Name ____________________

Teacher ____________________

ASCHAM GIRLS

NOVEMBER 2013

Year 11

MATHEMATICS

ASSESSMENT TEST

TIME: $1\frac{1}{2}$ HOURS (PLUS 5 MINUTES READING TIME)

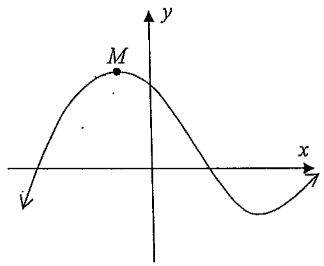
INSTRUCTIONS

- All questions should be attempted.
- Show all necessary working.
- Start each question in a **new booklet**.
- Write your name and teacher's name on each booklet.
- Approved calculators and templates may be used.

Section A

Multiple Choice: (5 marks) circle the correct answer

1. At the point M which statement would be true:



A. $f'(x)<0$ and $f''(x)<0$

B. $f'(x)=0$ and $f''(x)=0$

C. $f'(x)=0$ and $f''(x)>0$

D. $f'(x)=0$ and $f''(x)<0$

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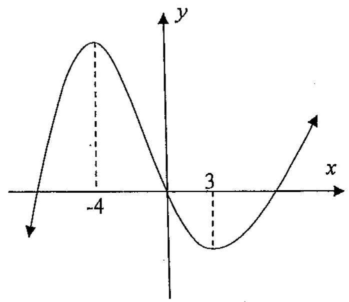
2. The graph of $y = f(x)$ is increasing when:



A. only when $x>3$

B. $-4<x<3$

C. $x>3$ *or* $x<-4$

D. $x=-4$

---

3. The circle $x^2+2x+y^2-6y+9=0$ has centre and radius respectively:

A. (1, -3) and radius = 1 B. (1,-3) and radius = 3

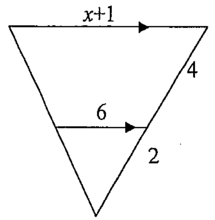
C. (-1, 3) and radius = 3 D. (-1,3) and radius = 1

---

4. The second derivative of $x^{\frac{1}{2}}$ is:

A. $\frac{1}{4}x^{\frac{-3}{2}}$ B. $\frac{4}{x^{\frac{3}{2}}}$ C. $4x\sqrt{x}$ D. $\frac{-1}{4x\sqrt{x}}$

5. The value of x is



A. 1

B. 17

C. 11

D. 1

Section B

QUESTION 6 **15 marks**

a) Differentiate with respect to x and leave in fully factored form where possible:

i) $\dfrac{3}{\sqrt{x}}$ **(1)** ii) $x^2(2x-1)^5$ **(4)**

b) Integrate with respect to x;

i) $\sqrt[3]{x}$ **(1)** ii) $\dfrac{6x^3-x^7}{x^3}$ **(2)**

iii) $x(2-3x)^2$ **(2)** iv) $\dfrac{3}{\sqrt{2x-1}}$ **(2)**

c) Evaluate: $\displaystyle\int_1^2\left(x+\frac{2}{x}\right)^2 dx$ **(3)**

QUESTION 7 **16 marks**

a). **(4)**

Given that $\dfrac{d^2y}{dx^2}=2x-1$

find y, given that there is a stationary point at $(3,0)$

b) A function is given by $f(x)=x^4-4x^3+20$.

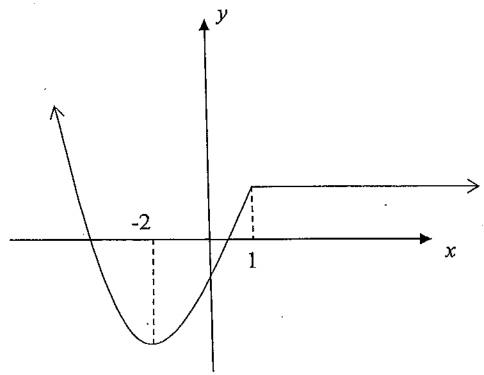
i) Find $f'(x)$ and $f''(x)$ **(2)**

ii) Find any stationary points and determine their nature. **(4)**

iii) Find any points of inflexion. **(2)**

iv) Sketch the graph **(2)**

d) The graph of $y=f(x)$ is sketched below. Copy the diagram onto your answer booklet and on the same diagram carefully sketch the graph of the derivative $y=f'(x)$. **(2)**



