BOARD OF STUDIES

NEW SOUTH WALES

2010

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

3380

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\quad x > 0$

Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{x}{\sqrt{1+3x^2}}\,dx$. **2**

(b) Evaluate $\int_0^{\frac{\pi}{4}} \tan x\,dx$. **3**

(c) Find $\int \frac{1}{x\left(x^2+1\right)}\,dx$. **3**

(d) Using the substitution $t=\tan\frac{x}{2}$, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$. **4**

(e) Find $\int \frac{dx}{1+\sqrt{x}}$. **3**

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z=5-i$.

(i) Find z^2 in the form $x+iy$. **1**

(ii) Find $z+2\overline{z}$ in the form $x+iy$. **1**

(iii) Find $\frac{i}{z}$ in the form $x+iy$. **2**

(b) (i) Express $-\sqrt{3}-i$ in modulus–argument form. **2**

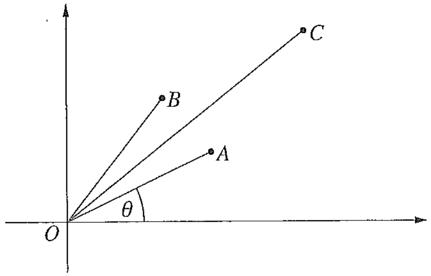
(ii) Show that $\left(-\sqrt{3}-i\right)^6$ is a real number. **2**

(c) Sketch the region in the complex plane where the inequalities $1\le|z|\le 2$ and $0\le z+\overline{z}\le 3$ hold simultaneously. **2**

Question 2 continues on page 5

(d) Let $z = \cos\theta + i\sin\theta$ where $0 < \theta < \frac{\pi}{2}$.

On the Argand diagram the point A represents z, the point B represents z^2 and the point C represents $z + z^2$.



Copy or trace the diagram into your writing booklet.

(i) Explain why the parallelogram $OACB$ is a rhombus. **1**

(ii) Show that $\arg\left(z + z^2\right) = \frac{3\theta}{2}$. **1**

(iii) Show that $\left|z + z^2\right| = 2\cos\frac{\theta}{2}$. **2**

(iv) By considering the real part of $z + z^2$, or otherwise, deduce that $\cos\theta + \cos 2\theta = 2\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$. **1**

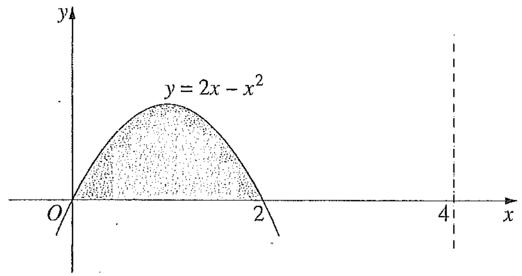
End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Sketch the graph $y = x^2 + 4x$. **1**

(ii) Sketch the graph $y = \frac{1}{x^2 + 4x}$. **2**

(b) The region shaded in the diagram is bounded by the x-axis and the curve $y = 2x - x^2$. **4**



The shaded region is rotated about the line $x = 4$.

Find the volume generated.

(c) Two identical biased coins are each more likely to land showing heads than showing tails. **2**

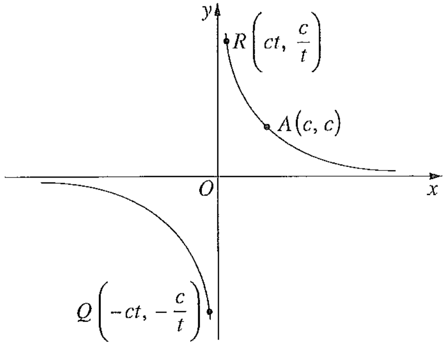
The two coins are tossed together, and the outcome is recorded. After a large number of trials it is observed that the probability that the two coins land showing a head and a tail is 0.48.

What is the probability that both coins land showing heads?

Question 3 continues on page 7

Question 3 (continued)

(d) The diagram shows the rectangular hyperbola $xy = c^2$, with $c > 0$.



The points $A(c, c)$, $R\left(ct, \frac{c}{t}\right)$ and $Q\left(-ct, -\frac{c}{t}\right)$ are points on the hyperbola, with $t \neq \pm 1$.

(i) The line ℓ_1 is the line through R perpendicular to QA. **2**

Show that the equation of ℓ_1 is

$$y = -tx + c\left(t^2 + \frac{1}{t}\right).$$

(ii) The line ℓ_2 is the line through Q perpendicular to RA. **1**

Write down the equation of ℓ_2.

(iii) Let P be the point of intersection of the lines ℓ_1 and ℓ_2. **2**

Show that P is the point $\left(\frac{c}{t^2}, ct^2\right)$.

(iv) Give a geometric description of the locus of P. **1**

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) (i) A curve is defined implicitly by $\sqrt{x} + \sqrt{y} = 1$. **2**

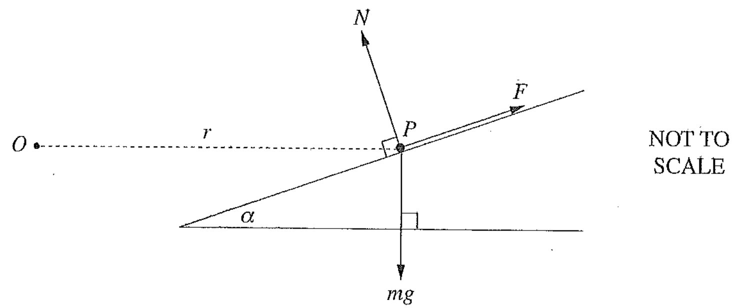
Use implicit differentiation to find $\frac{dy}{dx}$.

(ii) Sketch the curve $\sqrt{x} + \sqrt{y} = 1$. **2**

(iii) Sketch the curve $\sqrt{|x|} + \sqrt{|y|} = 1$. **1**

(b) A bend in a highway is part of a circle of radius r, centre O. Around the bend the highway is banked at an angle α to the horizontal.

A car is travelling around the bend at a constant speed v. Assume that the car is represented by a point P of mass m. The forces acting on the car are a lateral force F, the gravitational force mg and a normal reaction N to the road, as shown in the diagram.



(i) By resolving forces, show that $F = mg\sin\alpha - \frac{mv^2}{r}\cos\alpha$. **3**

(ii) Find an expression for v such that the lateral force F is zero. **1**

Question 4 continues on page 9

(c) Let k be a real number, $k \geq 4$.

Show that, for every positive real number b, there is a positive real number a such that $\frac{1}{a}+\frac{1}{b}=\frac{k}{a+b}$. **3**

(d) A group of 12 people is to be divided into discussion groups.

(i) In how many ways can the discussion groups be formed if there are 8 people in one group, and 4 people in another? **1**

(ii) In how many ways can the discussion groups be formed if there are 3 groups containing 4 people each? **2**

End of Question 4

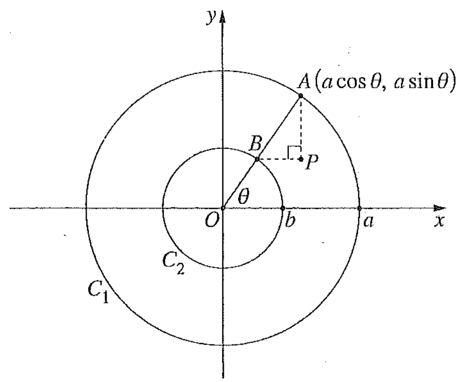
Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows two circles, C_1 and C_2, centred at the origin with radii a and b, where $a > b$.

The point A lies on C_1 and has coordinates $(a\cos\theta, a\sin\theta)$.

