BOARD OF STUDIES
NEW SOUTH WALES

2013
HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I Pages 2–7

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 8–15

60 marks

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 The polynomial $P(x) = x^3 - 4x^2 - 6x + k$ has a factor $x - 2$.

What is the value of k?

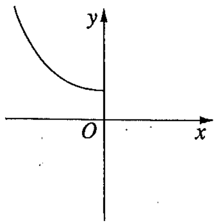
(A) 2

(B) 12

(C) 20

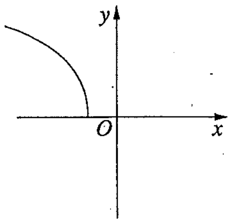
(D) 36

2 The diagram shows the graph $y = f(x)$.

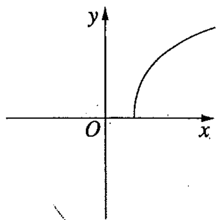


Which diagram shows the graph $y = f^{-1}(x)$?

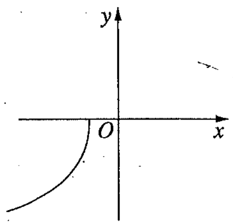
(A)



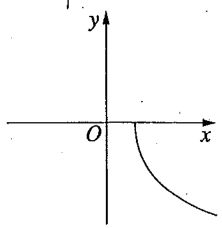
(B)



(C)

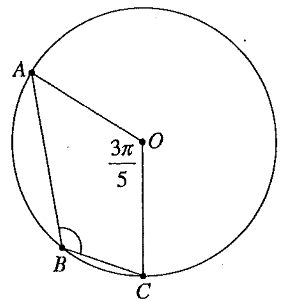


(D)



3 The points A, B and C lie on a circle with centre O, as shown in the diagram.

The size of $\angle AOC$ is $\frac{3\pi}{5}$ radians.



NOT TO SCALE

What is the size of $\angle ABC$ in radians?

(A) $\frac{3\pi}{10}$

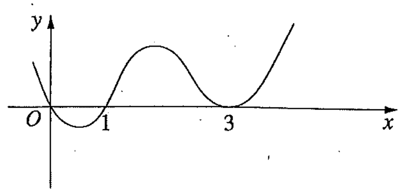
(B) $\frac{2\pi}{5}$

(C) $\frac{7\pi}{10}$

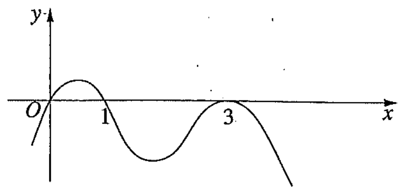
(D) $\frac{4\pi}{5}$

4 Which diagram best represents the graph $y = x\,(1-x)^3\,(3-x)^2$?

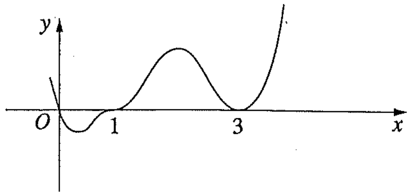
(A)



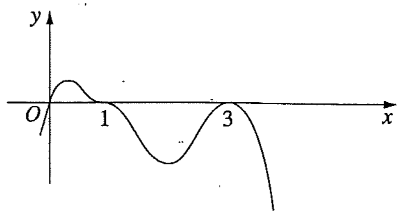
(B)



(C)



(D)



5 Which integral is obtained when the substitution $u = 1 + 2x$ is applied to $\int x\sqrt{1+2x}\,dx$?

(A) $\frac{1}{4}\int (u-1)\sqrt{u}\,du$

(B) $\frac{1}{2}\int (u-1)\sqrt{u}\,du$

(C) $\int (u-1)\sqrt{u}\,du$

(D) $2\int (u-1)\sqrt{u}\,du$

6 Let $|a| \le 1$. What is the general solution of $\sin 2x = a$?

(A) $x = n\pi + (-1)^n \frac{\sin^{-1} a}{2}$, n is an integer

(B) $x = \frac{n\pi + (-1)^n \sin^{-1} a}{2}$, n is an integer

(C) $x = 2n\pi \pm \frac{\sin^{-1} a}{2}$, n is an integer

(D) $x = \frac{2n\pi \pm \sin^{-1} a}{2}$, n is an integer

7 A family of eight is seated randomly around a circular table.

What is the probability that the two youngest members of the family sit together?