QUESTION 8 **11 marks**

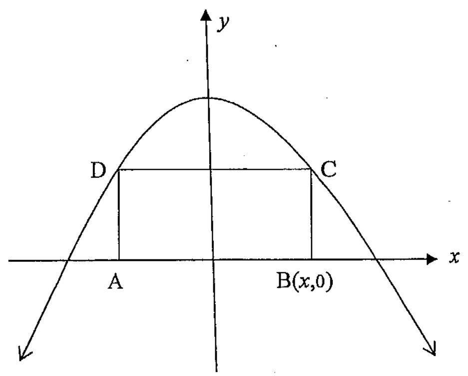
a) Find the coordinates of the focus and equation of the directrix of the parabola $(y-2)^2 = 6x+3$ **(3)**

b) The line $y = 4x - 1$ is a tangent to the parabola $y = x^2 + 3$.
   i) Find the point of contact **(2)**
   ii) Find the equation of the normal through the point of contact. **(2)**

c) Find the equation of the locus of the points P(x,y) whose distance from the point A(2,3) is twice its distance from the point B(-1,-2). **(4)**

QUESTION 9 **17 marks**

a) The parabola $y = 4 - x^2$ is sketched below. Between the parabola and the x-axis is inscribed the rectangle ABCD.



   i) Show that the area of the rectangle ABCD is given by

   $A = 8x - 2x^3$ **(2)**

   ii) Show that the rectangle has a maximum area when x has a value of $\frac{2\sqrt{3}}{3}$ **(3)**

   iii) Hence find the largest area of rectangle ABCD **(2)**

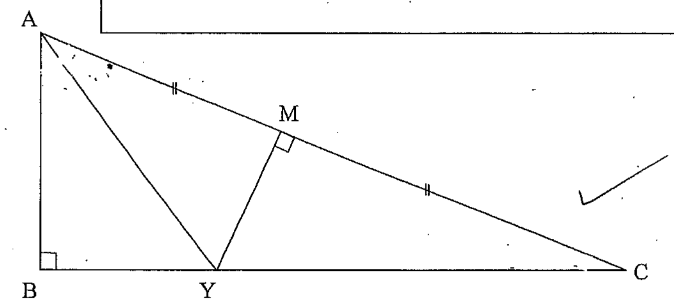
b) Consider the function $y = x^3 - 2px^2 + 3$. Find the value of p for which the function is increasing and concave up. **(4)**

c) **See next page**

Detach this page and place it in your Question 9 answer booklet.

c)

Do your working on this page.
Name: ______
Teacher: ______



The diagram shows a right angled triangle ABC with $\angle ABC = 90°$. The point M is the midpoint of AC, and Y is the point where the perpendicular to AC at M meets BC.

   i) Show that $\Delta AYM \equiv \Delta CYM$ **(3)**

   ii) Suppose it is also given that AY bisects $\angle BAC$. Find the size of $\angle YCM$ and hence find the exact ratio MY:AC. **(3)**

SOLUTIONS

Quest (6)

[SECTION A]

Qu.	
1	D
2	C
3	D
4	D
5	B

a) i) $\frac{3}{\sqrt{x}} = 3x^{-\frac{1}{2}}$

$\frac{d}{dx}\left(\frac{3}{\sqrt{x}}\right) = \frac{d}{dx}\left(3x^{-\frac{1}{2}}\right)$