The point B is the intersection of OA and C_2.

The point P is the intersection of the horizontal line through B and the vertical line through A.



(i) Write down the coordinates of B. **1**

(ii) Show that P lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. **1**

(iii) Find the equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at P. **2**

(iv) Assume that A is not on the y-axis. **2**

Show that the tangent to the circle C_1 at A, and the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at P, intersect at a point on the x-axis.

Question 5 continues on page 11

Question 5 (continued)

(b) Show that **2**

$$\int \frac{dy}{y(1-y)} = \ln\left(\frac{y}{1-y}\right) + c$$

for some constant c, where $0 < y < 1$.

(c) A TV channel has estimated that if it spends $\$x$ on advertising a particular program it will attract a proportion $y(x)$ of the potential audience for the program, where

$$\frac{dy}{dx} = ay(1-y)$$

and $a > 0$ is a given constant.

(i) Explain why $\frac{dy}{dx}$ has its maximum value when $y = \frac{1}{2}$. **1**

(ii) Using part (b), or otherwise, deduce that **3**

$$y(x) = \frac{1}{ke^{-ax}+1}$$

for some constant $k > 0$.

(iii) The TV channel knows that if it spends no money on advertising the program then the audience will be one-tenth of the potential audience. **1**

Find the value of the constant k referred to in part (c) (ii).

(iv) What feature of the graph $y = \frac{1}{ke^{-ax}+1}$ is determined by the result in part (c) (i)? **1**

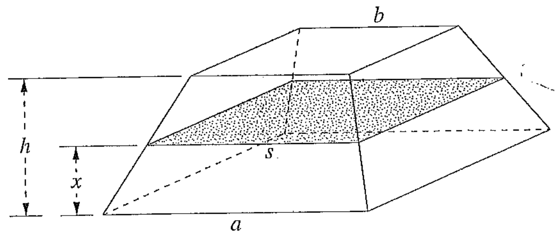
(v) Sketch the graph $y = \frac{1}{ke^{-ax}+1}$. **1**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the frustum of a right square pyramid. (A frustum of a pyramid is a pyramid with its top cut off.)

The height of the frustum is h m. Its base is a square of side a m, and its top is a square of side b m (with $a > b > 0$).



A horizontal cross-section of the frustum, taken at height x m, is a square of side s m, shown shaded in the diagram.

(i) Show that $s = a - \frac{(a-b)}{h}x$. **2**

(ii) Find the volume of the frustum. **2**

(b) A sequence a_n is defined by **3**

$$a_n = 2a_{n-1} + a_{n-2},$$

for $n \geq 2$, with $a_0 = a_1 = 2$.

Use mathematical induction to prove that

$$a_n = \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n \quad \text{for all } n \geq 0.$$

Question 6 continues on page 13

Question 6 (continued)

(c) (i) Expand $(\cos\theta + i\sin\theta)^5$ using the binomial theorem. **1**

(ii) Expand $(\cos\theta + i\sin\theta)^5$ using de Moivre's theorem, and hence show that **3**

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$

(iii) Deduce that $x = \sin\left(\frac{\pi}{10}\right)$ is one of the solutions to **1**

$$16x^5 - 20x^3 + 5x - 1 = 0.$$

(iv) Find the polynomial $p(x)$ such that $(x-1)\,p(x) = 16x^5 - 20x^3 + 5x - 1$. **1**

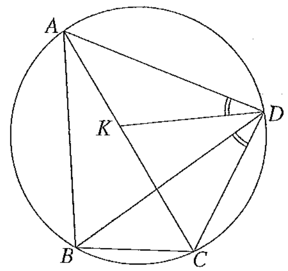
(v) Find the value of a such that $p(x) = (4x^2 + ax - 1)^2$. **1**

(vi) Hence find an exact value for $\sin\left(\frac{\pi}{10}\right)$. **1**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) In the diagram $ABCD$ is a cyclic quadrilateral. The point K is on AC such that $\angle ADK = \angle CDB$, and hence ΔADK is similar to ΔBDC.

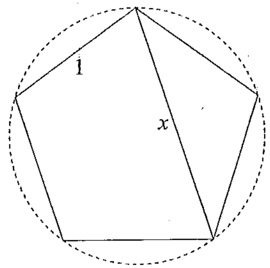


Copy or trace the diagram into your writing booklet.

(i) Show that $\triangle ADB$ is similar to $\triangle KDC$. **2**

(ii) Using the fact that $AC = AK + KC$, show that $BD \times AC = AD \times BC + AB \times DC$. **2**

(iii) A regular pentagon of side length 1 is inscribed in a circle, as shown in the diagram. **2**



Let x be the length of a chord in the pentagon.

Use the result in part (ii) to show that $x = \frac{1+\sqrt{5}}{2}$.

Question 7 continues on page 15

Question 7 (continued)

(b) The graphs of $y = 3x - 1$ and $y = 2^x$ intersect at $(1, 2)$ and at $(3, 8)$. **1**

Using these graphs, or otherwise, show that $2^x \geq 3x - 1$ for $x \geq 3$.

(c) Let $P(x) = (n-1)x^n - nx^{n-1} + 1$, where n is an odd integer, $n \geq 3$.

(i) Show that $P(x)$ has exactly two stationary points. **1**

(ii) Show that $P(x)$ has a double zero at $x = 1$. **1**

(iii) Use the graph $y = P(x)$ to explain why $P(x)$ has exactly one real zero other than 1. **2**

(iv) Let α be the real zero of $P(x)$ other than 1. **2**

Using part (b), or otherwise, show that $-1 < \alpha \leq -\frac{1}{2}$.

(v) Deduce that each of the zeros of $4x^5 - 5x^4 + 1$ has modulus less than or equal to 1. **2**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet.

Let

$$A_n = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx \quad \text{and} \quad B_n = \int_0^{\frac{\pi}{2}} x^2 \cos^{2n} x \, dx,$$

where n is an integer, $n \geq 0$. (Note that $A_n > 0$, $B_n > 0$.)

(a) Show that $n A_n = \frac{2n-1}{2} A_{n-1}$ for $n \geq 1$. **2**

(b) Using integration by parts on A_n, or otherwise, show that **1**

$$A_n = 2n \int_0^{\frac{\pi}{2}} x \sin x \cos^{2n-1} x \, dx \quad \text{for } n \geq 1.$$

(c) Use integration by parts on the integral in part (b) to show that **3**

$$\frac{A_n}{n^2} = \frac{(2n-1)}{n} B_{n-1} - 2B_n \quad \text{for } n \geq 1.$$

(d) Use parts (a) and (c) to show that **1**

$$\frac{1}{n^2} = 2\left(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n}\right) \quad \text{for } n \geq 1.$$

(e) Show that $\displaystyle\sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} - 2\frac{B_n}{A_n}$. **2**

(f) Use the fact that $\sin x \geq \frac{2}{\pi}x$ for $0 \leq x \leq \frac{\pi}{2}$ to show that **1**

$$B_n \leq \int_0^{\frac{\pi}{2}} x^2 \left(1 - \frac{4x^2}{\pi^2}\right)^n dx.$$

Question 8 continues on page 17

(g) Show that $\displaystyle\int_0^{\frac{\pi}{2}} x^2\left(1-\frac{4x^2}{\pi^2}\right)^n dx = \frac{\pi^2}{8(n+1)}\int_0^{\frac{\pi}{2}}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1} dx.$ **1**

(h) From parts (f) and (g) it follows that **2**

$$B_n \le \frac{\pi^2}{8(n+1)}\int_0^{\frac{\pi}{2}}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1} dx.$$

