Ascham School

Form 5
Preliminary Course Examination

Mathematics

August 2004

Time allowed: 3 hours

Instructions:

- All questions should be attempted
- All necessary working should be shown
- Marks may not be awarded for careless or badly presented work
- Do each question in a separate booklet
- Write your name and your teacher's name on each booklet
- Clearly label the front of each booklet with the number of the question
- Approved calculators may be used.

Question 1 (12 marks) Marks

a) Evaluate to 3 significant figures $\dfrac{8.76-5.3}{8.76\times 5.3}$ 2

b) If $x=13.2$ and $y=4\frac{1}{3}$ find correct to 2 decimal places the value of $\dfrac{x^2}{y^2}$ 2

c) Simplify leaving your answer as a power of 5

$5^3\times 125^{\frac{2}{3}}$ 2

d) Express with a rational denominator:

$\dfrac{1}{2\sqrt{3}-3}$ 2

e) The number of nurses on the staff of a hospital is reduced by 12%.
There are now 264 nurses. How many nurses were there originally? 2

f) Expand and simplify $4m^{-2}\div\frac{1}{2}m^{-1}$ 2

Question 2 (12 marks) **(start this question in a new booklet)** Marks

a) Expand and simplify

$3(x-y)-(x+y)^2$ 2

b) Factorise fully

$8x - 3ay - 2ax + 12y$ 2

c) Solve

$2x-\dfrac{x-1}{2}=2$ 3

d) Solve

$|2x-5|=2$ 2

e) Solve the following pair of simultaneous equations

$3x-y=6$
$2x+2y=5$ 3

Question 3 (12 marks) **(start this question in a new booklet)** **Marks**

a) If $\sin\theta = \frac{2}{7}$ and θ is acute, find the exact values of $\cos\theta$. 2

b) Solve for y, $\log_y\left(\frac{9}{16}\right) = 2$ 2

c) Sketch $y = \cos\theta$ for $0 \le x \le 360$ 1

d) Show that $\frac{\cos\theta}{1-\sin\theta} - \frac{\cos\theta}{1+\sin\theta} = 2\tan\theta$ 2

e) Given the function $f(x) = \begin{cases} x^2+1 & \text{for } x \ge 1 \\ 2 & \text{for } x < 1 \end{cases}$

i) Sketch the graph of $f(x)$ 2

ii) Find the value of $f(2)$ 1

f) On a number plane sketch the region where $y > x^2 - 1$ and $y \le x$ 2

Question 4 (12 marks) **(start this question in a new booklet)** **Marks**

A triangle ABC has vertices A(2,2) , B(5,7) and C(-3,1).

i) Draw a diagram with triangle ABC 1

ii) Find the length of BC 2

iii) Find the gradient of BC 2

iv) Show that the equation of BC is $4y = 3x + 13$ 2

v) Find the length of the perpendicular from A to BC 3

vi) Hence find the area of triangle ABC 2

Question 5 (12 marks) **(start this question in a new booklet)** **Marks**

a) For what values of x is the function $f(x) = \frac{x+2}{2x-1}$ undefined ? 1

b) What is the natural domain of the function $g(x) = \sqrt{4-2x}$? 2

c) Graph the function $y = |x+1|$ and state its <u>domain</u> and <u>range</u>. 4

d) Mary walks from point A to point B for 3.5 km on a bearing of 126°. She then changes direction and walks to point C, 2.3 km away on a bearing of 216°, and stops.

i) Draw a diagram showing all details of her journey. 1

ii) Find the distance AC. Give your answer correct to one decimal place. 2

iii) What is the bearing of point A from point C? Give your answer to the nearest minute. 2

Question 6 (12 marks) **(start this question in a new booklet)** **Marks**

a) Find the derivative of the following functions:

i) $f(x) = 2x^3 - \frac{x}{4} + 7$ 1

ii) $y = \sqrt{5-x^2}$ 2

iii) $y = \frac{x+3}{x^2+1}$ 2

iv) $y = \frac{x(x-2)^6}{2}$ 2

b) Given that $x = 2t^4 + 100t^3$, find $\frac{dx}{dt}$ and hence the values of t for which $\frac{dx}{dt} = 0$ 2

c) At which points are the tangents to the curve $y = x^3 - 6x^2 + 2x - 9$ parallel to the line $y = -7x + 4$? 3

