BOARD OF STUDIES

NEW SOUTH WALES

2010

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x\ , \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

Total marks – 84
Attempt Questions 1–7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Use the table of standard integrals to find $\int \frac{1}{\sqrt{4-x^2}}\,dx$. **1**

(b) Let $f(x)=\cos^{-1}\left(\frac{x}{2}\right)$. What is the domain of $f(x)$? **1**

(c) Solve $\ln(x+6)=2\ln x$. **3**

(d) Solve $\frac{3}{x+2}<4$. **3**

(e) Use the substitution $u=1-x$ to evaluate $\int_0^1 x\sqrt{1-x}\,dx$. **3**

(f) Five ordinary six-sided dice are thrown. **1**

What is the probability that exactly two of the dice land showing a four? Leave your answer in unsimplified form.

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) The derivative of a function $f(x)$ is given by **2**

$$f'(x)=\sin^2 x.$$

Find $f(x)$, given that $f(0)=2$.

(b) The mass M of a whale is modelled by

$$M=36-35.5e^{-kt},$$

where M is measured in tonnes, t is the age of the whale in years and k is a positive constant.

(i) Show that the rate of growth of the mass of the whale is given by the differential equation **1**

$$\frac{dM}{dt}=k(36-M).$$

(ii) When the whale is 10 years old its mass is 20 tonnes. **2**

Find the value of k, correct to three decimal places.

(iii) According to this model, what is the limiting mass of the whale? **1**

Question 2 continues on page 4

Question 2 (continued)

(c) Let

$$P(x) = (x+1)(x-3)\,Q(x) + ax + b,$$

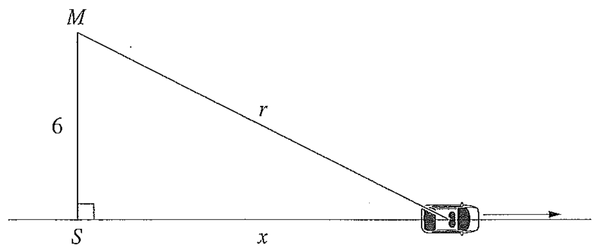
where $Q(x)$ is a polynomial and a and b are real numbers.
The polynomial $P(x)$ has a factor of $x - 3$.
When $P(x)$ is divided by $x + 1$ the remainder is 3.

(i) Find the values of a and b. **2**

(ii) Find the remainder when $P(x)$ is divided by $(x+1)(x-3)$. **1**

(d) A radio transmitter M is situated 6 km from a straight road. The closest point on the road to the transmitter is S. **3**

A car is travelling away from S along the road at a speed of 100 km h^{-1}. The distance from the car to S is x km and from the car to M is r km.



Find an expression in terms of x for $\frac{dr}{dt}$, where t is time in hours.

End of Question 2

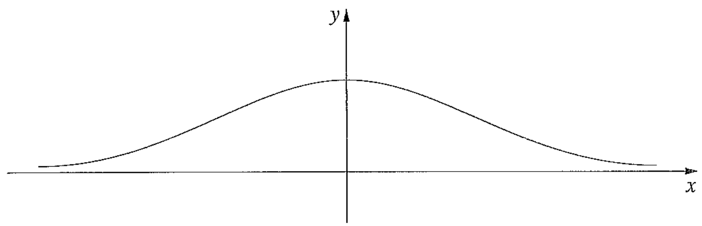
Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) At the front of a building there are five garage doors. Two of the doors are to be painted red, one is to be painted green, one blue and one orange.

(i) How many possible arrangements are there for the colours on the doors? **1**

(ii) How many possible arrangements are there for the colours on the doors if the two red doors are next to each other? **1**

(b) Let $f(x) = e^{-x^2}$. The diagram shows the graph $y = f(x)$.



(i) The graph has two points of inflexion. **3**

Find the x coordinates of these points.

(ii) Explain why the domain of $f(x)$ must be restricted if $f(x)$ is to have an inverse function. **1**

(iii) Find a formula for $f^{-1}(x)$ if the domain of $f(x)$ is restricted to $x \geq 0$. **2**

(iv) State the domain of $f^{-1}(x)$. **1**

(v) Sketch the curve $y = f^{-1}(x)$. **1**

(vi) (1) Show that there is a solution to the equation $x = e^{-x^2}$ between $x = 0.6$ and $x = 0.7$. **1**

(2) By halving the interval, find the solution correct to one decimal place. **1**

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) A particle is moving in simple harmonic motion along the x-axis.

Its velocity v, at x, is given by $v^2 = 24 - 8x - 2x^2$.