(A) $\dfrac{6!2!}{7!}$

(B) $\dfrac{6!}{7!2!}$

(C) $\dfrac{6!2!}{8!}$

(D) $\dfrac{6!}{8!2!}$

8 The angle θ satisfies $\sin\theta = \dfrac{5}{13}$ and $\dfrac{\pi}{2} < \theta < \pi$.

What is the value of $\sin 2\theta$?

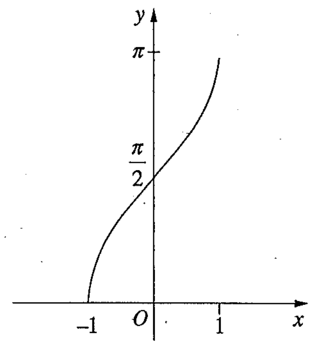
(A) $\dfrac{10}{13}$

(B) $-\dfrac{10}{13}$

(C) $\dfrac{120}{169}$

(D) $-\dfrac{120}{169}$

9 The diagram shows the graph of a function.



Which function does the graph represent?

(A) $y = \cos^{-1} x$

(B) $y = \dfrac{\pi}{2} + \sin^{-1} x$

(C) $y = -\cos^{-1} x$

(D) $y = -\dfrac{\pi}{2} - \sin^{-1} x$

10 Which inequality has the same solution as $|x+2| + |x-3| = 5$?

(A) $\dfrac{5}{3-x} \geq 1$

(B) $\dfrac{1}{x-3} - \dfrac{1}{x+2} \leq 0$

(C) $x^2 - x - 6 \leq 0$

(D) $|2x-1| \geq 5$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) The polynomial equation $2x^3 - 3x^2 - 11x + 7 = 0$ has roots α, β and γ. **1**

Find $\alpha\beta\gamma$.

(b) Find $\displaystyle\int \frac{1}{\sqrt{49 - 4x^2}}\,dx$. **2**

(c) An examination has 10 multiple-choice questions, each with 4 options. In each question, only one option is correct. For each question a student chooses one option at random. **2**

Write an expression for the probability that the student chooses the correct option for exactly 7 questions.

(d) Consider the function $f(x) = \dfrac{x}{4 - x^2}$.

(i) Show that $f'(x) > 0$ for all x in the domain of $f(x)$. **2**

(ii) Sketch the graph $y = f(x)$, showing all asymptotes. **2**

Question 11 continues on page 9

(e) Find $\displaystyle\lim_{x \to 0} \frac{\sin\frac{x}{2}}{3x}$. **1**

(f) Use the substitution $u = e^{3x}$ to evaluate $\displaystyle\int_0^{\frac{1}{3}} \frac{e^{3x}}{e^{6x} + 1}\,dx$. **3**

(g) Differentiate $x^2 \sin^{-1} 5x$. **2**

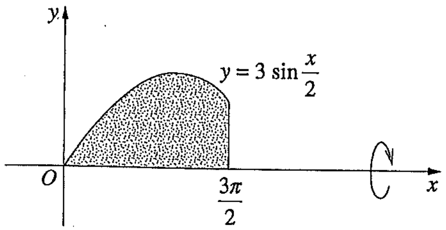
End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Write $\sqrt{3}\cos x - \sin x$ in the form $2\cos(x+\alpha)$, where $0 < \alpha < \frac{\pi}{2}$. **1**

(ii) Hence, or otherwise, solve $\sqrt{3}\cos x = 1 + \sin x$, where $0 < x < 2\pi$. **2**

(b) The region bounded by the graph $y = 3\sin\frac{x}{2}$ and the x-axis between $x = 0$ and $x = \frac{3\pi}{2}$ is rotated about the x-axis to form a solid. **3**



Find the exact volume of the solid.