Question 7 (12 marks) **(start this question in a new booklet)** **Marks**

a) Find the minimum value of the expression $x^2+4x-21$

b) Solve $x^4-5x^2+4=0$ 3

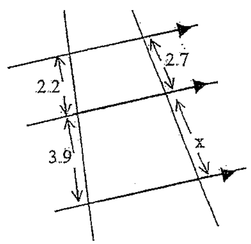
c) Find the values of p for which the equation $3x^2-2px+3=0$ has no real roots. 3

d) Given α and β are the roots of the equation $x^2+2x+4=0$, find:

- i) $\alpha+\beta$ 1
- ii) $\alpha\beta$ 1
- iii) $(\alpha+1)(\beta+1)$ 2
- iv) $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$ 2

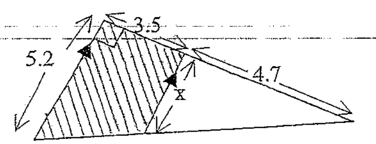
Question 8 (12 marks) **(start this question in a new booklet)** **Marks**

a) Find the value of x , giving reasons. (answer to 1 decimal place) 2



b) In the following diagram:

- i) Find the value of x , giving reasons. (answer to 1 decimal place) 2
- ii) Find the shaded area. (answer to 1 decimal place) 2

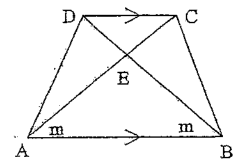


(Question 8 continued on the next page)

(Question 8 continued)

c) In the diagram below AB || DC and ∠CAB = ∠ABD = m.

- i) Show that CE = DE 2
- ii) Prove that ΔABC ≡ ΔBAD 2
- iii) Hence show that ∠DAC = ∠CBD 2



Question 9 (12 marks) **(start this question in a new booklet)** **Marks**

a) For what values of x is $x^2 \geq (x+1)(x+2)$

b) Simplify $\dfrac{16^{n+2}\times 4^{n-1}}{64^n}$ 2

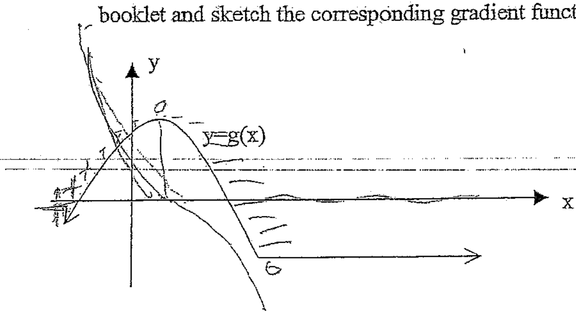
c) The first 3 terms of an arithmetic sequence are 48, 45 and 42

- i) Find an expression for the sum of the first n terms 2
- ii) Determine the least value of n so that the sum of the first n terms is negative. 2

d) Find the limiting sum of the series

$$2+\frac{2}{\sqrt{2}+1}+\frac{2}{(\sqrt{2}+1)^2}+\dots$$ 2

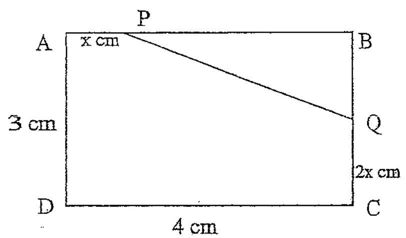
e) The graph of the curve $y = g(x)$ is drawn below. Copy the graph into answer booklet and sketch the corresponding gradient function. 2



Question 10 (12 marks) **(start this question in a new booklet)** **Marks**

a) If $ax+\frac{b}{x^2}=y$ cuts the x-axis at (2,0) and the gradient of the tangent at that point is 1, find a and b. 4

b)



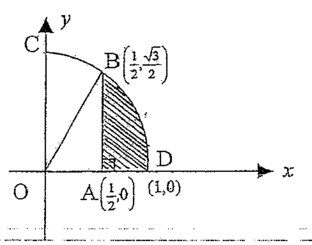
In the diagram above ABCD is a rectangle with CD $=4$ cm and AD $=3$ cm. P and Q are points on AB and BC respectively. If AP $=\frac{1}{2}$QC $=x$ cm.