Use the substitution $x = \frac{\pi}{2}\sin t$ in this inequality to show that

$$B_n \le \frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}} \cos^{2n+3} t\, dt \le \frac{\pi^3}{16(n+1)}A_n.$$

(i) Use part (e) to deduce that **1**

$$\frac{\pi^2}{6} - \frac{\pi^3}{8(n+1)} \le \sum_{k=1}^{n}\frac{1}{k^2} < \frac{\pi^2}{6}.$$

(j) What is $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2}$? **1**

End of paper

2010 Higher School Certificate Solutions
Mathematics Extension 2

Question 1

(a) Method 1

$$\int \frac{x}{\sqrt{1+3x^2}}\,dx = \int x(1+3x^2)^{-\frac{1}{2}}\,dx$$
$$= \frac{1}{6}\int 6x(1+3x^2)^{-\frac{1}{2}}\,dx$$
$$= \frac{(1+3x^2)^{\frac{1}{2}}}{\frac{1}{2}\times 6} + c$$
$$= \frac{1}{3}(1+3x^2)^{\frac{1}{2}} + c$$
$$= \frac{1}{3}\sqrt{1+3x^2} + c$$

Method 2

Let $u = 1+3x^2$

$$du = 6x\,dx \Rightarrow \frac{1}{6}du = x\,dx$$
$$\int \frac{x}{\sqrt{1+3x^2}}\,dx = \int \frac{\frac{1}{6}du}{\sqrt{u}}$$
$$= \int \frac{1}{6}u^{-\frac{1}{2}}du$$
$$= \frac{1}{6}\times 2u^{\frac{1}{2}} + c$$
$$= \frac{1}{3}u^{\frac{1}{2}} + c$$
$$= \frac{1}{3}\sqrt{1+3x^2} + c$$

(b) Method 1

$$\int_0^{\frac{\pi}{4}} \tan x\,dx = -\int_0^{\frac{\pi}{4}} \frac{-\sin x}{\cos x}\,dx$$
$$= -\Big[\log_e|\cos x|\Big]_0^{\frac{\pi}{4}}$$
$$= -\left[\log_e\frac{1}{\sqrt{2}} - \log_e 1\right]$$
$$= \log_e\sqrt{2}$$
$$= \frac{1}{2}\log_e 2$$

Method 2

Let $u = \tan x$

$$du = \sec^2 x\,dx$$
$$= \left(1+\tan^2 x\right)dx,$$
$$\frac{du}{1+u^2} = dx.$$

When $x = 0$, $\quad t = \tan 0 = 0$

When $x = \frac{\pi}{4}$, $\quad t = \tan\frac{\pi}{4} = 1$

$$\int_0^{\frac{\pi}{4}} \tan x\,dx = \int_0^1 \frac{u}{1+u^2}\,du$$
$$= \frac{1}{2}\Big[\log_e|1+u^2|\Big]_0^1$$
$$= \frac{1}{2}\log_e 2$$

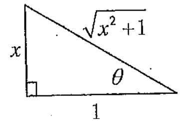
(c) Method 1

$$\frac{1}{x(x^2+1)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+1}$$
$$1 \equiv A(x^2+1) + x(Bx+C)$$
$$1 \equiv Ax^2 + A + Bx^2 + Cx$$
$$1 \equiv (A+B)x^2 + Cx + A$$

Equating coefficients gives:
$A = 1, \quad B = -1, \quad C = 0$

$$\int \frac{1}{x(x^2+1)}\,dx = \int \frac{1}{x} - \frac{x}{x^2+1}\,dx$$
$$= \log_e|x| - \frac{1}{2}\log_e|x^2+1| + c$$
$$= \log_e\left|\frac{x}{\sqrt{x^2+1}}\right| + c$$

Method 2



Let $x = \tan\theta$

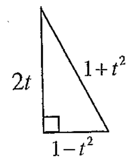
$$dx = \sec^2\theta\,d\theta$$
$$x^2+1 = \tan^2\theta + 1$$
$$= \sec^2\theta$$
$$\int \frac{1}{x(x^2+1)}\,dx = \int \frac{1}{\tan\theta\sec^2\theta}\sec^2\theta\,d\theta$$
$$= \int \frac{1}{\tan\theta}\,d\theta$$
$$= \int \frac{\cos\theta}{\sin\theta}\,d\theta$$
$$= \log_e|\sin\theta|$$
$$= \log_e\left|\frac{x}{\sqrt{x^2+1}}\right| + c$$

(d) Let $t = \tan\frac{x}{2}$

When $x = 0$, $\quad t = \tan\frac{0}{2} = 0$

When $x = \frac{\pi}{2}$, $\quad t = \tan\frac{\pi}{4} = 1$

$$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$$
$$= \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)dx$$
$$= \frac{1}{2}\left(1+t^2\right)dx$$
$$dx = \frac{2\,dt}{1+t^2}$$



$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx = \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}}\cdot\frac{2\,dt}{1+t^2}$$
$$= \int_0^1 \frac{2}{1+t^2+2t}\,dt$$
$$= \int_0^1 2(1+t)^{-2}\,dt$$
$$= \left[\frac{2(1+t)^{-1}}{-1}\right]_0^1$$
$$= -1+2$$
$$= 1$$

(e) Method 1

Let $u = \sqrt{x}$
$x = u^2$
$dx = 2u\,du$

or

Let $u = \sqrt{x}$
$du = \frac{1}{2\sqrt{x}}dx$
So $2u\,du = dx$

$$\int \frac{dx}{1+\sqrt{x}} = 2\int \frac{u\,du}{1+u}$$
$$= 2\int \left(\frac{1+u}{1+u} - \frac{1}{1+u}\right)du$$
$$= 2\int \left(1 - \frac{1}{1+u}\right)du$$
$$= 2(u - \log_e|1+u|) + c$$
$$= 2\sqrt{x} - 2\log_e\left(1+\sqrt{x}\right) + c$$

Method 2

Let $u = 1+\sqrt{x}$

$$du = \frac{1}{2\sqrt{x}}dx$$
$$2\sqrt{x}\,du = dx$$
$$2(u-1)\,du = dx$$
$$\int \frac{dx}{1+\sqrt{x}} = 2\int \frac{(u-1)\,du}{u}$$
$$= 2\int \left(1 - \frac{1}{u}\right)du$$
$$= 2(u - \log_e|u|) + d$$
$$= 2 + 2\sqrt{x} - 2\log_e\left(1+\sqrt{x}\right) + d$$

Note that $c = 2+d$ so that the answers using method 1 or 2 are equivalent.