$= -\frac{1}{2}(3)x^{-\frac{3}{2}}$

$= -\frac{3}{2x^{3/2}}$

$= -\frac{3}{2x\sqrt{x}}$ ✓

ii) $x^2(2x-1)^5$

$u = x^2 \quad v = (2x-1)^5$

$u' = 2x \quad v' = 5(2x-1)^4 . 2$

$= 10(2x-1)^4$

$\frac{d}{dx} x^2(2x-1)^5 = u'v + v'u$

$= 2x(2x-1)^5 + 10(2x-1)^4 . x^2$

$= 2x(2x-1)^5 + 10x^2(2x-1)^4$

$= 2x(2x-1)^4\left[(2x-1) + 5x\right]$

$= 2x(2x-1)^4(7x-1)$ ✓

b) i) $\sqrt[3]{x} = x^{\frac{1}{3}}$

$\int x^{1/3}\,dx = \left[\frac{3}{4}x^{4/3}\right] + C$

$= \frac{3}{4}x^{4/3} + C$

$= \frac{3}{4}\sqrt[3]{x^4} + C$ ✓

ii) $\frac{6x^3 - x^7}{x^3} = \frac{6x^3}{x^3} - \frac{x^7}{x^3}$

$= 6 - x^4$

$\int \frac{6x^3 - x^7}{x^3}\,dx = \int 6 - x^4\,dx$

$= \left[6x - \frac{x^5}{5}\right] + C$

$= 6x - \frac{1}{5}x^5 + C$ ✓

iii) $x(2-3x)^2 = x\left(4 - 12x + 9x^2\right)$

$= 4x - 12x^2 + 9x^3$

$\int x(2-3x)^2\,dx = \int 4x - 12x^2 + 9x^3\,dx$

$= \left[\frac{4x^2}{2} - \frac{12x^3}{3} + \frac{9x^4}{4}\right] + C$

$= 2x^2 - 4x^3 + \frac{9}{4}x^4 + C$ ✓

iv) $\frac{3}{\sqrt{2x-1}} = 3(2x-1)^{-\frac{1}{2}}$

$\int \frac{3}{\sqrt{2x-1}}\,dx = \int 3(2x-1)^{-\frac{1}{2}}\,dx$

$= 3\left[\frac{(2x-1)^{1/2}}{1/2 \cdot 2}\right] + C$

$= 3\left[(2x-1)^{1/2}\right] + C$

$= 3\sqrt{2x-1} + C$ ✓

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c) $\int_1^2 \left(x+\frac{2}{x}\right)^2 dx = \int_1^2 x^2 + 4 + \frac{4}{x^2}\, dx$

$= \int_1^2 x^2 + 4 + 4x^{-2}\, dx$

$= \left[\frac{x^3}{3} + 4x + \frac{4x^{-1}}{-1}\right]_1^2$

$= \left[\frac{1}{3}x^3 + 4x - \frac{4}{x}\right]_1^2$

$= \left[\left(\frac{1}{3}(2)^3+4(2)-\frac{4}{2}\right) - \left(\frac{1}{3}(1)^3+4(1)-\frac{4}{1}\right)\right]$

$= \left[\frac{26}{3} - \left(\frac{1}{3}\right)\right]$

$= \frac{26}{3} - \frac{1}{3}$

$= \frac{25}{3}$

$= 8\frac{1}{3}$

You may ask for an extra Writing Booklet if you need more space.

Quest (7)

a) $\frac{d^2y}{dx^2} = 2x-1$

$\frac{dy}{dx} = \int \frac{d^2y}{dx^2}\, dx$

$= \int 2x-1\, dx$

$= \left[\frac{\cancel{2}x^2}{\cancel{2}} - x\right] + C$

$\frac{dy}{dx} = x^2 - x + C$

$\frac{dy}{dx} = 0$ when $x = 3$

$0 = (3)^2 - (3) + C$

$0 = 9 - 3 + C$

$C + 6 = 0$

$c = -6$

$\therefore \frac{dy}{dx} = x^2 - x - 6$

4/4

$y = \int \frac{dy}{dx}\, dx$

$= \int x^2 - x - 6\, dx$

$= \left[\frac{x^3}{3} - \frac{x^2}{2} - 6x\right] + C$

$= \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + C$

sub ~~to~~ (3, 0)

$0 = \frac{1}{3}(3)^3 - \frac{1}{2}(3)^2 - 6(3) + C$

$= -\frac{27}{2} + C$

$c = \frac{27}{2}$

$c = 13\frac{1}{2}$

$\therefore y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + \frac{27}{2}$