(i) Find all values of x for which the particle is at rest. **1**

(ii) Find an expression for the acceleration of the particle, in terms of x. **1**

(iii) Find the maximum speed of the particle. **2**

(b) (i) Express $2\cos\theta + 2\cos\left(\theta + \frac{\pi}{3}\right)$ in the form $R\cos(\theta + \alpha)$, **3**

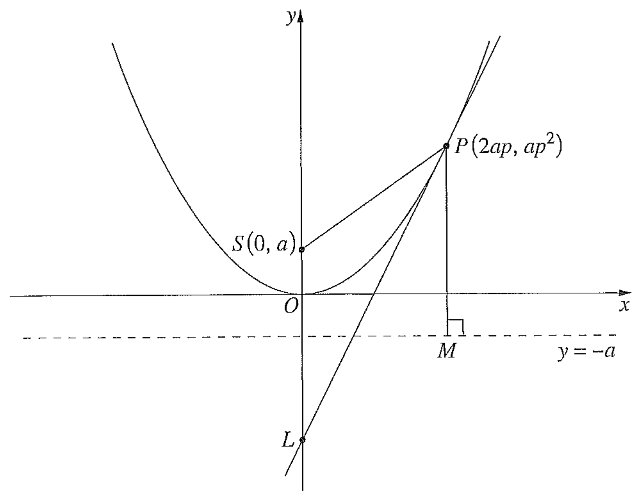
where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

(ii) Hence, or otherwise, solve $2\cos\theta + 2\cos\left(\theta + \frac{\pi}{3}\right) = 3$, **2**

for $0 < \theta < 2\pi$.

Question 4 continues on page 7

(c) The diagram shows the parabola $x^2 = 4ay$. The point $P(2ap, ap^2)$, where $p \neq 0$, is on the parabola. **3**



The tangent to the parabola at P, $y = px - ap^2$, meets the y-axis at L.

The point M is on the directrix, such that PM is perpendicular to the directrix.

Show that $SLMP$ is a rhombus.

End of Question 4

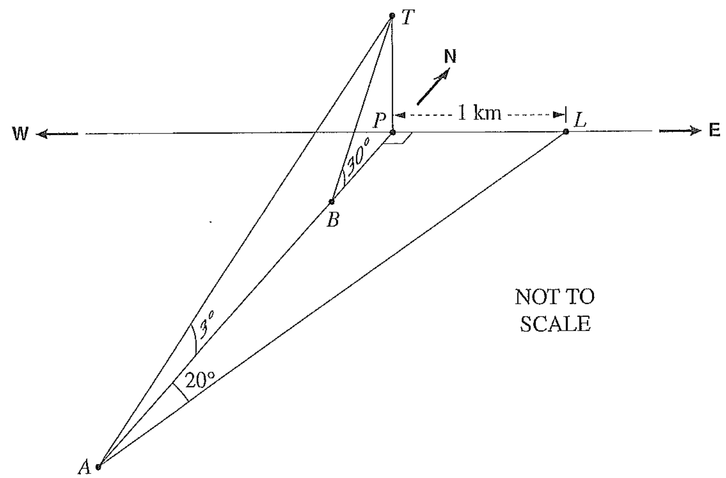
Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) A boat is sailing due north from a point A towards a point P on the shore line. The shore line runs from west to east.

In the diagram, T represents a tree on a cliff vertically above P, and L represents a landmark on the shore. The distance PL is 1 km.

From A the point L is on a bearing of 020°, and the angle of elevation to T is 3°.

After sailing for some time the boat reaches a point B, from which the angle of elevation to T is 30°.



(i) Show that $BP = \dfrac{\sqrt{3}\tan 3°}{\tan 20°}$. **3**

(ii) Find the distance AB. **1**

Question 5 continues on page 9

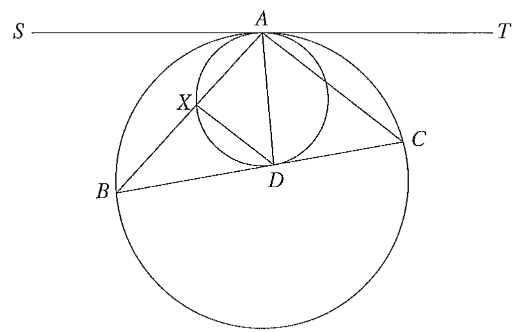
(b) Let $f(x) = \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$ for $x \neq 0$.

(i) By differentiating $f(x)$, or otherwise, show that $f(x) = \frac{\pi}{2}$ for $x > 0$. **3**

(ii) Given that $f(x)$ is an odd function, sketch the graph $y = f(x)$. **1**

(c) In the diagram, ST is tangent to both the circles at A.

The points B and C are on the larger circle, and the line BC is tangent to the smaller circle at D. The line AB intersects the smaller circle at X.



Copy or trace the diagram into your writing booklet.