Question 2

(a) (i) $z^2 = (5-i)^2$

$= 25 - 10i + i^2$

$= 24 - 10i$

(ii) $z + 2\bar{z} = 5 - i + 2(5+i)$

$= 5 - i + 10 + 2i$

$= 15 + i$

(iii) $\dfrac{i}{z} = \dfrac{i}{5-i}$

$= \dfrac{i}{5-i} \times \dfrac{5+i}{5+i}$

$= \dfrac{5i + i^2}{25 - i^2}$

$= \dfrac{-1+5i}{25+1}$

$= \dfrac{-1}{26} + \dfrac{5}{26}i$

(b) (i) $\left|-\sqrt{3} - 1\right| = \sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2}$

$= 2$

$Arg\left(-\sqrt{3} - i\right) = \dfrac{7\pi}{6}$ or $-\dfrac{5\pi}{6}$

but $Arg(z) = \theta$ where $-\pi < \theta \leq \pi$

$\therefore Arg\left(-\sqrt{3} - i\right) = -\dfrac{5\pi}{6}$

$-\sqrt{3} - i = 2\left(\cos\dfrac{-5\pi}{6} + i\sin\dfrac{-5\pi}{6}\right)$

(ii) $\left(-\sqrt{3} - i\right)^6 = \left[2\left(\cos\dfrac{7\pi}{6} + i\sin\dfrac{7\pi}{6}\right)\right]^6$

$= 2^6\left(\cos\dfrac{7\pi}{6}\times 6 + i\sin\dfrac{7\pi}{6}\times 6\right)$

$= 2^6(\cos 7\pi + i\sin 7\pi)$

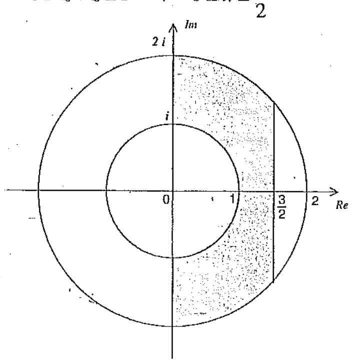
$= 2^6(\cos\pi + i\sin\pi)$

$= 64 \times (-1)$

$= -64$

(c) (i) $1 \leq |z| \leq 2 \Rightarrow 1 \leq x^2 + y^2 \leq 4$

$0 \leq z + \bar{z} \leq 3 \Rightarrow 0 \leq x \leq \dfrac{3}{2}$



(d) (i) $OA = \left|\vec{OA}\right| = |z| = 1$

$OB = \left|\vec{OB}\right| = \left|z^2\right| = |z|^2 = 1$

So OA=OB

OACB is a parallelogram with adjacent sides equal $\therefore$ it is a rhombus.

(ii) $Arg\left(z^2\right) = 2Arg(z)$

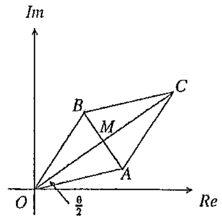
$= 2\theta$

$\therefore \angle BOA = \theta$

In a rhombus, the diagonals bisect the angles so $\angle COA = \dfrac{\theta}{2}$.

$Arg(z + z^2) = \theta + \dfrac{\theta}{2} = \dfrac{3\theta}{2}$.

(iii)



$\left|z + z^2\right| = OC$.

Let M be the midpoint of OC.

In a rhombus, the diagonals bisect each other at right angles.

In ΔOMA,

$\cos\dfrac{\theta}{2} = \dfrac{OM}{OA}$

$= \dfrac{OM}{1}$

$= OM$

$OC = 2OM$

So $OC = 2\cos\dfrac{\theta}{2}$

$\therefore \left|z + z^2\right| = 2\cos\dfrac{\theta}{2}$

(iv) $z + z^2 = 2\cos\dfrac{\theta}{2}\left(cis\dfrac{3\theta}{2}\right)$ from (ii) and (iii

$= 2\cos\dfrac{\theta}{2}\cos\dfrac{3\theta}{2} + i \times 2\cos\dfrac{\theta}{2}\sin\dfrac{3\theta}{2}$

$\text{Re}\left(z + z^2\right) = 2\cos\dfrac{\theta}{2}\cos\dfrac{3\theta}{2}$

Alternately,

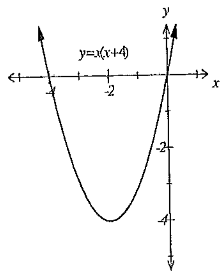
$z + z^2 = \cos\theta + i\sin\theta + \cos 2\theta + i\sin 2\theta$

$\text{Re}\left(z + z^2\right) = \cos\theta + \cos 2\theta$

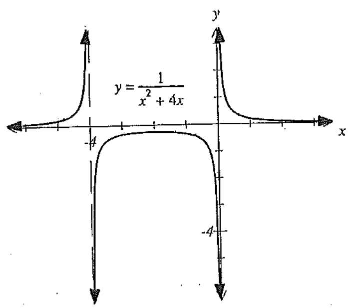
$\therefore \cos\theta + \cos 2\theta = 2\cos\dfrac{\theta}{2}\cos\dfrac{3\theta}{2}$

Question 3

(a) (i)



(ii)



(b) $V_{shell} = \pi R^2 H - \pi r^2 h$

$R = 4 - x,\ r = 4 - x - \delta x,\ h \approx H = y$

$V_{shell} = \pi y\left\{(4-x)^2 - (4 - x - \delta x)^2\right\}$

$= \pi y\{8 - 2x - \delta x\}\delta x$

$= 2\pi y\{4 - x\}\delta x$ as $(\delta x)^2 \approx 0$

$V_{solid} = \lim\limits_{\delta x \to 0} \sum\limits_{x=0}^{2} 2\pi y(4-x)\delta x$

$= 2\pi\int_0^2 \left(8x - 6x^2 + x^3\right)dx$

$= 2\pi\left[4x^2 - 2x^3 + \frac{x^4}{4}\right]_0^2$

$= 8\pi$

There are alternative methods.

(c) Let $p = P(H)$, $q = P(T)$, $q = 1 - p$

$(p+q)^2 = p^2 + 2pq + q^2$

$2pq = 0.48 \quad \therefore 2p(1-p) = 0.48$,

$p^2 - p + 0.24 = 0$

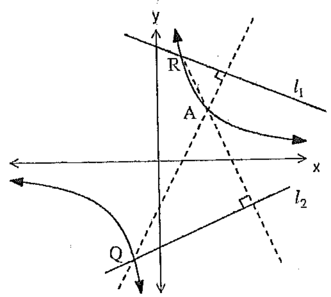
$(p - 0.6)(p - 0.4) = 0$

$\therefore p = 0.6$ or 0.4

but $p > q$, $\quad \therefore p = 0.6$

$P(HH) = 0.6 \times 0.6 = 0.36$.

(d) (i)



$m_{QA} = \dfrac{c + \frac{c}{t}}{c + ct}$

$= \dfrac{ct + c}{t} \times \dfrac{1}{c + ct}$

$= \dfrac{1}{t}$

$\therefore m_{l_1} = -t$.

Equation of l_1:

$$y-\frac{c}{t}=-t(x-ct)$$

$$y=-tx+ct^2+\frac{c}{t}$$

$$y=-tx+c\left(t^2+\frac{1}{t}\right)$$

(ii)

$$m_{RA}=\frac{c-\frac{c}{t}}{c-ct}=\frac{t-1}{t(1-t)}=-\frac{1}{t}$$

Equation of l_2:

$$y+\frac{c}{t}=t(x+ct)$$

$$y=tx+ct^2-\frac{c}{t}$$

$$y=tx+c\left(t^2-\frac{1}{t}\right).$$

(iii) Substitute for y in l_2

$$-tx+c\left(t^2+\frac{1}{t}\right)=tx+c\left(t^2-\frac{1}{t}\right)$$

$$-t^2x+ct^3+c=t^2x+ct^3-c$$

$$-2t^2x=-2c$$

$$x=\frac{c}{t^2}$$

Substitute:

$$y=-t\left(\frac{c}{t^2}\right)+c\left(t^2+\frac{1}{t}\right)$$

$$=ct^2$$

$\therefore$ P is $\left(\frac{c}{t^2},ct^2\right)$.

(iv) The locus of P is the first quadrant portion of the rectangular hyperbola which defines the original curve. Since the parameter is t^2 rather than t, there are no points in the third quadrant.

Alternate answer:

Since $\frac{c}{t^2}\times ct^2=c^2$, the locus of P is also the rectangular hyperbola $xy=c^2$. However, since the parameter is t^2 rather than t, the locus has no points in the third quadrant.

