^2$

$y^3 + 4y^2 + 4y = 9y^2 + 12y + 1$

$y^3 - 5y^2 - 8y - 1 = 0$

or $\underline{\underline{x^3 - 5x^2 - 8x - 1 = 0.}}$

(ii) $\alpha^2 + \beta^2 + \gamma^2$

$= \underline{\underline{\sum \alpha^2 = 5}}$ $\quad$ Test. $(\sum\alpha)^2 - 2(\sum\alpha\beta)$

$= 9 - 2\times 2$

$\underline{\underline{= 5.}}$

(iii) $\sum\alpha^3 = \alpha^3 + \beta^3 + \gamma^3$

If α is a root, then $\alpha^3 = 3\alpha^2 - 2\alpha + 1$

Similarly for β^3, γ^3

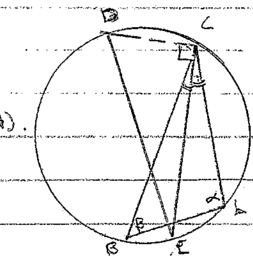
$\therefore \alpha^3 + \beta^3 + \gamma^3 = 3(\alpha^2 + \beta^2 + \gamma^2) - 2(\alpha+\beta+\gamma) + 3$

$\sum\alpha^3 = 3(5) - 2(3) + 3$

$= 15 - 6 + 3$

$\underline{\underline{= 12.}}$

(d).



$B\hat{C}E = E\hat{C}A$ (given, equal midpt. subtend equal angles)

$\therefore D\hat{C}E = \pi/2$, $\quad B\hat{C}A = \pi - (\alpha+\beta)$

$\therefore B\hat{C}E = \frac{1}{2} B\hat{C}A = \frac{\pi}{2} - \frac{(\alpha+\beta)}{2}$

$\therefore D\hat{C}A = \pi/2 + \pi/2 - \frac{(\alpha+\beta)}{2}$

~~Construct~~ DB. Construct DB.

$\therefore\ D\hat{B}A = \pi - O\hat{C}A$ (opp. cyclic quad)

$= \pi - \pi + \frac{\alpha+\beta}{2}$

$= \frac{\alpha+\beta}{2}$

But $\angle CBA = \beta$ (given)

$\therefore\ D\hat{B}C = \frac{\alpha+\beta}{2} - \beta$ (Angle sum)

$= \frac{\alpha-\beta}{2}$

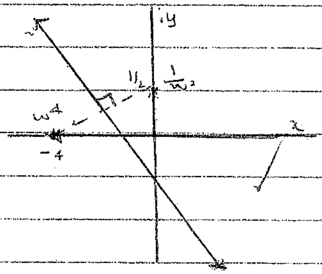
$\therefore\ D\hat{E}C = \frac{\alpha-\beta}{2}$ (Angle subtended on the same arc BE).

5. (a)(i). $w = 1 - i$

$= \sqrt{2}\,\text{cis}(-\pi/4)$.

(ii). $w^4 = 4\,\text{cis}(-\pi)$ $\quad w^{-2} = \frac{1}{2}\text{cis}(\pi/2)$

$= -4$ $\quad = i\frac{1}{2}$



(iii). $|z - w^4| = |z - \frac{1}{w^2}|$ $\quad (0, 1/2) \cap (-4, 0)$

midpoint $(-2, \frac{1}{4})$

$m = \frac{1/2}{4}$

$= \frac{1}{8}$.

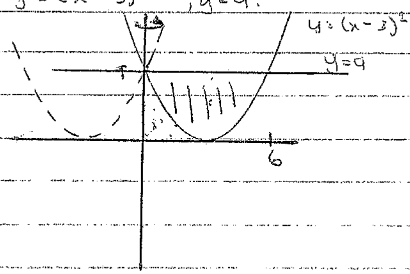
$\therefore m_2 = -8$.

$\therefore$ Eqn: $y - \frac{1}{4} = -8(x-2)$

$4y - 1 = -32x + 64$.

$32x + 4y - 65 = 0$.

(b) $y = (x-3)^2$, $y = 9$.