(i) Explain why $\angle AXD = \angle ABD + \angle XDB$. **1**

(ii) Explain why $\angle AXD = \angle TAC + \angle CAD$. **1**

(iii) Hence show that AD bisects $\angle BAC$. **2**

End of Question 5

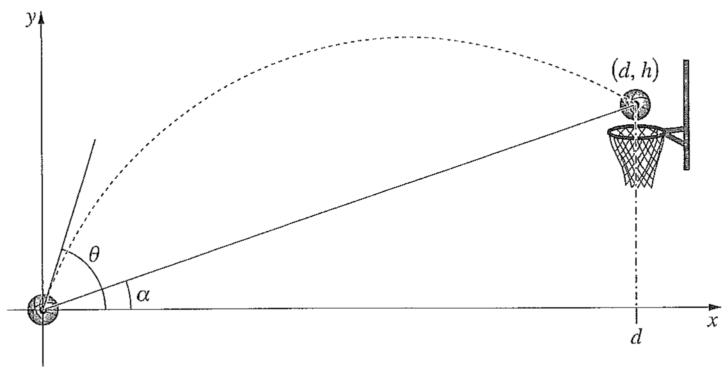
Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $\cos(A-B) = \cos A \cos B(1 + \tan A \tan B)$. **1**

(ii) Suppose that $0 < B < \frac{\pi}{2}$ and $B < A < \pi$. **1**

Deduce that if $\tan A \tan B = -1$, then $A - B = \frac{\pi}{2}$.

(b) A basketball player throws a ball with an initial velocity v m s^{-1} at an angle θ to the horizontal. At the time the ball is released its centre is at $(0, 0)$, and the player is aiming for the point (d, h) as shown on the diagram. The line joining $(0, 0)$ and (d, h) makes an angle α with the horizontal, where $0 < \alpha < \theta < \frac{\pi}{2}$.



Assume that at time t seconds after the ball is thrown its centre is at the point (x, y), where

$$x = vt\cos\theta$$
$$y = vt\sin\theta - 5t^2.$$

(You are NOT required to prove these equations.)

Question 6 continues on page 11

(i) If the centre of the ball passes through (d, h) show that **3**

$$v^2 = \frac{5d}{\cos\theta\sin\theta - \cos^2\theta\tan\alpha}.$$

(ii) (1) What happens to v as $\theta \to \alpha$? **1**

(2) What happens to v as $\theta \to \frac{\pi}{2}$? **1**

(iii) For a fixed value of α, let $F(\theta) = \cos\theta\sin\theta - \cos^2\theta\tan\alpha$. **2**

Show that $F'(\theta) = 0$ when $\tan 2\theta \tan\alpha = -1$.

(iv) Using part (a) (ii) or otherwise show that $F'(\theta) = 0$ when $\theta = \frac{\alpha}{2} + \frac{\pi}{4}$. **1**

(v) Explain why v^2 is a minimum when $\theta = \frac{\alpha}{2} + \frac{\pi}{4}$. **2**

End of Question 6

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) Prove by induction that **3**

$$47^n + 53 \times 147^{n-1}$$

is divisible by 100 for all integers $n \geq 1$.

(b) The binomial theorem states that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n.$$

(i) Show that $2^n = \sum_{k=0}^{n} \binom{n}{k}$. **1**

(ii) Hence, or otherwise, find the value of **1**

$$\binom{100}{0} + \binom{100}{1} + \binom{100}{2} + \ldots + \binom{100}{100}.$$

(iii) Show that $n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$. **2**

Question 7 continues on page 13

(c) (i) A box contains n identical red balls and n identical blue balls. A selection of r balls is made from the box, where $0 \leq r \leq n$. **1**

Explain why the number of possible colour combinations is $r + 1$.

(ii) Another box contains n white balls labelled consecutively from 1 to n. A selection of $n - r$ balls is made from the box, where $0 \leq r \leq n$. **1**

Explain why the number of different selections is $\binom{n}{r}$.

(iii) The n red balls, the n blue balls and the n white labelled balls are all placed into one box, and a selection of n balls is made. **3**

Using part (b), or otherwise, show that the number of different selections is $(n+2)2^{n-1}$.

End of paper

2010 Higher School Certificate Solutions
Mathematics Extension 1

Question 1

(a) $$\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\left(\frac{x}{2}\right) + C.$$

(b) For $f(x) = \cos^{-1}\left(\frac{x}{2}\right)$,

The domain is given by:

$$-1 \le \frac{x}{2} \le 1$$

so $-2 \le x \le 2$.

(c) Since $\ln(x+6) = 2\ln x$

$$\ln(x+6) = \ln x^2$$
$$x+6 = x^2$$
$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$
$$x = -2, 3 \quad \text{but } x > 0$$
$$\therefore x = 3.$$

(d) *METHOD 1* – Algebraic:

$$\frac{3}{x+2} < 4$$
$$(x+2)^2 \times \frac{3}{x+2} < 4 \times (x+2)^2$$
$$3(x+2) < 4(x^2+4x+4)$$
$$3x+6 < 4x^2+16x+16x$$
$$4x^2+13x+10 > 0$$
$$(4x+5)(x+2) > 0$$
$$x < -2, \quad x > -\frac{5}{4}.$$

METHOD 2 – Critical Point:

$$\frac{3}{x+2} < 4, \quad x \ne -2$$
$$3 < 4(x+2) \quad \text{if } x > -2$$
$$3 < 4x+8$$
$$-5 < 4x$$
$$-\frac{5}{4} < x$$

x	-3	$-1\frac{1}{2}$	-1
f(x)	true	false	true

So $x < -2$, $x > -\frac{5}{4}$.