Question 4

(a) (i) Differentiate with respect to x

$$\sqrt{x}+\sqrt{y}=1$$

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\times\frac{dy}{dx}=0$$

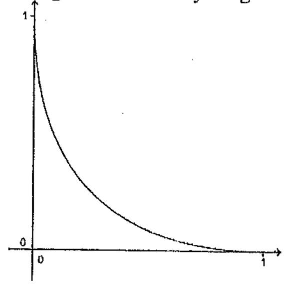
$$\frac{dy}{dx}=-\frac{1}{2\sqrt{x}}\div\frac{1}{2\sqrt{y}}$$

$$=-\sqrt{\frac{y}{x}}$$

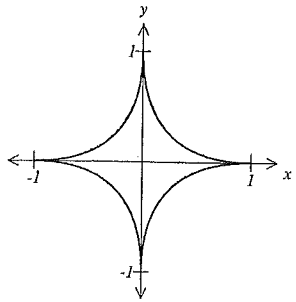
$$=\frac{\sqrt{x}-1}{\sqrt{x}},\quad x\neq 0$$

(ii) $\sqrt{x}+\sqrt{y}=1$

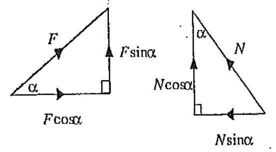
(1, 0) and (0, 1) are obvious endpoints. The gradient is always negative.



(iii) $\sqrt{|x|}+\sqrt{|y|}=1$



(b) (i)



Vertically the forces are in balance,

$\therefore N\cos\alpha+F\sin\alpha=mg$ ①

Horizontally there is a net centripetal force of $\frac{mv^2}{r}$,

$\therefore N\sin\alpha-F\cos\alpha=\frac{mv^2}{r}$ ②

Multiply ① by $\sin\alpha$, ② by $\cos\alpha$ and subtract:

$$F\sin^2\alpha+F\cos^2\alpha=mg\sin\alpha-\frac{mv^2}{r}\cos\alpha$$

$$\text{ie } F=mg\sin\alpha-\frac{mv^2}{r}\cos\alpha$$

(ii) When $F=0$:

$$0=mg\sin\alpha-\frac{mv^2}{r}\cos\alpha$$

$$\frac{mv^2}{r}\cos\alpha=mg\sin\alpha$$

$$v^2=\frac{gr\sin\alpha}{\cos\alpha}$$

$$v=\sqrt{gr\tan\alpha}$$

(c)

$$\text{As }\frac{1}{a}+\frac{1}{b}=\frac{k}{a+b}$$

$$\frac{a+b}{ab}=\frac{k}{a+b}$$

$$(a+b)^2=abk$$

$$a^2+(2b-bk)a+b^2=0$$

$$\Delta_{\text{for }a}=(2b-bk)^2-4.1.b^2$$

$$=b^2\left[4-4k+k^2-4\right]$$

$$=k(k-4)b^2$$

$$\geq 0\quad\text{since } k\geq 0 \text{ and } b\in\mathbb{R}$$

$$a=\frac{(k-2)b\pm\sqrt{\Delta}}{2}\quad\therefore a\in\mathbb{R}.$$

(d) (i) ${}^{12}C_4={}^{12}C_8=495$

(ii) $\frac{{}^{12}C_4\times{}^{8}C_4\times{}^{4}C_4}{3!}=5775$

Question 5

(a) (i) $B(b\cos\theta,b\sin\theta)$

(ii)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{(a\cos\theta)^2}{a^2}+\frac{(b\sin\theta)^2}{b^2}$$

$$=\cos^2\theta+\sin^2\theta$$

$$=1$$

$\therefore P$ satisfies the equation of the ellipse, so P lies on the ellipse.

(iii) Method 1

$$\frac{d}{dx}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=\frac{d}{dx}(1)$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

$$m_{\text{tangent}}=-\frac{b^2}{a^2}\times\frac{a\cos\theta}{b\sin\theta}$$

$$=-\frac{b\cos\theta}{a\sin\theta}$$

$\therefore$ equation of tangent at P is

$$y-b\sin\theta=-\frac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)$$

$$ay\sin\theta-ab\sin^2\theta=-bx\cos\theta+ab\cos^2\theta$$

$$bx\cos\theta+ay\sin\theta=ab(\cos^2\theta+\sin^2\theta)$$

$$bx\cos\theta+ay\sin\theta=ab$$

Method 2

$$\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}$$

$$=\frac{dy}{d\theta}\div\frac{dx}{d\theta}$$

$$=\frac{b\cos\theta}{-a\sin\theta}$$

$$=-\frac{b\cos\theta}{a\sin\theta}$$

Then proceed as in Method 1

(iv) From (iii) the tangent at A is:

$ax\cos\theta + ay\sin\theta = a^2 \quad (\text{as } b = a)$

$x\cos\theta + y\sin\theta = a$ ①

and the tangent at P is

$bx\cos\theta + ay\sin\theta = ab$ ②

① × b

$bx\cos\theta + by\sin\theta = ab$ ③

② − ③

$y\sin\theta(a-b) = 0$

$\therefore \; y = 0$ as $(a > b, \theta \neq 0)$

Intersection is on the x-axis as $y = 0$.

(b) Method 1: Differentiation of RHS

$$\frac{d}{dx}\left[\ln\left(\frac{y}{1-y}\right)+c\right] = \frac{d}{dx}\left[\ln y - \ln(1-y)\right]$$

$$= \frac{1}{y} - \frac{-1}{1-y}$$

$$= \frac{1}{y} + \frac{1}{1-y}$$

$$= \frac{1-y+y}{y(1-y)}$$

$$= \frac{1}{y(1-y)}$$

Method 2: Partial Fractions of LHS

$$\int \frac{dy}{y(1-y)} = \int\left(\frac{A}{y} + \frac{B}{1-y}\right)dy$$

$\therefore 1 \equiv A(1-y) + By$

when $y = 0$, $A = 1$

when $y = 1$, $B = 1$

$$\int \frac{dy}{y(1-y)} = \int\left(\frac{1}{y} + \frac{1}{1-y}\right)dy$$

$$= \ln|y| - \ln|1-y| + c$$

$$= \ln\left(\frac{y}{1-y}\right) + c.$$

[because $0 < y < 1$]

(c) (i) As $ay(1-y)$ is a concave down quadratic in y as $a > 0$, then the maximum value will exist on the axis of symmetry, $y = -\dfrac{1}{2(-1)} = \dfrac{1}{2}$.

Many other methods would be acceptable.

(ii) Method 1 – Separation of variables

$$\int \frac{dy}{y(1-y)} = \int a\,dx$$

$$\ln\left(\frac{y}{1-y}\right) + c = ax$$

$$\ln\left(\frac{y}{1-y}\right) = ax - c$$

$$\frac{y}{1-y} = e^{ax-c}$$

$$\frac{1-y}{y} = e^{-ax+c}$$

$$\frac{1}{y} - 1 = e^{-ax+c}$$

$$\frac{1}{y} = e^{-ax+c} + 1$$

$$y = \frac{1}{e^c e^{-ax} + 1}$$

Letting $k = e^{-c}$

$$\therefore y = \frac{1}{ke^{-ax} + 1}.$$

Method 2 – Taking reciprocals

$$\frac{dx}{dy} = \frac{1}{ay(1-y)}$$

$$x = \int \frac{dy}{ay(1-y)}$$

$$= \frac{1}{a}\int \frac{dy}{y(1-y)}$$

$$\therefore x = \frac{1}{a}\ln\left(\frac{y}{1-y}\right) + c_2$$

$$\therefore ax + c_3 = \ln\left(\frac{y}{1-y}\right) \quad [c_3 = -ac_2]$$

$$\text{OR} \therefore ax - c_4 = \ln\left(\frac{y}{1-y}\right) \quad [c_4 = ac_2]$$

$$\therefore e^{ax+c_3} = \frac{y}{1-y} \quad \text{OR} \quad \frac{y}{1-y} = e^{ax-c_4}$$

then proceed as in Method 1.