(c) A cup of coffee with an initial temperature of 80°C is placed in a room with a constant temperature of 22°C. **3**

The temperature, T°C, of the coffee after t minutes is given by

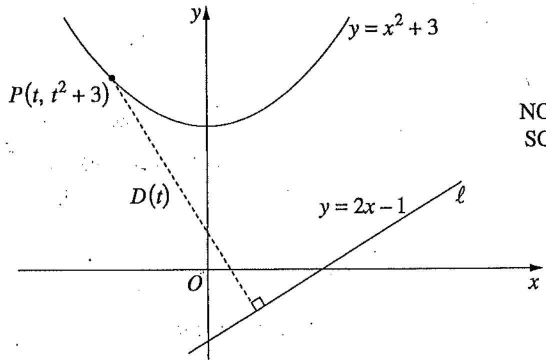
$$T = A + Be^{-kt},$$

where A, B and k are positive constants. The temperature of the coffee drops to 60°C after 10 minutes.

How long does it take for the temperature of the coffee to drop to 40°C? Give your answer to the nearest minute.

Question 12 continues on page 11

(d) The point $P(t, t^2+3)$ lies on the curve $y = x^2 + 3$. The line ℓ has equation $y = 2x - 1$. The perpendicular distance from P to the line ℓ is $D(t)$.



(i) Show that $D(t) = \dfrac{t^2 - 2t + 4}{\sqrt{5}}$. **2**

(ii) Find the value of t when P is closest to ℓ. **1**

(iii) Show that, when P is closest to ℓ, the tangent to the curve at P is parallel to ℓ. **1**

(e) A particle moves along a straight line. The displacement of the particle from the origin is x, and its velocity is v. The particle is moving so that $v^2 + 9x^2 = k$, where k is a constant. **2**

Show that the particle moves in simple harmonic motion with period $\frac{2\pi}{3}$.

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A spherical raindrop of radius r metres loses water through evaporation at a rate that depends on its surface area. The rate of change of the volume V of the raindrop is given by

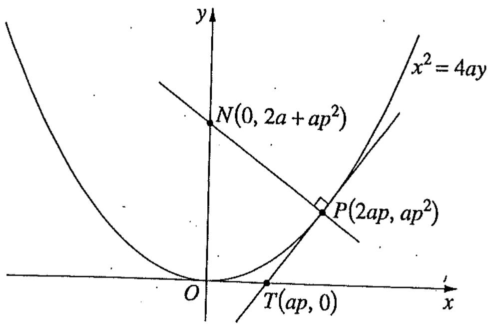
$$\frac{dV}{dt} = -10^{-4}A,$$

where t is time in seconds and A is the surface area of the raindrop. The surface area and the volume of the raindrop are given by $A = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$ respectively.

(i) Show that $\frac{dr}{dt}$ is constant. **1**

(ii) How long does it take for a raindrop of volume 10^{-6} m^3 to completely evaporate? **2**

(b) The point $P(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$. The tangent to the parabola at P meets the x-axis at $T(ap, 0)$. The normal to the tangent at P meets the y-axis at $N(0, 2a + ap^2)$.



The point G divides NT externally in the ratio $2:1$.

(i) Show that the coordinates of G are $(2ap, -2a - ap^2)$. **2**

(ii) Show that G lies on a parabola with the same directrix and focal length as the original parabola. **2**

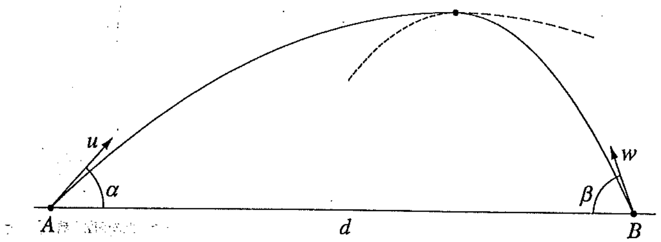
Question 13 continues on page 13

Question 13 (continued)

(c) Points A and B are located d metres apart on a horizontal plane. A projectile is fired from A towards B with initial velocity u m s^{-1} at angle α to the horizontal.