(e) $$u = 1-x \quad \Rightarrow \quad x = 1-u.$$

$$\frac{du}{dx} = -1$$
$$\therefore du = -dx$$
$$\therefore dx = -du$$

When $x = 1$, $u = 0$

$x = 0$, $u = 1$

So $$\int_0^1 x\sqrt{1-x}\,dx = \int_1^0 (1-u)u^{\frac{1}{2}}.-du$$
$$= -\int_1^0 u^{\frac{1}{2}} - u^{\frac{3}{2}}\,du$$
$$= -\left[\frac{2u^{\frac{3}{2}}}{3} - \frac{2u^{\frac{5}{2}}}{5}\right]_1^0$$
$$= -\left[(0) - \left(\frac{2}{3} - \frac{2}{5}\right)\right]$$
$$= \frac{4}{15}.$$

(f) $${}^5C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3.$$

Question 2

(a) $$f'(x) = \sin^2 x$$
$$f(x) = \int \sin^2 x\,dx$$

Using $\cos 2x = 1 - 2\sin^2 x$

$$f(x) = \int \frac{1-\cos 2x}{2}\,dx$$
$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

$f(0) = 2$:

$$2 = \frac{1}{2}(0) - \frac{1}{4}(0) + C$$
$$\therefore C = 2$$

Thus $f(x) = \frac{1}{2}x - \frac{1}{4}\sin 2x + 2$.

(b) (i) $$M = 36 - 35.5e^{-kt} \Rightarrow 35.5e^{-kt} = 36 - M$$
$$\frac{dM}{dt} = 0 - k \times -35.5e^{-kt}$$
$$= k \times 35.5e^{-kt}$$
$$= k(36-M).$$

(ii) When $t = 10$, $M = 20$

$$M = 36 - 35.5e^{-kt}$$
$$20 = 36 - 35.5e^{-k\times 10}$$
$$35.5e^{-10k} = 16$$
$$e^{-10k} = \frac{16}{35.5}$$
$$-10k = \ln\left(\frac{16}{35.5}\right)$$
$$k = \ln\left(\frac{16}{35.5}\right) \div -10$$
$$= 0.07969439...$$
$$= 0.080 \quad (3\text{ dp}).$$

(iii) $$M = 36 - 35.5e^{-kt}$$
$$M = 36 - \frac{35.5}{e^{kt}}$$

as $t \to \infty$, $\frac{35.5}{e^{kt}} \to 0$

$\therefore M \to 36$

$\therefore$ the limiting mass is 36 tonnes.

(c) (i) $P(x) = (x+1)(x-3)Q(x) + ax + b$

Given $P(3) = 0$:

$$P(3) = 0 + a(3) + b = 0$$
$$3a + b = 0$$

and $P(-1) = 8$:

$$P(-1) = 0 + a(-1) + b = 8$$
$$-a + b = 8$$

Solving simultaneously:

$3a + b = 0$ ①

$-a + b = 8$ ②

① – ②

$4a = -8$

$a = -2$

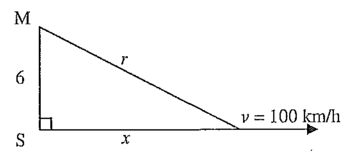
$b = 6$

$\therefore$ $a = -2$ and $b = 6$.

(ii) When $P(x) = (x+1)(x-3)Q(x) + ax + b$ is divided by $(x+1)(x-3)$, the remainder will be $ax+b$.

$\therefore$ the remainder is $-2x+6$.

(d)



$$r^2 = x^2 + 36$$
$$r = \sqrt{x^2+36} = \left(x^2+36\right)^{\frac{1}{2}}, \quad r > 0$$

$$\frac{dr}{dx}=\frac{1}{2}\left(x^2+36\right)^{-\frac{1}{2}}.2x$$
$$=\frac{x}{\sqrt{x^2+36}}$$
$$\frac{dr}{dt}=\frac{dr}{dx}.\frac{dx}{dt}$$
$$=\frac{x}{\sqrt{x^2+36}}\times 100$$
$$=\frac{100x}{\sqrt{x^2+36}}.$$

Question 3

(a) (i) $\frac{5!}{2!}=60.$

(ii) The two doors that must be together can be considered to be a single item. Therefore, there are four items to be arranged. $4!=24$.

(b) (i) $f(x)=e^{-x^2}$

$f'(x)=-2xe^{-x^2}$

$f''(x)=-2e^{-x^2}+(-2x^{-x^2}\times -2x)$

$=-2e^{-x^2}(1-2x^2)$

$f''(x)=0$ when $-2e^{-x^2}(1-2x^2)=0.$

ie $(1-2x^2)=0$ as $-2e^{-x^2}<0$ for all x

$2x^2=1$

$x^2=\frac{1}{2}$

$x=\pm\frac{1}{\sqrt{2}}=\pm\frac{\sqrt{2}}{2}$

These must be the two values indicated by the question.

(ii) The domain of $f(x)$ must be restricted to be a one-to-one (or monotonic) function in order to have an inverse function $f^{-1}(x)$.