i) Show that the area of pentagon APQCD is given by the expression

$$6+\frac{11x}{2}-x^2$$ 2

ii) Hence find the maximum area of the rectangle 2

c) The diagram below shows the first quadrant of the circle $x^2+y^2=1$. Points A and D have coordinates $\left(\frac{1}{2},0\right)$ and (1,0) respectively. AB is perpendicular to the x axis with point B having coordinates $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$



i) Find the exact value of $\angle$ COB 2

ii) Show that the exact value of the shaded area ABD is

$$\frac{4\pi-3\sqrt{3}}{24}$$ 2

QUESTION 1

(a) $\dfrac{8.76-5.3}{8.76\times5.3}$

$= \dfrac{3.46}{46.428}$

$= 0.0745$ (3 sig. fig.)

(b) $\dfrac{x^2}{y^2} = \dfrac{(13.2)^2}{(4\frac{1}{3})^2}$

$= \dfrac{174.24}{\frac{169}{9}}$

$= 9.27905$

$= 9.30$ (2 d.p.)

(c) $5^3 \times 125^{2/3}$

$= 5^3 \times (5^3)^{2/3}$

$= 5^3 \times 5^2$

$= 5^5$

(d) $\dfrac{1}{2\sqrt3-3} = \dfrac{1}{2\sqrt3-3} \times \dfrac{2\sqrt3+3}{2\sqrt3+3}$

$= \dfrac{2\sqrt3+3}{12-9}$

$= \dfrac{2\sqrt3+3}{3}$

(e) $88\% \rightarrow 264$

$1\% \rightarrow \dfrac{264}{88}$

$\therefore 100\% \rightarrow 100 \times \dfrac{264}{88}$

$= 300$

$\therefore$ there were 300 nurse

(f) $4m^{-2} \div \frac{1}{2} m^{-1}$

$= 8m^{-1}$

$= \dfrac{8}{m}$

QUESTION 2:

(a) $3(x-y)-(x+y)^2$

$= 3x-3y-(x^2+2xy+y^2)$

$= 3x-3y-x^2-2xy-y^2$

(b) $8x-3ay-2ax+12y$

$= 8x-2ax-3ay+12y$

$= 2x(4-a)-3y(a-4)$

$= (2x+3y)(\cancel{a-4})$ $2x(4-a)+3y(4-a)$

$(2x+3y)(4-a)$

(c) $2x - \dfrac{x-1}{2} = 2$

$4x - (x-1) = 4$

$4x - x + 1 = 4$

$3x + 1 = 4$

$3x = 3$

$x = 1$

$x = \frac{5}{3}$ $\quad -\frac{1}{2}$

(d) $|2x-5| = 2$

$2x-5 = 2$ OR $2x-5 = -2$

$2x = 7$ $\qquad 2x = 3$

$x = 3\frac{1}{2}$ OR $x = 3/2$

(e) $3x - y = 6$ --- ①

$2x + 2y = 5$ --- ②

① × 2: $6x - 2y = 12$

② + ③: $8x = 17$

$x = \dfrac{17}{8}$

Subst: $x = \dfrac{17}{8}$ into ②

$2 \times \dfrac{17}{8} + 2y = 5$

$2y = 5 - \dfrac{34}{8}$

$2y = \dfrac{6}{8}$

$y = \dfrac{3}{8}$

$\therefore$ sol'n is $\left.\begin{array}{l} x = \frac{17}{8} \\ y = \frac{3}{8} \end{array}\right\}$

QUESTION 3

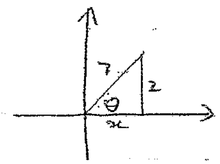
(a)

$7^2 = x^2 + 2^2$

$49 = x^2 + 4$

$x^2 = 45$

$x = \sqrt{45}$



$\therefore \cos\theta = \dfrac{\sqrt{45}}{7}$

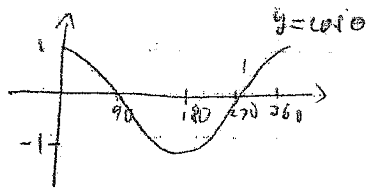
$= \dfrac{3\sqrt{5}}{7}$

(b) $\log_y\left(\frac{9}{16}\right) = 2$

$\therefore \frac{9}{16} = y^2$

$y = \pm\frac{3}{2}$

(c)