(iii) Given $x = 0$, $y = \dfrac{1}{10}$,

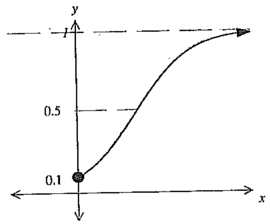
$$\frac{1}{10} = \frac{1}{ke^0 + 1} = \frac{1}{k+1}$$

$k + 1 = 10$

$k = 9.$

(iv) The feature of the graph is the point of inflexion at $y = ½$.

(v)

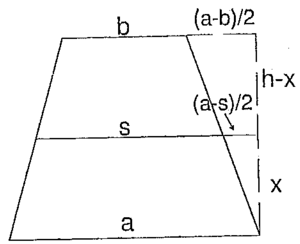


$x \to \infty$, $y \to \dfrac{1}{0^+ + 1} \to 1^-$

$x = 0$, $y = 0{\cdot}1$

Question 6

(a) (i)



Corresponding sides of similar triangles are in proportion.

Hence:

$$\frac{\frac{a-s}{2}}{\frac{a-b}{2}} = \frac{x}{h}$$

$$\frac{a-s}{a-b} = \frac{x}{h}$$

$$a - s = \frac{a-b}{h}x$$

$$s = a - \frac{a-b}{h}x$$

(ii)

$$V = \int_0^h s^2\,dx$$

$$= \int_0^h \left(a - \frac{(a-b)}{h}x\right)^2 dx$$

$$= \int_0^h \left(a^2 - \frac{2a(a-b)}{h}x + \frac{(a-b)^2}{h^2}x^2\right)dx$$

$$= \left[a^2x - \frac{a(a-b)}{h}x^2 + \frac{(a-b)^2}{h^2}\frac{x^3}{3}\right]_0^h$$

$$= a^2h - \frac{a(a-b)}{h}h^2 + \frac{(a-b)^2}{h^2}\frac{h^3}{3}$$

$$= \frac{h}{3}\left(3a^2 - 3a^2 + 3ab + a^2 - 2ab + b^2\right)$$

$$= \frac{h}{3}\left(a^2 + ab + b^2\right)$$

(b) Required to prove that:

$a_n = \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n$ for all $n \geq 0$.

When $n = 0$:

$a_0 = \left(1+\sqrt{2}\right)^0 + \left(1-\sqrt{2}\right)^0$

$= 1 + 1$

$= 2$

When $n = 1$:

$a_1 = \left(1+\sqrt{2}\right)^1 + \left(1-\sqrt{2}\right)^1$

$= 1 + \sqrt{2} + 1 - \sqrt{2}$

$= 2.$

$\therefore$ the formula holds true for $n = 0, 1$.

When $n = 2$:

$a_2 = (1+\sqrt{2})^2 + (1-\sqrt{2})^2$

$= 1 + 2\sqrt{2} + 2 + 1 - 2\sqrt{2} + 2$

$= 6$

Using the formula:

$a_2 = 2a_1 + a_0 = 4 + 2 = 6$

$\therefore$ the formula holds true for $n = 2$.

Assume the formula is true for $n = k-1$ and $n = k$ (where $k \geq 1$ (k integer)).

That is, assume

$a_k = \left(1+\sqrt{2}\right)^k + \left(1-\sqrt{2}\right)^k$ and

$a_{k-1} = \left(1+\sqrt{2}\right)^{k-1} + \left(1-\sqrt{2}\right)^{k-1}$.

From the definition of the sequence,
$a_{k+1} = 2a_k + a_{k-1}$.

$$a_{k+1} = 2\left(\left(1+\sqrt{2}\right)^k + \left(1-\sqrt{2}\right)^k\right) + \left(1+\sqrt{2}\right)^{k-1} + \left(1-\sqrt{2}\right)^{k-1}$$

Factorising,

$$\begin{aligned} a_{k+1} &= \left(1+\sqrt{2}\right)^{k-1}\left(1+2\left(1+\sqrt{2}\right)\right) + \left(1-\sqrt{2}\right)^{k-1}\left(1+2\left(1-\sqrt{2}\right)\right) \\ &= \left(1+\sqrt{2}\right)^{k-1}\left(1+2\sqrt{2}+2\right) + \left(1-\sqrt{2}\right)^{k-1}\left(1-2\sqrt{2}+2\right) \\ &= \left(1+\sqrt{2}\right)^{k-1}\left(1+\sqrt{2}\right)^2 + \left(1-\sqrt{2}\right)^{k-1}\left(1-\sqrt{2}\right)^2 \\ &= \left(1+\sqrt{2}\right)^{k+1} + \left(1-\sqrt{2}\right)^{k+1} \end{aligned}$$

Hence the formula is true for $n = k+1$.

Therefore, by the principle of mathematical induction, the formula is true for all integers $n \geq 0$.

(c) (i)

$$\begin{aligned} (\cos\theta + i\sin\theta)^5 &= \cos^5\theta + 5\cos^4\theta(i\sin\theta) \\ &\quad + 10\cos^3\theta(i\sin\theta)^2 \\ &\quad + 10\cos^2\theta(i\sin\theta)^3 \\ &\quad + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5 \\ &= \cos^5\theta + 5i\cos^4\theta\sin\theta \\ &\quad - 10\cos^3\theta\sin^2\theta \\ &\quad - 10i\cos^2\theta\sin^3\theta \\ &\quad + 5\cos\theta\sin^4\theta + i\sin^5\theta \end{aligned}$$

(ii) By de Moivre's theorem,
$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$.
equating imaginary parts,

$$\begin{aligned} \sin 5\theta &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \\ &= 5\left(1-\sin^2\theta\right)^2\sin\theta - 10\left(1-\sin^2\theta\right)\sin^3\theta + \sin^5\theta \\ &= 5\sin\theta\left(1-2\sin^2\theta+\sin^4\theta\right) - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\ &= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\ &= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta \end{aligned}$$

(iii) Substitute $\theta = \frac{\pi}{10}$ into the identity proven in part (ii):
$16\sin^5\frac{\pi}{10} - 20\sin^3\frac{\pi}{10} + 5\sin\frac{\pi}{10} = \sin\left(5\times\frac{\pi}{10}\right)$
That is,
$16\sin^5\frac{\pi}{10} - 20\sin^3\frac{\pi}{10} + 5\sin\frac{\pi}{10} = 1$
$16\sin^5\frac{\pi}{10} - 20\sin^3\frac{\pi}{10} + 5\sin\frac{\pi}{10} - 1 = 0$
Therefore, $x = \sin\frac{\pi}{10}$ is one of the solutions to $16x^5 - 20x^3 + 5x - 1 = 0$.

(iv) By long division or by inspection,
$16x^5 - 20x^3 + 5x - 1$
$= (x-1)\left(16x^4 + 16x^3 - 4x^2 - 4x + 1\right)$
That is,
$p(x) = 16x^4 + 16x^3 - 4x^2 - 4x + 1$.

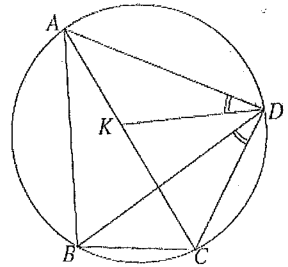
(v) $16x^4 + 16x^3 - 4x^2 - 4x + 1 = \left(4x^2 + ax - 1\right)^2$
$= 16x^4 + 8ax^3 + \ldots$
Equating the coefficients of x^3 gives $a = 2$.