b) $f(x) = x^4 - 4x^3 + 20$

i) $f'(x) = 4x^3 - 12x^2$

$f''(x) = 12x^2 - 24x$

ii) stationary points when $f'(x) = 0$

$4x^3 - 12x^2 = 0$

$4x^2(x-3) = 0$

$\therefore x = 0$ or $x = 3$

$y = 20$ $\quad y = -7$

$(0, 20)$ $\quad (3, -7)$

Test $f'(x)$

x	-0.1	0	0.1
$f'(x)$	-0.124	0	-0.116
slope	\	-	\

x	2.9	3	3.1
$f'(x)$	-17.7	0	3.844
slope	\	-	/

$\therefore$ horizontal point of inflexion at $(0, 20)$

minimum turning point at $(3, -7)$

iii) points of inflexion when $f''(x) = 0$ and there is a change in concavity

$f''(x) = 12x^2 - 24x$

$12x^2 - 24x = 0$

$12x(x-2) = 0$

$\therefore x = 0$ or $x = 2$

$y = 20$ $\quad y = 4$

$(0, 20)$ ~~is~~ a horizontal point of inflexion from $f'(x)$ $\quad (2, 4)$

test $f''(x)$

x	1.9	2	2.1
$f''(x)$	-2.2	0	2.5
concavity	$\cap$	-	$\cup$

$\therefore$ point of inflexion at $(2, 4)$ due to change in concavity.

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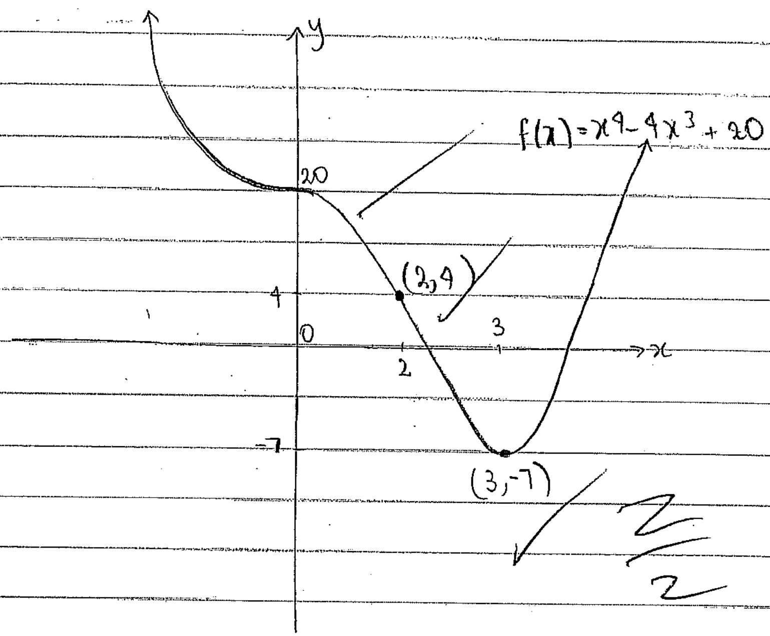
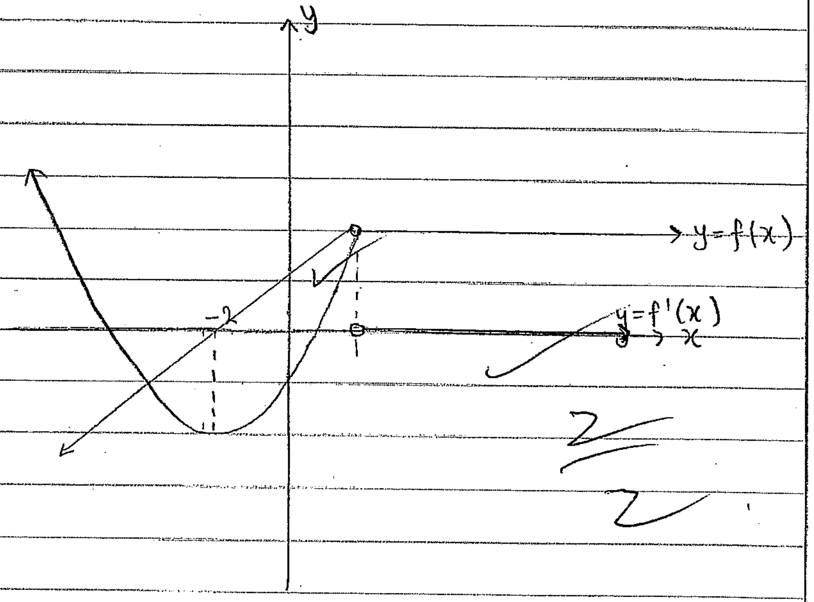
iv) Sketch



d)



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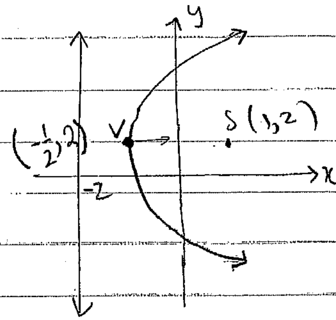
Quest 8

a) $(y-2)^2 = 6x+3$

$(y-2)^2 = 6\left(x+\frac{1}{2}\right)$



focal length:

$4a = 6$

$a = \frac{6}{4}$

$\therefore a = \frac{3}{2}$

$\therefore$ focus ~~is~~ $(1,2)$

directrix $x = -2$

3

b) $y = 4x-1$ tangent to $y = x^2+3$

i) $y = 4x-1$ ... ①

$y = x^2+3$ ... ②

① = ②

$4x-1 = x^2+3$

$x^2-4x+3+1=0$

$x^2-4x+4=0$

$(x-2)^2=0$

$\therefore x = 2$

point of contact when ~~x~~ $x=2$

when $x=2$

$y = 4(2)-1$

$= 8-1$

$= 7$

$\therefore$ point of contact is $(2,7)$

2

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ii) $y = x^2+3$

$m_{tangent} = y'$

$y' = 2x$

$\therefore$ at $(2,7)$

$y' = 2(2)$

$y' = 4$

$\therefore m_{tangent} = 4$

$\therefore m_{normal} = -\frac{1}{4}$

equation of the normal through $(2,7)$

$y - y_1 = m(x-x_1)$

$y-7 = -\frac{1}{4}(x-2)$

$4y-28 = -x+2$

$x+4y-28-2=0$

$\therefore x+4y-30=0$

2

c) $P(x,y)$ $A(2,3)$ $B(-1,-2)$

reverse it!

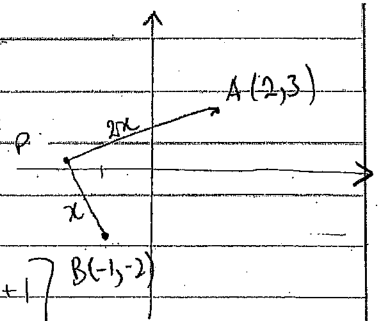
~~$2PA = PB$~~ $PA = 2PB$

~~$2\sqrt{(y_2-y_1)^2+(x_2-x_1)^2} = \sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$~~

~~$2\sqrt{(y-3)^2+(x-2)^2} = \sqrt{(y+2)^2+(x+1)^2}$~~

~~$4[(y-3)^2+(x-2)^2] = 4[(y+2)^2+(x+1)^2]$~~

~~$4[y^2-6y+9+x^2-4x+4] = 4[y^2+4y+4+x^2+2x+1]$~~



$PA = 2 \times PB$

$\sqrt{(x-2)^2+(y-3)^2} = 2\times\sqrt{(x+1)^2+(y+2)^2}$

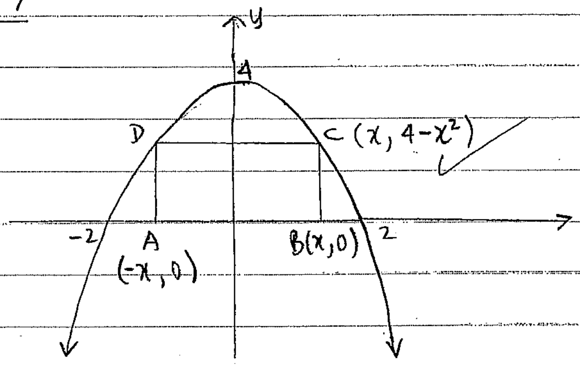
$x^2-4x+4+y^2-6y+9 = 4[x^2+2x+1+y^2+4y+4]$

$x^2-4x+y^2-6y+13 = 4x^2+8x+4y^2+16y+20$

$0 = 3x^2+12x+3y^2+22y+7$

Quest 9

a)



i) $A = \text{length} \times \text{width}$

$= 2x(4 - x^2)$

$= 8x - 2x^3$

$\therefore A = 8x - 2x^3$ as required

2/2

ii) max area when $\frac{dA}{dx} = 0$

$\frac{dA}{dx} = 8 - 6x^2$

$8 - 6x^2 = 0$

$6x^2 = 8$

$x^2 = \frac{8}{6}$

$x^2 = \frac{4}{3}$

$x = \pm\sqrt{\frac{4}{3}}$

$= \sqrt{\frac{4}{3}}$ only (x is a length

$x = \frac{2}{\sqrt{3}}$

$x = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{3} \quad \therefore x = \frac{2\sqrt{3}}{3}$