$V = 2\pi\int_0^6 rh\,dx$

$r = x,\ h = 9 - y$

$V = 2\pi\int_0^6 x(9-y)\,dx$

$= 2\pi\int_0^6 x(9-(x-3)^2)\,dx$

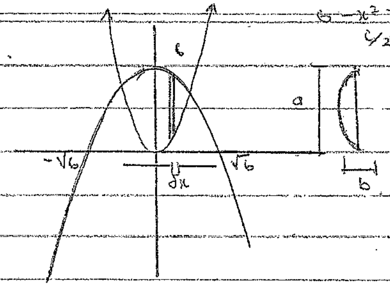
$= 2\pi\int_0^6 9x - x^3 + 6x^2 - 9x$

$= 2\pi\left[-\frac{x^4}{4} + 2x^3\right]_0^6$

$= 2\pi[-324 + 432]$

$= 216\pi$

(c).



$3b = a$.

$b = a/3$.

$\therefore$ Area $= \frac{\pi ab}{2}$

$= \frac{\pi a^2}{6}$

But $a = (y_2 - y_1)$

$= 6 - x^2 - 2x^2$

$a = 6 - 3x^2$

$\therefore V = \frac{\pi}{6}\int_{-\sqrt{2}}^{\sqrt{2}}(6-3x^2)^2\,dx$

$= \pi/3\int_0^{\sqrt{2}}(36 - 36x^2 + 9x^4)\,dx$

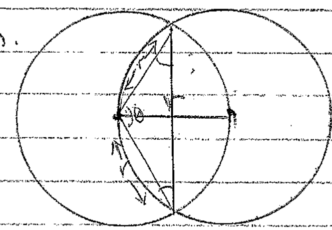
$= \pi/3\left[36x - \frac{36x^3}{3} + \frac{9x^5}{5}\right]_0^{\sqrt{2}}$

$= \pi/3\left[36x - 12x^3 + \frac{9x^5}{5}\right]_0^{\sqrt{2}}$

$= \pi/3\left[36\sqrt{2} - 24\sqrt{2} + \frac{36\sqrt{2}}{5}\right]$

$= \frac{96\pi\sqrt{2}}{15} = \frac{32\sqrt{2}\pi}{5}$

6. (a).



Area $= 2\times\frac{1}{2}r^2(\theta - \sin\theta)$

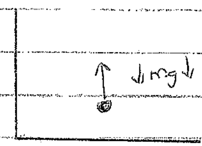
$\theta = 2\pi/3$.

$= r^2\left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right)\right)$

$= r^2\left(\frac{2\pi}{3} - \sqrt{3}/2\right)$

$= \frac{r^2}{6}(4\pi - 3\sqrt{3})$

(b).



(i). $m = 3$ $\quad R = \frac{v^2}{2}$.

$mg = 18N$.

$m\ddot{y} = -(mg + R)$

$3\ddot{y} = -(18 + \frac{v^2}{2})$

$\ddot{y} = -(6 + \frac{v^2}{6})$

$= -\frac{1}{6}(36 + v^2)$

(ii) $v\frac{dv}{dy} = -\frac{1}{6}(36 + v^2)$

$\frac{dv}{dy} = \frac{-(36+v^2)}{6v}$

$\frac{dy}{dv} = \frac{-6v}{36 + v^2}$

$\int dy = \int \frac{-6v}{36+v^2} \cdot dv$

$= -3 \int \frac{2v}{36+v^2} \cdot dv$

$= -3 \ln|36+v^2| + c.$

At $y=0,\ v=6$

$\therefore c = 3\ln|72|$

$\therefore y = 3\ln\left|\frac{72}{36+v^2}\right|$

$\frac{y}{3} = \ln\left|\frac{72}{36+v^2}\right|$

$e^{y/3} = \frac{72}{36+v^2}$

$36+v^2 = \frac{72}{e^{y/3}}$

$v^2 = 72e^{-y/3} - 36$

$\underline{v^2 = 36(2e^{-y/3} - 1)}$

(iii). Max Height $y=H,\ v=0.$ $\Longrightarrow$ $y = 3\ln\left|\frac{72}{36+v^2}\right|$