At the same time, another projectile is fired from B towards A with initial velocity w m s^{-1} at angle β to the horizontal, as shown on the diagram.

The projectiles collide when they both reach their maximum height.



The equations of motion of a projectile fired from the origin with initial velocity V m s^{-1} at angle θ to the horizontal are

$$x = Vt\cos\theta \quad \text{and} \quad y = Vt\sin\theta - \frac{g}{2}t^2. \qquad \text{(Do NOT prove this.)}$$

(i) How long does the projectile fired from A take to reach its maximum height? **2**

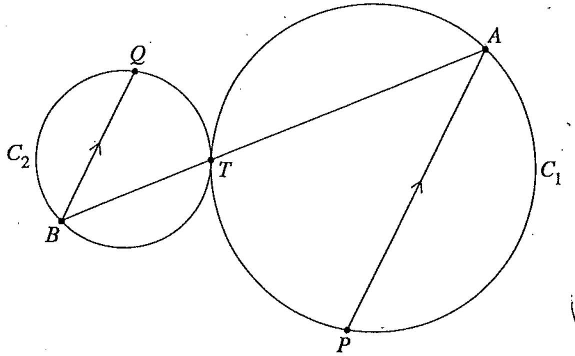
(ii) Show that $u\sin\alpha = w\sin\beta$. **1**

(iii) Show that $d = \frac{uw}{g}\sin(\alpha + \beta)$. **2**

Question 13 continues on page 14

Question 13 (continued)

(d) The circles C_1 and C_2 touch at the point T. The points A and P are on C_1. The line AT intersects C_2 at B. The point Q on C_2 is chosen so that BQ is parallel to PA. **3**



Copy or trace the diagram into your writing booklet.

Prove that the points Q, T and P are collinear.

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that for $k > 0$, $\frac{1}{(k+1)^2} - \frac{1}{k} + \frac{1}{k+1} < 0$. **1**

(ii) Use mathematical induction to prove that for all integers $n \geq 2$, **3**

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}.$$

(b) (i) Write down the coefficient of x^{2n} in the binomial expansion of $(1+x)^{4n}$. **1**

(ii) Show that $\left(1+x^2+2x\right)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^{2n-k}(x+2)^{2n-k}$. **2**

(iii) It is known that **3**

$$x^{2n-k}(x+2)^{2n-k} = \binom{2n-k}{0} 2^{2n-k} x^{2n-k} + \binom{2n-k}{1} 2^{2n-k-1} x^{2n-k+1} + \cdots + \binom{2n-k}{2n-k} 2^0 x^{4n-2k}.$$

(Do NOT prove this.)

Show that

$$\binom{4n}{2n} = \sum_{k=0}^{n} 2^{2n-2k} \binom{2n}{k}\binom{2n-k}{k}.$$

(c) The equation $e^t = \frac{1}{t}$ has an approximate solution $t_0 = 0.5$.

(i) Use one application of Newton's method to show that $t_1 = 0.56$ is another approximate solution of $e^t = \frac{1}{t}$. **2**

(ii) Hence, or otherwise, find an approximation to the value of r for which the graphs $y = e^{rx}$ and $y = \log_e x$ have a common tangent at their point of intersection. **3**

End of paper

2013 Higher School Certificate Solutions Mathematics Extension 1

SECTION I

Summary

1	C	4	D	7	A	9	B
2	D	5	A	8	D	10	C
3	C	6	B				

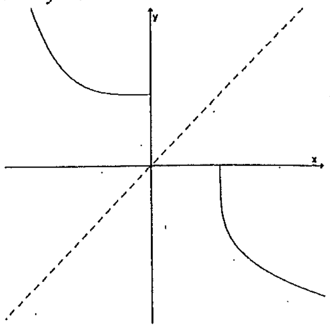
1 (C) $P(2)=0$

and $P(2)=8-16-12+k$

thus $0=-20+k$

hence $k=20$.