(iii) $y=e^{-x^2}$ (restricted domain $x\geq 0$)

Interchange x and y:

i.e. $x=e^{-y^2}$

$\ln x=\ln(e^{-y^2})$

$\ln x=-y^2\ln(e)$

$\ln(x)=-y^2$ since $\ln(e)=1$

$y^2=-\ln(x)$

$y=\pm\sqrt{\ln(x^{-1})}$ or $\pm\sqrt{\ln\left(\frac{1}{x}\right)}$

reject $-\sqrt{\ln\left(\frac{1}{x}\right)}$ since $x\geq 0$

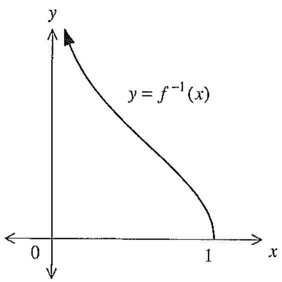
$\therefore y=\sqrt{\ln\left(\frac{1}{x}\right)}$

(iv) For $f(x)=e^{-x^2}$, $f(0)=1$

$\therefore$ range for $f(x)$ is: $0<y\leq 1$

$\therefore$ domain for $f^{-1}(x)$ is $0<x\leq 1$.

(v)



(vi) (1) Rearrange the equation to

$e^{-x^2}-x=0$

Let $f(x)=e^{-x^2}-x$

$f(0.6)=e^{-0.36}-0.6=0.097\cdots$

$f(0.7)=e^{-0.49}-0.7=-0.087\cdots$

Since $f(0.6)>0$ and $f(0.7)<0$, the solution of $x=e^{-x^2}$ must lie between $x=0.6$ and $x=0.7$.

(vi) (2) $f\left(\frac{0.6+0.7}{2}\right)=f(0.65)$

$=e^{-0.4225}-0.65$

$=0.00540\cdots>0$

$\therefore$ the solution must lie between $x=0.65$ and $x=0.7$

$\therefore$ the solution is $x=0.7$ (1 dp).

Question 4

(a) (i) The particle is at rest when $v=0$.

$24-8x-2x^2=0$

$2x^2+8x-24=0$

$x^2+4x-12=0$

$(x+6)(x-2)=0$

$\therefore$ At rest when $x=-6$ and $x=2$.

(ii) $a=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$=\frac{d}{dx}\left(\frac{1}{2}(24-8x-2x^2)\right)$

$=\frac{d}{dx}(12-4x-x^2)$

$=-2x-4$

(iii) Maximum speed when $a=0$.

$-2x-4=0$

$-2x=4$

$x=-2$

When $x=-2$,

$v^2=24-8(-2)-2(-2)^2$

$=32$

$v=\pm\sqrt{32}$

$\therefore$ Maximum speed is $\sqrt{32}=4\sqrt{2}$.

(b) (i) $2\cos\theta+2\cos\left(\theta+\frac{\pi}{3}\right)$

$=2\left(\cos\theta+\cos\left(\theta+\frac{\pi}{3}\right)\right)$

$=2\left(\cos\theta+\cos\theta\cos\frac{\pi}{3}-\sin\theta\sin\frac{\pi}{3}\right)$

$=2\left(\cos\theta+\cos\theta\times\frac{1}{2}-\sin\theta\times\frac{\sqrt{3}}{2}\right)$

$=2\cos\theta+\cos\theta-\sqrt{3}\sin\theta$

$=3\cos\theta-\sqrt{3}\sin\theta$

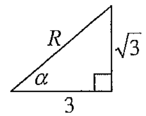
Now,

$R\cos(\theta+\alpha)=R\cos\theta\cos\alpha-R\sin\theta\sin\alpha$

Compare with $3\cos\theta-\sqrt{3}\sin\theta$:

$R\cos\alpha=3$ $\quad$ $R\sin\alpha=\sqrt{3}$

$\cos\alpha=\frac{3}{R}$ $\quad$ $\sin\alpha=\frac{\sqrt{3}}{R}$



$\therefore\ \tan\alpha=\frac{\sqrt{3}}{3}$

$R^2=\left(\sqrt{3}\right)^2+3^2$

$\alpha=\frac{\pi}{6}$ $\quad$ $R=\sqrt{12}$

$=2\sqrt{3}$

$2\cos\theta+2\cos\left(\theta+\frac{\pi}{3}\right)=2\sqrt{3}\cos\left(\theta+\frac{\pi}{6}\right)$

(ii) $2\sqrt{3}\cos\left(\theta+\frac{\pi}{6}\right)=3$

$$\cos\left(\theta+\frac{\pi}{6}\right)=\frac{3}{2\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$$

$$\cos\left(\theta+\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$

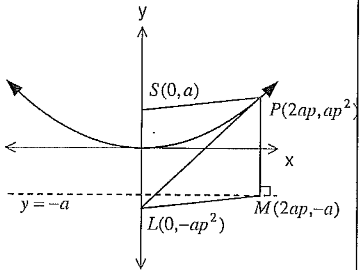
$$\theta+\frac{\pi}{6}=\frac{\pi}{6} \quad \text{or} \quad \theta+\frac{\pi}{6}=2\pi-\frac{\pi}{6}$$

$$\theta=0 \qquad\qquad \theta=\frac{5\pi}{3}$$

but the domain is $0<\theta<2\pi$,

$\therefore\ \theta=\frac{5\pi}{3}$ is the only solution.