$H = 3\ln\left|\frac{72}{36}\right|$

$= 3\ln 2.$

$\doteqdot 2.079$

(iv). $\frac{dv}{dt} = -\frac{1}{6}(v^2+36)$

$\frac{dt}{dv} = \frac{-6}{v^2+36}$

$t = -6\int \frac{dv}{v^2+36}$

$= \frac{-6}{6}\tan^{-1}\left(\frac{v}{6}\right) + c$

$t = -\tan^{-1}\left(\frac{v}{6}\right) + c.$

At $t=0,\ v=6.$

$\therefore 0 = -\tan^{-1}(1) + c$

$c = \tan^{-1}(1)$

$= \pi/4.$

$\therefore t = -\tan^{-1}\left(\frac{v}{6}\right) + \pi/4.$

To Max Height $v=0,\ t=T$

$T = \pi/4$ secs.

$\therefore$ Total time $= \pi/2$ secs.

$\therefore \pi/2 = -\tan^{-1}\left(\frac{v}{6}\right) + \pi/4.$

$\pi/4 = -\tan^{-1}\left(\frac{v}{6}\right)$

$\tan(-\pi/4) = \frac{v}{6}$

$-1 = v \quad v = -6.$

Motion: 0, $\downarrow 18$, $\uparrow \frac{v^2}{2}$

$m\ddot{y} = 18 - \frac{v^2}{2}$

$3\ddot{y} = \frac{36-v^2}{2}$

$\ddot{y} = \frac{1}{6}(36-v^2)$

$v\frac{dv}{dx} = \frac{1}{6}(36-v^2)$

$\frac{dx}{dv} = \frac{6v}{36-v^2}$

$\therefore \int_0^{3\ln 2} 1\, dx = -3\int_0^v \frac{-2v}{36-v^2}\, dv$

$[x]_0^{3\ln 2} = -3\left[\ln|36-v^2|\right]_0^v$

$3\ln 2 = -3\left[\ln(36-v^2) - \ln 36\right]$

$-\ln 2 = \ln\left(\frac{36-v^2}{36}\right)$

$\therefore \frac{1}{2} = \frac{36-v^2}{36} \Rightarrow v^2 = 18$

$v = 3\sqrt{2}$ m/s

7.(a)(i) $P(z) = z^6 + 1$

$P(i) = (i)^6 + 1$

$= -1 + 1$

$= 0.$

$\therefore i$ is root

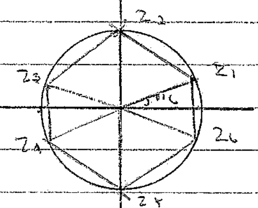
(ii). $z^6 = -1$

$z = \text{cis}\left(\frac{2k\pi + \pi}{6}\right)$

$z_1 = \text{cis}\,\pi/6,\ \text{cis}\,\pi/2,\ \text{cis}\,\frac{5\pi}{6},\ \text{cis}\,\frac{7\pi}{6},\ \text{cis}\left(\frac{9\pi}{6}\right),\ \text{cis}\,\frac{11\pi}{6},$

$\text{cis}\,\frac{-5\pi}{6},\ \text{cis}(-\pi/2),\ \text{cis}(-\pi/6)$

$\underline{z_1 = \text{cis}\,\pi/6,\ z_2 = i,\ z_3 = \text{cis}\,5\pi/6,\ z_4 = \text{cis}\,\frac{-5\pi}{6},\ z_5 = -i,\ z_6 = \text{cis}\,{-\pi}/6}$



(iii). $(z^2+1)(z^4 - z^2 + 1) = z^6 + 1$

LHS $= z^6 - z^4 + z^2 + z^4 - z^2 + 1$

$= z^6 + 1$

$=$ RHS.