(vi) From part (iii) $x = \sin\frac{\pi}{10}$ is one of the solutions to $16x^5 - 20x^3 + 5x - 1 = 0$
From parts (iv) and (v),
$16x^5 - 20x^3 + 5x - 1 = 0$
$(x-1)\left(4x^2 + 2x - 1\right)^2 = 0$
Therefore $x = 1$ or $x = \frac{-2 \pm \sqrt{20}}{8}$.

Thus $\sin\frac{\pi}{10} = \frac{-2 \pm \sqrt{20}}{8}$, since $\sin\frac{\pi}{10} > 0$ and $\sin\frac{\pi}{10} \neq 1$.
That is $\sin\frac{\pi}{10} = \frac{-1+\sqrt{5}}{4}$.

Question 7

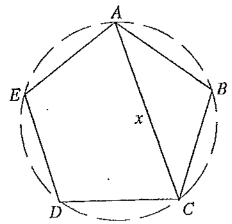
(a) (i)



In Δ's ADB, KDC
$\angle ADB = \angle KDC$
(given angle + common angle)
$\angle ABD = \angle KCD$ (standing on arc AD)
$\therefore \Delta ABD\ |||\ \Delta KCD$ (equi-angular)
$\frac{AB}{KC} = \frac{AD}{KD} = \frac{BD}{CD}$ (sides in proportion)

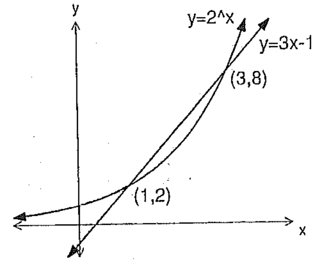
(ii) Since $\Delta ADK\ |||\ \Delta BDC$
$\frac{AD}{BD} = \frac{AK}{BC} = \frac{DK}{DC}$
$\therefore AD \times BC = BD \times AK$
from (i) $AB \times CD = BD \times KC$
adding we get:
$AB \times CD + AD \times BC = BD \times KC + BD \times AK$
$AB \times CD + AD \times BC = BD \times (KC + AK)$
$\therefore BD \times AC = AD \times BC + AB \times DC$

(iii)



$AB = BC = CD = 1$
$AC = AD = BD = x$
$BD \times AC = AD \times BC + AB \times DC$
$x \times x = x \times 1 + 1 \times 1$
$x^2 = x + 1$
$x^2 - x - 1 = 0$
$x = \frac{1 \pm \sqrt{5}}{2}$
$x = \frac{1+\sqrt{5}}{2}$ since $x > 0$

(b)

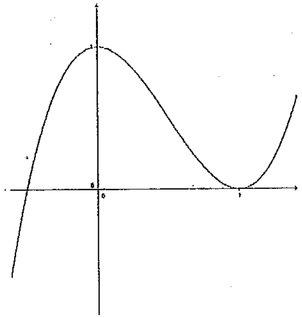


By inspection
$2^x \geq 3x - 1$ for $x \geq 3$

(c) (i) $P(x) = (n-1)x^n - nx^{n-1} + 1$
$P'(x) = n(n-1)x^{n-1} - n(n-1)x^{n-2}$
$= n(n-1)x^{n-2}[x-1]$
$= 0$ when $x = 0, 1$
∴ There are two stationary points at (0,1) and (1,0)

(ii) $P(1) = (n-1)1^n - n.1^{n-1} + 1$
$= 0$
$P'(1) = n(n-1)1^{n-1} - n(n-1)1^{n-2}$
$= 0$
$\therefore x = 1$ is a double zero.

(iii) as $x \to \infty$, $P(x) \to \infty$
as $x \to -\infty$, $P(x) \to -\infty$



There is a double zero at $x = 1$ and only two turning points. There is another turning point at (0, 1). Thus there must be a zero in the domain $-\infty < x < 0$.

(iv) $P(-1) = (n-1)(-1)^n - n(-1)^{n-1} + 1$

$= (n-1)(-1) - n(1) + 1$ [n is odd]

$= -n + 1 - n + 1$

$= -2n + 2$

< 0 [$n \geq 3$]

$$P\left(-\frac{1}{2}\right) = (n-1)\left(-\frac{1}{2}\right)^n - n\left(-\frac{1}{2}\right)^{n-1} + 1$$

$$= (n-1)\frac{-1}{2^n} - n\frac{1}{2^{n-1}} + 1 \quad [n \text{ is odd}]$$

$$= \frac{1}{2^n}(-n + 1 - 2n) + 1$$

$$= \frac{-3n+1}{2^n} + 1$$

$$= \frac{2^n - 3n + 1}{2^n}$$

From (b)

$2^n \geq 3n - 1$ for $n \geq 3$

$2^n - 3n + 1 \geq 0$

$$\frac{2^n - 3n + 1}{2^n} \geq 0$$

$$P\left(-\frac{1}{2}\right) \geq 0, \quad n \geq 3$$

$P(x)$ is continuous and the change in sign indicates that the curve must cut the x-axis.

$$\therefore -1 < \alpha \leq -\frac{1}{2}$$

(v) $P(x)$ is of degree 5, so 5 zeros. Three of them are 1, 1 and α. Since the coefficients are real, the remaining 2 zeros are complex conjugates, say β and $\overline{\beta}$. The product of the roots is

$$1.1.\alpha.\beta.\overline{\beta} = -\frac{1}{4}$$

$$\alpha.|\beta|^2 = -\frac{1}{4}$$

But $-1 < \alpha \leq -\frac{1}{2}$, from part (iv).

On inspection:

$$\frac{1}{4} < |\beta|^2 \leq \frac{1}{2}$$

$$\frac{1}{2} < |\beta| \leq \frac{1}{\sqrt{2}}$$

It follows that all zeros have a modulus less than or equal to 1.

Question 8

(a) $u = \cos^{2n-1} x$

$$\frac{du}{dx} = -(2n-1).\cos^{2n-2} x.\sin x$$

$v = \sin x$

$$\frac{dv}{dx} = \cos x$$

$$A_n = \int_0^{\frac{\pi}{2}} \cos x.\cos^{2n-1} x\,dx$$

$$= \left[\sin x.\cos^{2n-1} x\right]_0^{\frac{\pi}{2}} - (2n-1)\int_0^{\frac{\pi}{2}} \sin x.\cos^{2n-2} x.-\sin x\,dx$$

$$= 0 + (2n-1)\int_0^{\frac{\pi}{2}} \sin^2 x.\cos^{2n-2} x\,dx$$

$$= (2n-1)\int_0^{\frac{\pi}{2}} (1-\cos^2 x).\cos^{2n-2} x\,dx$$

$$= (2n-1)\left[\int_0^{\frac{\pi}{2}} \cos^{2(n-1)} x\,dx - \int_0^{\frac{\pi}{2}} \cos^{2n} x\,d\right.$$

$$= (2n-1)A_{n-1} - (2n-1)A_n$$

$$2nA_n = (2n-1)A_{n-1}$$

$$nA_n = \frac{2n-1}{2}A_{n-1}$$

(b) $u = \cos^{2n} x$

$du = -2n\sin x.\cos^{2n-1} x.dx$

$v = x \Rightarrow dv = dx$

$$A_n = \left[x.\cos^{2n} x\right]_0^{\frac{\pi}{2}} + 2n\int_0^{\frac{\pi}{2}} x.\cos^{2n-1} x.\sin x.dx$$