(c)



by observation $M=(2ap,-a)$

when $x=0$, $y=p.0-ap^2$

$\therefore L=(0,-ap^2)$

METHOD 1

Midpoints of *MS* and *PL*.

$$M_{MS}=\left(\frac{2ap+0}{2},\frac{-a+a}{2}\right)$$

$$=(ap,0)$$

$$M_{PL}=\left(\frac{2ap+0}{2},\frac{ap^2+-ap^2}{2}\right)$$

$$=(ap,0)$$

$\therefore$ Midpoint of *MS* = Midpoint of *PL*.

$\therefore$ the diagonals bisect each other.

Gradients of *MS* and *PL*:

$$m_{MS}=\frac{-a-a}{2ap-0}$$

$$=\frac{-2a}{2ap}$$

$$=-\frac{1}{p}$$

Since equation of *PL* is

$y=px-ap^2$,

then $m_{PL}=p$.

$\therefore m_{MS}\times m_{PL}=-\frac{1}{p}\times p=-1$

$\therefore MS\perp PL$

$\therefore$ the diagonals are perpendicular.

$\therefore$ *SLMP* is a rhombus (diagonals bisect each other at right angles).

METHOD 2

Lengths of sides of quadrilateral.

$$SL=SO+OL$$

$$=a+\left|-ap^2\right|$$

$$=a+ap^2$$

$$MP=ap^2+\left|-a\right|$$

$$=ap^2+a$$

$$SP=\sqrt{(2ap-0)^2+(ap^2-a)^2}$$

$$=\sqrt{4a^2p^2+a^2p^4-2a^2p^2+a^2}$$

$$=\sqrt{a^2p^4+2a^2p^2+a^2}$$

$$=\sqrt{a^2(p^4+2p^2+1)}$$

$$=\sqrt{a^2(p^2+1)^2}$$

$$=a(p^2+1)$$

$$=ap^2+a$$

Similarly,

$$LM=\sqrt{(2ap-0)^2+(-a+ap^2)^2}$$

$$\vdots$$

$$=ap^2+a$$

$\therefore\ SL=MP=LM=SP=ap^2+a$

$\therefore$ *SLMP* is a rhombus (4 equal sides).

Question 5

(a) (i) Using ΔALP and ΔATP

$$\tan 20^\circ=\frac{1}{AP}$$

$$AP=\frac{1}{\tan 20^\circ}$$

$$\tan 3^\circ=\frac{PT}{AP}$$

$$=PT\tan 20^\circ$$

$$PT=\frac{\tan 3^\circ}{\tan 20^\circ}$$

Using ΔBPT

$$\tan 30^\circ=\frac{PT}{BP}$$

$$BP=\frac{PT}{\tan 30^\circ}$$

$$=\sqrt{3}PT$$

$$=\frac{\sqrt{3}\tan 3^\circ}{\tan 20^\circ} \text{ as required.}$$

(ii)

$$AP=\frac{PT}{\tan 3^\circ}$$

$$=\frac{\tan 3^\circ}{\tan 20^\circ}\div\tan 3^\circ$$

$$=\frac{1}{\tan 20^\circ}$$

$$AB=AP-BP$$

$$AB=\frac{1}{\tan 20^\circ}-\frac{\sqrt{3}\tan 3^\circ}{\tan 20^\circ}$$

$$\therefore AB=\frac{1-\sqrt{3}\tan 3^\circ}{\tan 20^\circ}.$$

(b) (i) $f(x)=\tan^{-1}x+\tan^{-1}\left(\frac{1}{x}\right)$

Consider only $x>0$.

$$f'(x)=\frac{1}{1+x^2}+\frac{1}{1+\left(\frac{1}{x}\right)^2}\times -x^{-2}$$

$$=\frac{1}{1+x^2}+\frac{-x^{-2}}{1+\left(\frac{1}{x}\right)^2}\times\frac{x^2}{x^2}$$

$$=\frac{1}{1+x^2}+\frac{-1}{1+x^2}$$

$$=0$$

$\therefore$ the graph of $f(x)$ is horizontal for $x>0$.

Choose an easy value such as $x=1$

$$f(1)=\tan^{-1}1+\tan^{-1}\left(\frac{1}{1}\right)$$

$$=\frac{\pi}{4}+\frac{\pi}{4}$$

$$=\frac{\pi}{2}$$

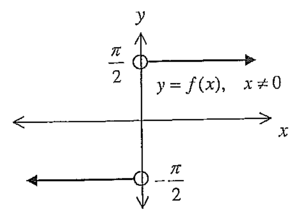
$\therefore\ f(x)=\frac{\pi}{2}$ for $x>0$.