(d) LHS $= \dfrac{\cos\theta}{1-\sin\theta} - \dfrac{\cos\theta}{1+\sin\theta}$

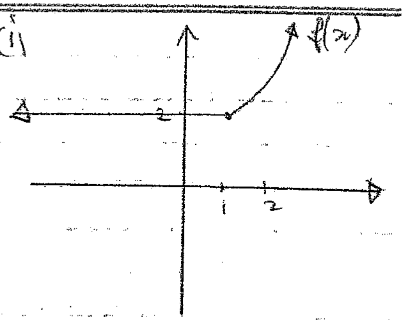
$= \dfrac{\cos\theta(1+\sin\theta) - \cos\theta(1-\sin\theta)}{1-\sin^2\theta}$

$= \dfrac{\cos\theta + \cos\theta\sin\theta - \cos\theta + \cos\theta\sin\theta}{\cos^2\theta}$

$= \dfrac{2\sin\theta\cos\theta}{\cos^2\theta}$

$= \dfrac{2\sin\theta}{\cos\theta} = 2\tan\theta$ = RHS

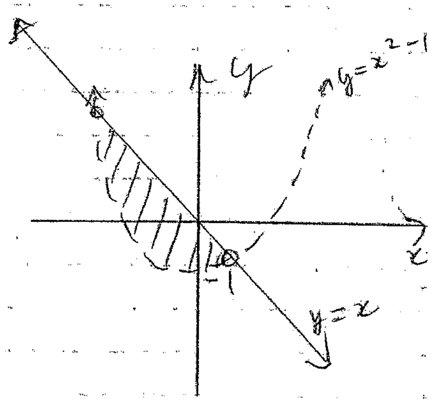
(e)(i)



(ii) $f(2) = 2^2 + 1$

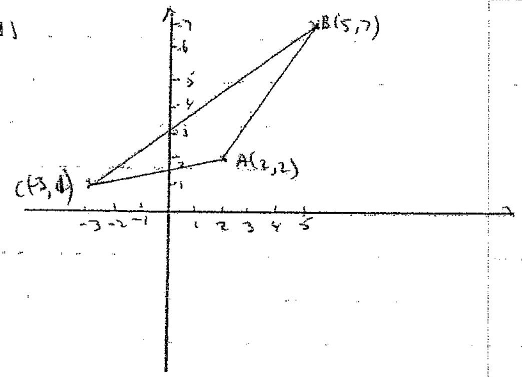
$= 5$

(f)



QUESTION 4

(i)



(ii) $BC = \sqrt{(-3-5)^2 + (-1-7)^2}$

$= \sqrt{64+36}$

$= \sqrt{100}$

$= 10$ units

(iii) Grad. $BC = \dfrac{7-1}{5-3}$

$= \dfrac{6}{8}$

$= \dfrac{3}{4}$

(iv) EQUATION$_{BC}$:

$y - 7 = \frac{3}{4}(x-5)$

$4y - 28 = 3x - 15$

$4y = 3x + 13$

(v) Perp. Length $= \dfrac{|ax_1 + by_1 + c_1|}{\sqrt{a^2+b^2}}$ …… $3x - 4y + 13 = 0$, $A(2,2)$