(iv). $z^4 - z^2 + 1 = 0.$

Let $u = z^2$

$u^2 - u + 1 = 0.$

$u = \frac{1 \pm \sqrt{1-4}}{2}$

$= \frac{1 \pm \sqrt{3i^2}}{2}$

$= \frac{1 \pm i\sqrt{3}}{2}.$

$\therefore z^2 = \frac{1 \pm i\sqrt{3}}{2}.$

$\Longrightarrow \therefore z = \text{cis}\,\pi/6 = \frac{\sqrt{3}}{2} + i\frac{1}{2}.$

$= \text{cis}\,5\pi/6 = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$

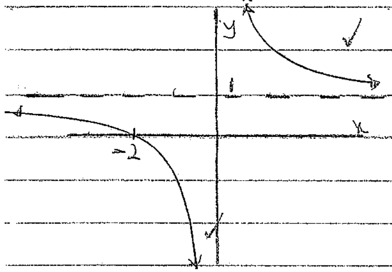
$= \text{cis}\,\frac{-5\pi}{6} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

$= \text{cis}\,{-\pi}/6 = \sqrt{3}/2 - i\,1/2$

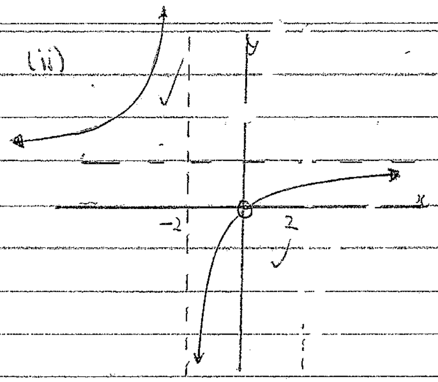
$\therefore z = \pm\frac{1}{2}(\sqrt{3} + i),\ \pm\frac{1}{2}(\sqrt{3} - i)$

(b) (i) $f(x) = \frac{x+2}{x}$

$= 1 + \frac{2}{x}$



(ii)



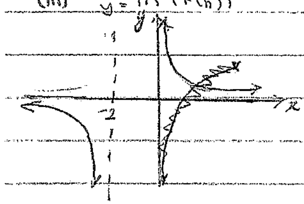
$y = \frac{1}{f(x)}$

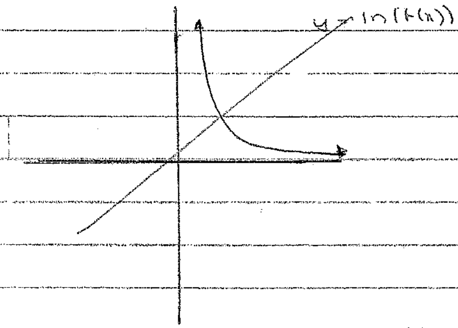
$= \frac{x}{x+2}$

$= \frac{x+2}{x+2} - \frac{2}{x+2}$

$= 1 - \frac{2}{x+2}$

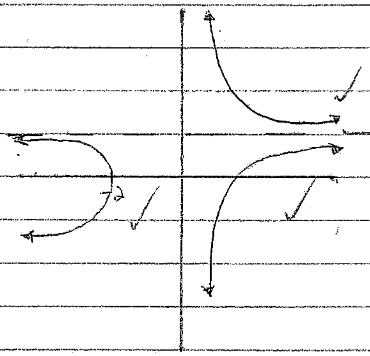
(iii) $y = \ln(f(x))$





(iv) $y^2 = f(x)$

$y = \pm\sqrt{f(x)}$



6. (a). $\int_0^\pi x\, f(\sin x)\,dx = \pi/_2 \int_0^\pi f(\sin x)\,dx$

Let $x = \pi - u$. At $x = \pi$, $u = 0$.

$dx = -du$ $\quad x = 0,\ u = \pi$

$I = \int_\pi^0 (\pi - u) f(\sin(\pi - u))\,dx$ $\qquad \sin\theta = \sin(\pi - \theta)$

$= \int_\pi^0 (\pi) f(\sin(u)) - u\, f(\sin(u))(-du)$

$= \int_0^\pi \pi f(\sin(u)) - u\, f(\sin u)\,du$ $\qquad$ Reverting to x.

$I = \int_0^\pi \pi f(\sin x) - x f(\sin x)\,dx$

$2I = \int_0^\pi \pi f(\sin x) \Rightarrow I = \pi/_2 \int_0^\pi f(\sin x)\,dx$ $\quad$ = RHS.