2 (D) The inverse would be a reflection in the line $y=x$.



3 (C) $\angle ABC=\frac{1}{2}\text{reflex}\angle AOC$

$$=\frac{1}{2}\left(2\pi-\frac{3\pi}{5}\right)$$

$$=\pi-\frac{1}{2}\times\frac{3\pi}{5}$$

$$=\frac{7\pi}{10}.$$

4 (D)
- There is a single root at $x=0$ (cuts x-axis),
- a triple root at $x=1$ (horizontal point of inflection on x-axis),
- a double root at $x=3$ (touches x-axis),
- the leading term is $-x^6$, which has a negative coefficient, when $x\to\pm\infty,\ y\to-\infty$.

The only match is (D).

5 (A) $u=1+2x \quad \therefore \quad x=\frac{1}{2}(u-1)$

$\frac{du}{dx}=2 \quad \therefore \quad dx=\frac{1}{2}du$

$$\int x\sqrt{1+2x}\,dx=\int\frac{1}{2}(u-1)\sqrt{u}\,\frac{1}{2}du$$

$$=\frac{1}{4}\int(u-1)\sqrt{u}\ du$$

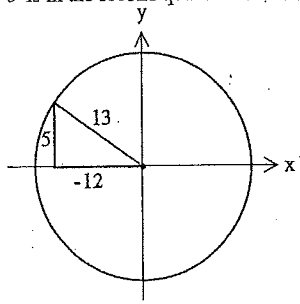
6 (B) $2x=n\pi+(-1)^n\sin^{-1}(a)$

$$x=\frac{n\pi+(-1)^n\sin^{-1}(a)}{2}.$$

7 (A) 8 elements can be arranged in 7! ways around a circle. The 2 youngest can be arranged in 2! ways. The remaining 6 members can be arranged in 6! ways.

The result is $\frac{6!2!}{7!}$.

8 (D) θ is in the second quadrant $\therefore \cos\theta<0$.



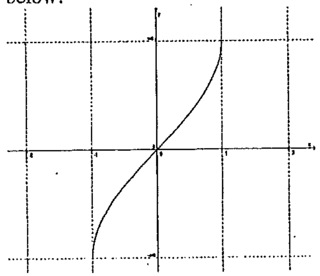
$\sin\theta=\frac{5}{13}$ and $\cos\theta=-\frac{12}{13}$

$\sin 2\theta=2\sin\theta\cos\theta$

$$=2\times\frac{5}{13}\times-\frac{12}{13}$$

$$=-\frac{120}{169}.$$

9 (B) The graph of $y=\sin^{-1}x$ is shown below:



Therefore the required function is $y=\sin^{-1}x$ translated in the positive y-direction by $\frac{\pi}{2}$ units.

Thus $y=\frac{\pi}{2}+\sin^{-1}x$.

10 (C) For the equation $|x+2|+|x-3|=5$, we need to consider three regions:

$x<-2,\quad -2\le x\le 3\quad$ and $\quad x>3$

If $x<-2$, $-(x+2)-(x-3)=5$

$$-2x=4$$

$$x=-2$$

If $-2\le x\le 3$, $(x+2)-(x-3)=5$

$$5=5$$

therefore always true for $-2\le x\le 3$

If $x>3$, $(x+2)+(x-3)=5$

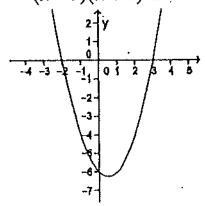
$$2x=6$$

$$x=3$$

Thus $|x+2|+|x-3|=5$ has the solution: $-2\le x\le 3$ and this is the same as

$$x^2-x-6\le 0$$

$$(x-3)(x+2)\le 0$$



that has solution $-2\le x\le 3$.