test for max in $\frac{d^2A}{dx^2}$

$\frac{d^2A}{dx^2} = -12x$

when $x = \frac{2\sqrt{3}}{3}$

$\frac{d^2A}{dx^2} = -12\left(\frac{2\sqrt{3}}{3}\right)$

$= -13.8\ldots$

< 0

concave down $\therefore$ maximum

3/3

$\therefore$ maximum area when x has a value of $\frac{2\sqrt{3}}{3}$

iii) largest area when $x = \frac{2\sqrt{3}}{3}$

$A = 8x - 2x^3$

$= 8\left(\frac{2\sqrt{3}}{3}\right) - 2\left(\frac{2\sqrt{3}}{3}\right)^3$

$= \frac{16\sqrt{3}}{3} - 2\left(\frac{8(\sqrt{3})^3}{27}\right)$

$= 6.158\ldots$

$\doteqdot 6.16\ u^2$ (2dp)

$\therefore$ larges area is approximately $6.16\ u^2$ (2dp)

2/2

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b) $y = x^3 - 2px^2 + 3$

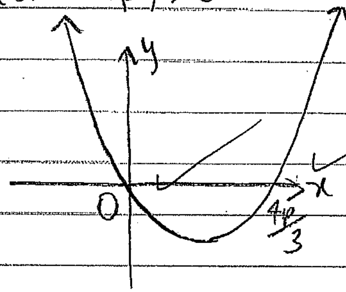
~~increasing when $y' > 0$~~
~~$y' = 3x^2 - 4px$~~
~~$x(3x - 4p) > 0$~~
~~$\therefore 3x^2 - 4px > 0$~~
~~concave up when $y'' > 0$~~
~~$y'' = 6x - 4p$~~
~~$6x - 4p > 0$~~
~~but $y = x^3 - 2px^2 + 3$~~
~~using discriminant y~~

increasing when $y' > 0$

$y' = 3x^2 - 4px$

$3x^2 - 4px > 0$

$x(3x - 4p) > 0$



$\therefore x < 0$ or $x > \frac{4p}{3}$

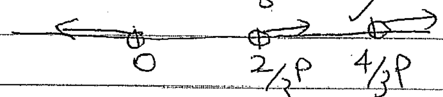
concave ~~down~~ up when $y'' > 0$

$y'' = 6x - 4p$

$6x - 4p > 0$

$6x > 4p$

$x > \frac{2}{3}p$



$\therefore$ for increasing & concave down

$x > \frac{4}{3}p$

~~$4p < 3x$~~

$3x > 4p$

$\therefore p < \frac{3x}{4}$ for increasing & concave down

$\frac{4}{4}$

You may ask for an extra Writing Booklet if you need more space.

Start here.

i) In ΔAYM & ΔCYM

$AM = MC$ (M is the midpoint of AC)

$\angle YMA = \angle YMC = 90°$ ($YM \perp AC$)

YM is a common side

$\therefore \Delta AYM \equiv \Delta CYM$ (SAS)

$\frac{3}{3}$

ii) let $\angle BAY = \alpha$

$\therefore \angle YAM = \alpha$ (AY bisects $\angle BAC$)

$\therefore \angle YCM = \angle YAM$ (matching $\angle$s in congruent Δs)

$= \alpha$

$\therefore$ in ΔABC

$\angle ABC + \angle BAC + \angle ACB = 180°$ ($\angle$ sum of Δ)

$90° + 2\alpha + \alpha = 180°$

$90° + 3\alpha = 180°$

$3\alpha = 90$

$\alpha = 30°$

$\therefore \angle YCM = 30°$

In ΔCYM:

$\tan\angle YCM = \frac{YM}{MC}$

$\tan 30° = \frac{YM}{MC}$

$\frac{1}{\sqrt{3}} = \frac{YM}{MC}$

$\frac{3}{3}$

$2MC = AC$ (M is the midpoint of AC)

$\therefore \frac{YM}{2MC} = \frac{1}{\sqrt{3}} \times \frac{1}{2}$

$\frac{YM}{2MC} = \frac{1}{2\sqrt{3}}$

$\therefore \frac{YM}{AC} = \frac{1}{2\sqrt{3}}$

$\therefore MY : AC = 1 : 2\sqrt{3}$