(a) (ii). $\int_0^\pi x \cos^2 x\,dx$

$\int_0^\pi x(1 - \sin^2 x)\,dx$

$= \int_0^\pi x - \int_0^\pi x \sin^2 x\,dx$

$= \left[\frac{x^2}{2}\right]_0^\pi - \int_0^\pi x \sin^2 x\,dx$

$= \frac{\pi^2}{2} -$ $\qquad f(x) = \sin x$.

$= \frac{\pi^2}{2} - \pi/_2 \int_0^\pi \sin^2 x\,dx$

$= \frac{\pi^2}{2} - \frac{\pi}{4} \int_0^\pi 1 - \cos 2x\,dx$ $\qquad \cos 2x = 1 - 2\sin^2 x$

$\qquad \sin^2 x = \frac{1 - \cos 2x}{2}$

$= \frac{\pi^2}{2} - \frac{\pi}{4}\left[x - \frac{\sin 2x}{2}\right]_0^\pi$

$= \frac{\pi^2}{2} - \frac{\pi}{4}[\pi]$

$= \frac{\pi^2}{4}$

Test: $\int x \sin^2 x$

$= \frac{1}{2}\int_0^\pi x(1 - \cancel{x}\cos 2x)$

$= \frac{1}{2}\int_0^\pi x - x\cos 2x$

$= \left[\frac{x^2}{4}\right] - \frac{1}{2}\int x \cos 2x\,dx$

$u = x \quad v = \frac{\sin 2x}{2}$

$u' = 1 \quad v' = \cos 2x$

$= \left[\frac{x^2}{4}\right]_0^\pi - \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^\pi + \frac{1}{4}\int \sin 2x$

$= \frac{\pi^2}{4} - \left[\frac{\cos 2x}{8}\right]_0^\pi$

$= \frac{\pi^2}{4} - 0$.

$= \frac{\pi^2}{4}$.

(b) ~~$a + b + c = 1$, $a + b + c \geq 3\sqrt[3]{abc}$.~~

~~$a^2 + b^2 \geq 2ab$ (AP ≥ GP)~~

~~$a^2 + c^2 \geq 2ac$~~

~~$c^2 + b^2 \geq 2bc$~~

~~$\therefore 2(a^2 + b^2 + c^2) \geq 2(ab + bc + ca)$~~

~~But $(a^2 + b^2 + c^2) = a^2 + b^2 + c^2 + 2(a + b + c)$~~

~~$\therefore a^2 + b^2 + c^2 \geq 3(ab + bc + ca)$~~

~~Since $a + b + c = 1$~~

~~Then $ab + bc + ca \leq 1/3$~~

$a + \frac{1}{a} \geq 2$.

$(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) = 1 + 1 + \left(\frac{a}{b} + \frac{b}{a}\right) \geq 2 + 2$.

$\therefore (a + b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4$

$\therefore (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 1 + 1 + 1 + \left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{c}{a} + \frac{a}{c}\right) \geq 9$.

$\therefore (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$

But $a + b + c = 1$ $\quad \therefore \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$

(iii). $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)$

$\left(\frac{1}{ab}-\frac{1}{a}-\frac{1}{b}+1\right)\left(\frac{1}{c}-1\right)$

$\frac{1}{abc}-\frac{1}{ac}-\frac{1}{bc}+\frac{1}{c}-\frac{1}{ab}+\frac{1}{a}+\frac{1}{b}-1.$

$\frac{1}{abc}-\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right)+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ ✓

$\frac{a+b+c}{3}\geq\sqrt[3]{abc}$

$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 9.$ ✓

$\frac{(a+b+c)^3}{27}\geq abc$

$\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}$

$\therefore abc\leq\frac{1}{27}$

$=\frac{a+b+c}{abc}$ ✓

$\frac{1}{abc}\geq 27$

$=\frac{1}{abc}\geq 27.$ ✓

$\therefore \frac{1}{abc}-\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right)+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 27-27+9.$

$\therefore \geq 9.$ ✓

$\therefore$ Min value $=9.$

V. Good!