SECTION II

Question 11

(a) $a=2, b=-3, c=-11, d=7$

$$\alpha\beta\gamma=-\frac{d}{a}$$

$$=-\frac{7}{2}$$

(b) *Method 1:*

Let $u = 2x$

$$\frac{du}{dx} = 2 \qquad \therefore dx = \frac{1}{2}du$$

$$\int \frac{1}{\sqrt{49-4x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{7^2-u^2}}\,du$$

$$= \frac{1}{2}\sin^{-1}\frac{u}{7} + C$$

$$= \frac{1}{2}\sin^{-1}\frac{2x}{7} + C.$$

$$\left[\text{alternatively } -\frac{1}{2}\cos^{-1}\frac{2x}{7} + C_1\right]$$

OR

Method 2:

From the table of Standard Integrals:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{\sqrt{49-4x^2}}\,dx = \int \frac{1}{\sqrt{4\left(\frac{49}{4}-x^2\right)}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{49}{4}-x^2\right)}}\,dx$$

$$= \frac{1}{2}\sin^{-1}\frac{x}{\frac{7}{2}} + C$$

$$= \frac{1}{2}\sin^{-1}\frac{2x}{7} + C.$$

(c) Binomial distribution with 10 trials and $p = \frac{1}{4}$

$$P(7\text{ correct}) = {}^{10}C_7\left(\frac{1}{4}\right)^7\left(\frac{3}{4}\right)^3$$

$$= \frac{405}{131072}$$

$$\approx 0.0031.$$

(d) (i)

$$f(x) = \frac{x}{4-x^2}$$

$$f'(x) = \frac{1.(4-x^2) - x(-2x)}{(4-x^2)^2}$$

$$= \frac{4-x^2+2x^2}{(4-x^2)^2}$$

$$= \frac{4+x^2}{(4-x^2)^2}$$

$$> 0$$

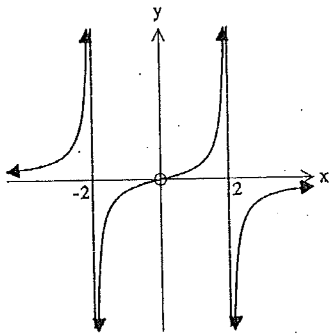
since $4+x^2 > 0$ and $(4-x^2)^2 > 0$.

(ii) Vertical asymptotes at $x = \pm 2$ and

as $x \to \infty$, $y \to 0^-$

as $x \to -\infty$, $y \to 0^+$

when $x = 0$, $y = 0$



(e)

$$\lim_{x\to\infty}\frac{\sin\frac{x}{2}}{3x} = \frac{1}{6}\lim_{x\to\infty}\frac{\sin\frac{x}{2}}{\frac{x}{2}}$$

$$= \frac{1}{6}\times 1$$

$$= \frac{1}{6}.$$

(f) For $u = e^{3x}$

$$\frac{du}{dx} = 3e^{3x} \qquad \therefore \frac{1}{3}du = e^{3x}dx$$

When $x = \frac{1}{3}$, $u = e$

$x = 0$, $u = 1$

$$\int_0^{\frac{1}{3}} \frac{e^{3x}}{e^{6x}+1}\,dx = \frac{1}{3}\int_1^e \frac{1}{u^2+1}\,du$$

$$= \frac{1}{3}\left[\tan^{-1}u\right]_1^e$$

$$= \frac{1}{3}\left[\tan^{-1}e - \tan^{-1}1\right]$$

$$= \frac{1}{3}\left(\tan^{-1}e - \frac{\pi}{4}\right).$$

(g) Let $y = \sin^{-1}5x$

$= \sin^{-1}u$ where $u = 5x$

then $\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$ and $\frac{du}{dx} = 5$