$= \dfrac{|3\times2 - 4\times2 + 13|}{\sqrt{3^2+(-4)^2}}$

$= \dfrac{|6-8+13|}{\sqrt{9+16}} = \dfrac{11}{5}$

(vi) AREA of ΔABC

$= \frac{1}{2} \times \frac{11}{5} \times 10\ u^2$

$= 11\ u^2$

QUESTION 5

(a) $f(x) = \dfrac{x+2}{2x-1}$

undefined for: $2x-1=0$

$x = \frac{1}{2}$

(b) $g(x) = \sqrt{4-2x}$

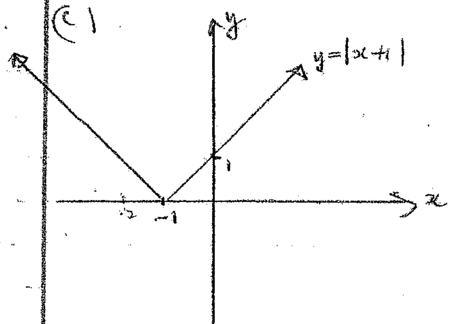
$4-2x \geq 0$

$2x \leq 4$

$x \leq 2$

$\therefore$ domain is $x \leq 2$

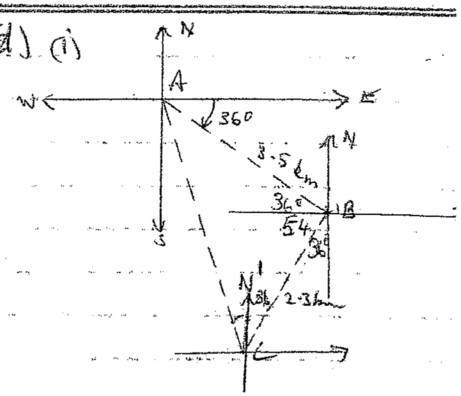
(c)



Domain: $x \in R$

Range: $y \geq 0$

(d) (i)



(ii) $\angle ABC = 90°$

$\therefore AC^2 = 3.5^2 + 2.3^2$

$AC = 4.2$ km. (1 d.p.)

(iii) Need to find $N'CA$

$\tan A = \dfrac{2.3}{3.5}$

$A = 33° 19'$ (nearest minute)

$\therefore N'CA = 180 - (90 + 36 + 33°19')$

$= 20° 41'$

$\therefore$ Bearing is $360° - 20°41$

QUESTION 6:

(a) (i) $f(x) = 2x^3 - \frac{x}{4} + 7$

$f'(x) = 6x^2 - \frac{1}{4}$

(ii) $y = \sqrt{5-x^2}$

$y' = \frac{1}{2}(5-x^2)^{-\frac{1}{2}} \times -2x$

$= -x(5-x^2)^{-\frac{1}{2}}$

$= \dfrac{-x}{\sqrt{5-x^2}}$

(iii) $y = \dfrac{x+3}{x^2+1}$

$y' = \dfrac{(x^2+1)\times 1 - (x+3)\times 2x}{(x^2+1)^2}$

$= \dfrac{x^2+1-2x^2-6x}{(x^2+1)^2}$

$= \dfrac{-x^2-6x+1}{(x^2+1)^2}$

(iv) $y = \dfrac{x(x-2)^6}{2}$

$\dfrac{dy}{dx} = \frac{1}{2} \times \left[x \times 6(x-2)^5 + 1 \times (x-2)^6\right]$

$= \frac{1}{2}\left[(x-2)^5(6x + (x-2))\right]$

$= \frac{1}{2}\left[(x-2)^5(7x-2)\right]$

(b) $x = 2t^4 + 100t^3$

$\frac{dx}{dt} = 8t^3 + 300t^2$

$0 = 8t^3 + 300t^2$

$= 4t^2(2t + 75)$

$\therefore t = 0$ OR $t = -37\frac{1}{2}$

(c) $y = -7x + 4$ has gradient of -7

for $y = x^3 - 6x^2 + 2x - 9$

$\frac{dy}{dx} = 3x^2 - 12x + 2$

$\therefore 3x^2 - 12x + 2 = -7$ ... parallel.

$3x^2 - 12x + 9 = 0$

$3(x^2 - 4x + 3) = 0$

$3(x-3)(x-1) = 0$

$\therefore x = 3$ OR $x = 1$

for $x = 3$: $y = 3^3 - 6(9) + 6 - 9$

$= -30$

$\therefore (3, -30)$

for $x = 1$: $y = 1^2 - 6(1) + 2 - 9$

$= -12$

$\therefore (1, -12)$

QUESTION 7:

(a) $x^2 + 4x - 21$

$x = \frac{-b}{2a}$

$= \frac{-4}{2}$

$= -2$

Subst $x = -2$: $(-2)^2 + 4(-2) - 21$

$= 4 - 8 - 21$

$= -25$

$\therefore$ min value is -25

(b) $x^4 - 5x^2 + 4 = 0$ ... let $u = x^2$

$u^2 - 5u^2 + 4 = 0$

$(u-4)(u-1) = 0$

$u = 4$ OR $u = 1$

$x^2 = 4$ OR $x^2 = 1$

$\therefore x = \pm 2$ OR $x = \pm 1$

(c) $3x^2 - 2px + 3 = 0$

has no real roots when $\Delta < 0$

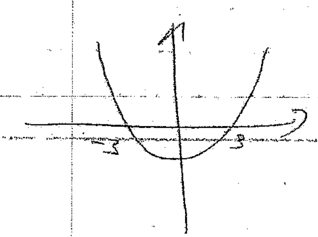
$b^2 - 4ac < 0$

$(-2p)^2 - 4 \times 3 \times 3 < 0$

$4p^2 < 36$

$p^2 < 9$

$p^2 - 9 < 0$



$\therefore -3 < p < 3$

(d) $x^2+2x+4=0$

(i) $\alpha+\beta = \frac{-b}{a}$

$= \frac{-2}{1}$

$= -2$

(ii) $\alpha\beta = \frac{c}{a}$

$= \frac{4}{1}$

$= 4$

(iii) $(\alpha+1)(\beta+1)$

$= \alpha\beta + \alpha + \beta + 1$

$= 4 - 2 + 1$

$= \underline{\underline{3}}$

(iv) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

$= \frac{\alpha^2+\beta^2}{\alpha^2\beta^2}$

$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$

$= \frac{(-2)^2 - 2\times 4}{4^2}$

$= \frac{4-8}{16} = -\frac{1}{4}$

QUESTION 8

(a) $\frac{x}{2.7} = \frac{3.9}{2.2}$ ... ratio of intercepts on // lines

$x = 4.8$ (1 d.p.)

(b) (i) $\frac{x}{5.2} = \frac{4.7}{8.2}$ (corresp. sides of m Δs)

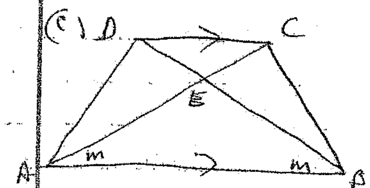
$x = 3.0$ (1 d.p.)

(ii) Shaded area

$= \frac{1}{2}[5.2\times 8.2 - 2.9804\times 4.7]$

$= 14.3\ m^2$ (1 d.p.)

(c)



(i) $\angle CDE = \angle DBA = m$ (alt. ∠'s // lines)

$\angle DCE = \angle CAB = m$ (alt, // lines)

∴ Δ DEC is isosceles

∴ DE = CE

(ii) In Δ's ABC & BAD

AB is common

$\angle CAB = \angle DBA = m$ (given)

$CA = CE + EA = DE + EA = DB$ (sum of equal sides of isosceles Δ's DEC & BEA)

∴ ΔABC ≡ ΔBAD (SAS)

(iii) $\angle DAB = \angle CBA$ -- corresp. angles of congruent triangles

$\angle DAC = \angle DAB - m^\circ$ (adjacent angles)

$\angle CBD = \angle CBA - m^\circ$ (adjacent angles)

$\therefore \angle DAB = \angle CBA$.

QUESTION 9

(a) $x^2 \geq (x+1)(x+2)$

$x^2 \geq x^2 + 3x + 2$

$0 \geq 3x + 2$

$3x \leq -2$

$x \leq -\frac{2}{3}$

(b) $\dfrac{16^{n+2} \times 4^{n-1}}{64^n} = \dfrac{(4^2)^{n+2} \times 4^{n-1}}{4^{3n}}$

$= \dfrac{4^{2n+4} \times 4^{n-1}}{4^{3n}}$

$= \dfrac{4^{3n+3}}{4^{3n}}$

$= 4^3$

$= 64$

(c) 48, 45, 42

(i) A.P. with $a = 48$

$d = -3$

$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

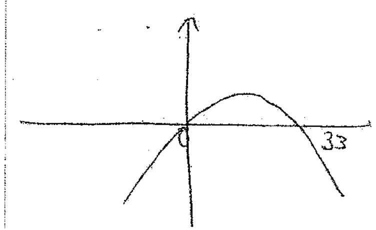
$= \frac{n}{2}\left(96 + (n-1)\times -3\right)$