$$= 0 + 2n\int_0^{\frac{\pi}{2}} x.\sin x.\cos^{2n-1} x.dx$$

$$= 2n\int_0^{\frac{\pi}{2}} x.\sin x.\cos^{2n-1} x.dx$$

(c) Let $u = \sin x\cos^{2n-1} x$

$$\frac{du}{dx} = \cos^{2n} x - (2n-1)\cos^{2n-2} x\sin^2 x$$

$$= \cos^{2n} x - (2n-1)\cos^{2n-2} x\left(1-\cos^2 x\right)$$

$$= 2n\cos^{2n} x - (2n-1)\cos^{2n-2} x$$

Let $dv = x$

$$v = \frac{x^2}{2}$$

$$A_n = 2n\int_0^{\frac{\pi}{2}} x\sin x\cos^{2n-1} x.dx$$

$$= 2n\left\{\left[\frac{x^2}{2}.\sin x\cos^{2n-1} x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{x^2}{2}.\left(2n\cos^{2n} x - (2n-1)\cos^{2n-2} x\right).dx\right\}$$

$$= 2n\left\{0 - n\int_0^{\frac{\pi}{2}} x^2\cos^{2n} x.dx + \frac{2n-1}{2}\int_0^{\frac{\pi}{2}} x^2\cos^{2n-2} x.dx\right\}$$

$$= 2n\left\{-nB_n + \frac{2n-1}{2}B_{n-1}\right\}$$

$$= -2n^2B_n + (2n-1)nB_{n-1}$$

$$\frac{A_n}{n^2} = \frac{2n-1}{n}B_{n-1} - 2B_n$$

(d) $$\frac{A_n}{n^2} = \frac{2n-1}{n}B_{n-1} - 2B_n$$

$$\frac{1}{n^2} = \frac{2n-1}{nA_n}B_{n-1} - \frac{2B_n}{A_n}$$

$$= \frac{2n-1}{\left(\frac{2n-1}{2}\right)A_{n-1}}B_{n-1} - \frac{2B_n}{A_n} \quad \text{from (a)}$$

$$\frac{1}{n^2} = \frac{2B_{n-1}}{A_{n-1}} - \frac{2B_n}{A_n}$$

$$= 2\left(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n}\right)$$

(e) Using the formula from (d)

$$\sum_{k=1}^{n}\frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2}$$

$$= 2\left(\frac{B_0}{A_0} - \frac{B_1}{A_1}\right) + 2\left(\frac{B_1}{A_1} - \frac{B_2}{A_2}\right) + \ldots + 2\left(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n}\right)$$

$$= 2\left(\frac{B_0}{A_0} - \frac{B_n}{A_n}\right)$$

$$B_0 = \int_0^{\frac{\pi}{2}} x^2dx = \left[\frac{x^3}{3}\right]_0^{\frac{\pi}{2}} = \frac{\pi^3}{24}$$

$$A_0 = \int_0^{\frac{\pi}{2}} dx = \left[x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$\sum_{k=1}^{n}\frac{1}{k^2} = 2\left(\frac{B_0}{A_0} - \frac{B_n}{A_n}\right)$$

$$= 2\left(\frac{\frac{\pi^3}{24}}{\frac{\pi}{2}} - \frac{B_n}{A_n}\right)$$

$$= \frac{\pi^2}{6} - 2\frac{B_n}{A_n}$$

(f) Given $\sin x \geq \frac{2}{\pi}x$ for $0 \leq x \leq \frac{\pi}{2}$

$$\sin^2 x \geq \frac{4x^2}{\pi^2}$$

$$B_n = \int_0^{\frac{\pi}{2}} x^2\cos^{2n} x\,dx$$

$$= \int_0^{\frac{\pi}{2}} x^2(1-\sin^2 x)^n\,dx$$

$$\leq \int_0^{\frac{\pi}{2}} x^2\left(1-\frac{4x^2}{\pi^2}\right)^n dx$$

(g)

$u = x \quad dv = x\left(1-\frac{4x^2}{\pi^2}\right)^n dx$

$du = dx \quad v = \frac{-\pi^2}{8(n+1)}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1}$

$$\int_0^{\frac{\pi}{2}} x^2\left(1-\frac{4x^2}{\pi^2}\right)^n dx = \int_0^{\frac{\pi}{2}} x.x\left(1-\frac{4x^2}{\pi^2}\right)^n dx$$

$$= \left[\frac{-\pi^2 x}{8(n+1)}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1}\right]_0^{\frac{\pi}{2}}$$

$$+\frac{\pi^2}{8(n+1)}\int_0^{\frac{\pi}{2}}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1} dx$$

$$= 0+\frac{\pi^2}{8(n+1)}\int_0^{\frac{\pi}{2}}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1} dx$$

$$= \frac{\pi^2}{8(n+1)}\int_0^{\frac{\pi}{2}}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1} dx$$

(h)

Let $x = \frac{\pi}{2}\sin t, \quad dx = \frac{\pi}{2}\cos t.dt$

When $x = 0, \quad t = 0$

When $x = \frac{\pi}{2}, \quad t = \frac{\pi}{2}$

$$B_n \le \frac{\pi^2}{8(n+1)}\int_0^{\frac{\pi}{2}}\left(1-\sin^2 t\right)^{n+1}.\frac{\pi}{2}\cos t\, dt$$

$$= \frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\left(1-\sin^2 t\right)^{n+1}.\cos t\, dt$$

$$= \frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\left(\cos^2 t\right)^{n+1}.\cos t\, dt$$

$$= \frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\cos^{2n+3} t\, dt$$

$$\le \frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\cos^{2n} t\, dt \quad \text{as } \cos t \le 1$$

$$= \frac{\pi^3}{16(n+1)} A_n$$

$$\therefore B_n \le \frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\cos^{2n+3} t\, dt \le \frac{\pi^3}{16(n+1)} A_n$$

(i)

From (e)

$$\sum_{k=1}^{n}\frac{1}{k^2} = \frac{\pi^2}{6} - 2\frac{B_n}{A_n}$$

$$2\frac{B_n}{A_n} = \frac{\pi^2}{6} - \sum_{k=1}^{n}\frac{1}{k^2}$$

From (h)

$$B_n \le \frac{\pi^3}{16(n+1)} A_n$$

$$2B_n \le \frac{\pi^3}{8(n+1)} A_n$$

$$\frac{2B_n}{A_n} \le \frac{\pi^3}{8(n+1)} \quad \text{since } A_n > 0$$

Combining $\frac{\pi^2}{6} - \sum_{k=1}^{n}\frac{1}{k^2} \le \frac{\pi^3}{8(n+1)}$

$$\frac{\pi^2}{6} - \frac{\pi^3}{8(n+1)} \le \sum_{k=1}^{n}\frac{1}{k^2}$$

Also from (e)

$$\sum_{k=1}^{n}\frac{1}{k^2} = \frac{\pi^2}{6} - 2\frac{B_n}{A_n} \quad \text{and} \quad \frac{B_n}{A_n} > 0$$

$$\therefore \sum_{k=1}^{n}\frac{1}{k^2} < \frac{\pi^2}{6}$$

$$\therefore \frac{\pi^2}{6} - \frac{\pi^3}{8(n+1)} \le \sum_{k=1}^{n}\frac{1}{k^2} < \frac{\pi^2}{6}$$

(j)

As $n \to \infty, \quad \frac{\pi^3}{8(n+1)} \to 0$

From (i)

$$\frac{\pi^2}{6} - 0 \le \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2} < \frac{\pi^2}{6}$$

$$\therefore \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2} = \frac{\pi^2}{6}$$

End of Extension 2 solutions