(ii) When $x=-1$

$$f(-1)=\tan^{-1}(-1)+\tan^{-1}\left(\frac{1}{-1}\right)$$

$$=\left(-\frac{\pi}{4}\right)+\left(-\frac{\pi}{4}\right)$$

$$=-\frac{\pi}{2}$$



(c) (i) The exterior angle of a triangle (ΔXBD) is equal to the sum of the interior opposite angles.

(ii) $\angle TAC + \angle CAD = \angle TAD$ and $\angle TAD = \angle AXD$ (Alternate segment theorem - The angle between a tangent and a chord is equal to the angle in the alternate segment.)
$\therefore \angle AXD = \angle TAC + \angle CAD$

(iii) Let $\angle XBD = \alpha$, $\angle XDB = \beta$
$\therefore \angle AXD = \alpha + \beta$ (see part (i))
Hence, $\angle TAD = \alpha + \beta$ (see part (ii))
Now, $\angle TAC = \angle ABC = \alpha$
(Alternate segment theorem)
$\therefore \angle CAD = \alpha + \beta - \alpha = \beta$
Now, $\angle XAD = \angle XDB = \beta$
(Alternate segment theorem)
$\therefore \angle CAD = \angle XAD$
Hence, AD bisects $\angle BAC$.

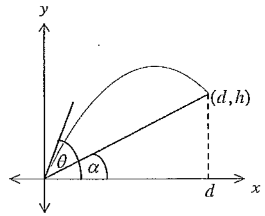
Question 6

(a) (i) $RHS = \cos A \cos B(1 + \tan A \tan B)$

$$= \cos A \cos B\left(1 + \frac{\sin A\ \sin B}{\cos A\ \cos B}\right)$$
$$= \cos A \cos B + \sin A \sin B$$
$$= \cos(A - B)$$
$$= LHS$$

(ii) If $\tan A \tan B = -1$ then
$\cos(A-B) = \cos A \cos B(1-1)$ [from (i)]
$\cos(A-B) = 0$
i.e. $A - B = \frac{\pi}{2}, \frac{3\pi}{2} \ldots$
but $B < A < \pi$ [subtract B]
thus $0 < A - B < \pi - B$ with $0 < B < \frac{\pi}{2}$
$\therefore A - B = \frac{\pi}{2}$ is the only solution

(b) (i)



$x = vt\cos\theta$ ①
$y = vt\sin\theta - 5t^2$ ②

from ① $t = \dfrac{x}{v\cos\theta}$
substitute in ②

$$y = v\left(\frac{x}{v\cos\theta}\right)\sin\theta - 5\left(\frac{x}{v\cos\theta}\right)^2$$
$$= \frac{x\sin\theta}{\cos\theta} - \frac{5x^2}{v^2\cos^2\theta}$$

(d, h) lies on the directory path,

substituting $h = \dfrac{d\sin\theta}{\cos\theta} - \dfrac{5d^2}{v^2\cos^2\theta}$

$$\frac{5d^2}{v^2\cos^2\theta} = \frac{d\sin\theta}{\cos\theta} - h$$
$$\frac{5d^2}{v^2} = d\sin\theta\cos\theta - h\cos^2\theta$$
$$\frac{v^2}{5d^2} = \frac{1}{d\sin\theta\cos\theta - h\cos^2\theta}$$
$$v^2 = \frac{5d^2}{d\sin\theta\cos\theta - h\cos^2\theta}$$
$$= \frac{5d^2}{d\left(\sin\theta\cos\theta - \frac{h}{d}\cos^2\theta\right)}$$
$$= \frac{5d}{\sin\theta\cos\theta - \frac{h}{d}\cos^2\theta}$$
$$= \frac{5d}{\cos\theta\sin\theta - \cos^2\theta\tan\alpha}$$

where $\tan\alpha = \dfrac{h}{d}$.

(ii) (1) When $\theta = \alpha$,

$$v^2 = \frac{5d}{\cos\alpha\sin\alpha - \cos^2\alpha\tan\alpha}$$

$$v^2 = \frac{5d}{\cos\alpha\sin\alpha - \cos^2\alpha\dfrac{\sin\alpha}{\cos\alpha}}$$
$$= \frac{5d}{\cos\alpha\sin\alpha - \cos\alpha\sin\alpha}$$

as $\theta \to \alpha$, $\cos\alpha\sin\alpha - \cos\alpha\sin\alpha \to 0$
$\therefore v \to \infty$

(ii) (2) When $\theta = \dfrac{\pi}{2}$,

$$v^2 = \frac{5d}{\cos\frac{\pi}{2}\sin\frac{\pi}{2} - \cos^2\frac{\pi}{2}\tan\alpha}$$
$$= \frac{5d}{(0\times 1) - (0\times\tan\alpha)}$$

as $\theta \to \dfrac{\pi}{2}$, $\cos\theta\sin\theta - \cos^2\theta\tan\theta \to 0$
$\therefore v \to \infty$