$= \frac{n}{2}(99 - 3n)$

(ii) Want $\frac{n}{2}(99-3n) < 0$

$n(99-3n) < 0$

$\therefore n > 33$



(d) $a = 2$

$r = \frac{2}{\sqrt{2}+1}$

$S_\infty = \frac{a}{1-r}$

$= \frac{2}{1-\left(\frac{1}{\sqrt{2}+1}\right)}$

$= \frac{2}{\frac{\sqrt{2}+1-1}{\sqrt{2}+1}}$

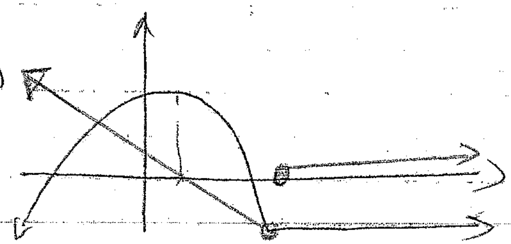
$= \frac{2(\sqrt{2}+1)}{\sqrt{2}}$

$= \frac{2(\sqrt{2}+1)}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{4+2\sqrt{2}}{2}$

$= 2+\sqrt{2}$

(e)



QUESTION 10

(a) $y = ax + \frac{b}{x^2}$

$y' = a - 2bx^{-3}$

$= a - \frac{2b}{x^3}$

at $(2,0)$: $\text{GRADIENT}_{\text{TAN}} = 1$

$\therefore\ 1 = a - \frac{2b}{8}$

$8 = 8a - 2b$

$4a - b = 4$ ---- (1)

Subst $(2,0)$ into $y = ax + \frac{b}{x^2}$

$0 = 2a + \frac{b}{4}$

$0 = 8a + b$ ---- (2)

(1) + (2): $12a = 4$

$a = \frac{1}{3}$

Subst: $a = \frac{1}{3}$ into (2): $0 = 8 \times \frac{1}{3} + b$

$b = -\frac{8}{3}$

$\therefore\ \left.\begin{array}{l} a = \frac{1}{3} \\ b = -8/3 \end{array}\right\}$

(b) (i) Area of APQCD = Area ABCD − Area Δ PBQ

$$= (3\times4) - \tfrac{1}{2}\times(4-x)(3-2x)$$
$$= 12 - \tfrac{1}{2}(12-8x-3x+2x^2)$$
$$= 12 - \tfrac{1}{2}(12-11x+2x^2)$$
$$= 12 - 6 + \tfrac{11}{2}x - x^2$$
$$= 6 + \tfrac{11}{2}x - x^2$$

(ii) $x = \frac{-b}{2a}$ — $a=-1$, $b=\frac{11}{2}$, $c=6$

$$= \frac{-\frac{11}{2}}{-2}$$
$$= \frac{11}{4}$$

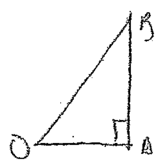
∴ Max when $x = \frac{11}{4}$ since $a<0$

Subst $x=\frac{11}{4}$ into $6+\frac{11}{2}x - x^2$

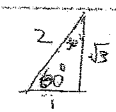
$$= 6 + \tfrac{11}{2}\times\tfrac{11}{4} - \left(\tfrac{11}{4}\right)^2$$
$$= 459/16$$
$$= 34\tfrac{11}{16}$$

∴ max area is $34\frac{11}{16}$ u^2

(c) In Δ OAB



$$\tan BOA = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$



$\therefore \angle BOA = 60^0$

Also: $\angle COA = \angle COB + \angle BOA = 90^0$

$$\angle COB + 60^0 = 90^0$$
$$\therefore \angle COB = 30^0$$

(ii) Area ABD $= \frac{1}{4}$(Area of circle $x^2+y^2=1$) − (Sector COB + Δ BOA)

$$= \tfrac{1}{4}\times\pi\times1^2 - \left(\tfrac{30}{360}\times\pi\times1^2 + \tfrac{1}{2}\times\tfrac{1}{2}\times\tfrac{\sqrt{3}}{2}\right)$$
$$= \frac{\pi}{4} - \left(\frac{\pi}{12} + \frac{\sqrt{3}}{8}\right)$$
$$= \frac{6\pi - 2\pi + 3\sqrt{3}}{24}$$
$$= \frac{4\pi - 3\sqrt{3}}{24} \quad u^2.$$