(iii) (α and $\tan\alpha$ are constants)
Using the double angle formulas:

$$F(\theta) = \cos\theta\sin\theta - \cos^2\theta\tan\alpha$$
$$= \frac{1}{2}\sin 2\theta - \frac{1}{2}(\cos 2\theta + 1)\tan\alpha$$
$$F'(\theta) = 2\times\frac{1}{2}\cos 2\theta - -2\times\frac{1}{2}\sin 2\theta\tan\alpha$$
$$= \cos 2\theta + \sin 2\theta\tan\alpha$$

$F'(\theta) = 0$ when
$\cos 2\theta + \sin 2\theta\tan\alpha = 0$ [$\div\cos 2\theta$]
$1 + \tan 2\theta\tan\alpha = 0$
$\tan 2\theta\tan\alpha = -1$

(iv) From (a)(ii)
if $\tan A\tan B = -1$ then $A - B = \dfrac{\pi}{2}$
if $\tan 2\theta\tan\alpha = -1$ then

$$2\theta - \alpha = \frac{\pi}{2}$$
$$2\theta = \frac{\pi}{2} + \alpha$$
$$\theta = \frac{\alpha}{2} + \frac{\pi}{4}$$

(v) $F'(\theta) = 0$ when $\theta = \dfrac{\alpha}{2} + \dfrac{\pi}{4}$

Let $v^2 = \dfrac{5d}{F(\theta)}$

$(v^2)' = \dfrac{-5dF'(\theta)}{[F(\theta)]^2}$ [quotient rule]

when $F'(\theta) = 0$, $(v^2)' = 0$
$\therefore v^2$ is a max or min when
$\theta = \dfrac{\alpha}{2} + \dfrac{\pi}{4} = \dfrac{1}{2}\left(\alpha + \dfrac{\pi}{2}\right)$

as $\theta \to \dfrac{\pi}{2}$, $v \to \infty$ and
$\theta \to \alpha$, $v \to \infty$
These are obvious maxima.
$\therefore v^2$ must be a minimum at $\dfrac{\alpha}{2} + \dfrac{\pi}{4}$

Question 7

(a) When $n = 1$, $47^1 + 53\times 147^{1-1} = 100$.
This is divisible by 100.
$\therefore$ the statement is true for $n = 1$.

Assume the statement true for $n = k$,
Assume $47^k + 53\times 147^{k-1} = 100M$
(where M is a positive integer).

When $n = k + 1$,
$47^{k+1} + 53\times 147^k$
$= 47\times 47^k + 53\times 147\times 147^{k-1}$
$= 47\times 47^k + 53\times(100 + 47)\times 147^{k-1}$
$= 47\times 47^k + 53\times 47\times 147^{k-1} + 5300\times 147^{k-1}$
$= 47\times[47^k + 53\times 147^{k-1}] + 5300\times 147^{k-1}$
$= 47\times 100M + 100\times 53\times 147^{k-1}$
$= 100\times[47M + 53\times 147^{k-1}]$
$= 100N$
(where N *is* an integer defined by $47M + 53\times 147^{k-1}$).
$\therefore$ the statement is true for $n = k + 1$.

Hence, if the statement is true for $n = 1$ and $n = k + 1$, it must be true for $n = 1+1 = 2$. By Mathematical Induction, it is true for all positive integers n.

(b) (i) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$

Substitute $x = 1$:

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$
$$= \sum_{k=0}^{n}\binom{n}{k}.$$

(ii) Substitute $n = 100$ in (i):

$$\binom{100}{0} + \binom{100}{1} + \cdots + \binom{100}{100} = \sum_{k=0}^{100}\binom{100}{k}$$
$$= 2^{100}.$$

(iii) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$

$$\frac{d}{dx}(1+x)^n = \binom{n}{1} + 2\binom{n}{2}x + \cdots + n\binom{n}{n}x^{n-1}$$

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + \cdots + n\binom{n}{n}x^{n-1}$$

substitute $x = 1$

$$n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n}$$
$$= \sum_{k=1}^{n} k\binom{n}{k}.$$

(c) (i) The selection of r balls could contain r red balls and 0 blue balls right through to 0 red balls and r blue balls. Thus there are $r + 1$ combinations.

(ii) The selection of $n-r$ balls from a possible n is given by

$$\binom{n}{n-r} = \binom{n}{n-(n-r)}$$
$$= \binom{n}{r}$$

(iii) The selection of n balls is made up of r balls from the red and blue balls and $n-r$ balls from the labeled white balls. Using (i) and (ii)

$$\text{combination} = (r+1)\binom{n}{r}$$

$$\text{Total selections} = \sum_{r=0}^{n}(r+1)\binom{n}{r}$$
$$= \sum_{r=0}^{n} r\binom{n}{r} + \sum_{r=0}^{n}\binom{n}{r}$$
$$= \sum_{r=0}^{n} r\binom{n}{r} + 2^n$$
$$= \sum_{r=1}^{n} r\binom{n}{r} + 2^n$$
$$= n2^{n-1} + 2^n$$
$$= n2^{n-1} + 2.2^{n-1}$$
$$= (n+2)2^{n-1}$$

End of Extension 1 solutions