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-u^2}}\times 5$$

$$= \frac{5}{\sqrt{1-u^2}}$$

$$= \frac{5}{\sqrt{1-25x^2}}$$

$$\frac{d}{dx}\left(x^2\sin^{-1}5x\right) = 2x\sin^{-1}5x + x^2\cdot\frac{5}{\sqrt{1-25x^2}}$$

$$= 2x\sin^{-1}5x + \frac{5x^2}{\sqrt{1-25x^2}}.$$

Question 12

(a) (i) $\sqrt{3}\cos x - \sin x \equiv 2\cos(x+\alpha)$

$\equiv 2\cos x\cos\alpha - 2\sin x\cos\alpha$

Equating coefficients of $\cos x$:

$$\cos\alpha = \frac{\sqrt{3}}{2} \text{ and } \sin\alpha = \frac{1}{2}$$

$$\therefore \alpha = \frac{\pi}{6}$$

$$\therefore \sqrt{3}\cos x - \sin x \equiv 2\cos\left(x+\frac{\pi}{6}\right).$$

(ii)

$$\sqrt{3}\cos x = 1 + \sin x$$

$$\sqrt{3}\cos x - \sin x = 1$$

$$2\cos\left(x+\frac{\pi}{6}\right) = 1$$

$$\cos\left(x+\frac{\pi}{6}\right) = \frac{1}{2}$$

$$x + \frac{\pi}{6} = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$$

$$x = \frac{\pi}{6} \text{ or } \frac{3\pi}{2}.$$

(b) $V = \pi\int_a^b y^2\,dx$ where $y^2 = 9\sin^2\frac{x}{2}$

Consider: $\cos 2\theta = 1 - 2\sin^2\theta$

$$\sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta)$$

$$\sin^2\frac{x}{2} = \tfrac{1}{2}(1-\cos x)$$

$$\frac{y^2}{9} = \tfrac{1}{2}(1-\cos x)$$

$$y^2 = \tfrac{9}{2}(1-\cos x)$$

$$V = \frac{9}{2}\pi\int_0^{\frac{3\pi}{2}}(1-\cos x)\,dx$$

$$= \frac{9\pi}{2}\left[x - \sin x\right]_0^{\frac{3\pi}{2}}$$

$$= \frac{9\pi}{2}\left(\frac{3\pi}{2} - (-1) - (0-0)\right)$$

$$= \frac{9\pi}{2}\left(\frac{3\pi}{2}+1\right)\text{ units}^3.$$

(c) $T = A + Be^{-kt}$

$A = 22$ (ambient temperature is 22°C)

$\therefore T = 22 + Be^{-kt}$

When $t = 0$, $T = 80$

$80 = 22 + Be^0$

$\therefore B = 58$

When $t=10$, $T=60$

$$60=22+58e^{-10k}$$
$$38=58e^{-10k}$$
$$e^{-10k}=\frac{38}{58}$$
$$-10k=\ln\frac{19}{29}$$
$$k=-\frac{1}{10}\ln\frac{19}{29}=0.04228\ldots$$

Now set $T=40$ and solve for t.

$$40=22+58e^{-kt}$$
$$18=58e^{-kt}$$
$$e^{-kt}=\frac{9}{29}$$
$$-kt=\ln\frac{9}{29}$$
$$t=-\ln\frac{9}{29}\div k$$
$$t=27.67062306$$
$$=28 \text{ minutes} \quad (\text{nearest minute}).$$

(d) (i) $D(t)=\dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

where the line is $2x-y-1=0$

and the point is (t,t^2+3)

$$D(t)=\frac{|2t-1(t^2+3)-1|}{\sqrt{2^2+1^2}}$$
$$=\frac{|-t^2+2t-4|}{\sqrt{5}}$$
$$=\frac{t^2-2t+4}{\sqrt{5}}$$

since $t^2-2t+4=(t-1)^2+3$
$>0.$

(ii) *Method 1:*

$$\frac{dD}{dt}=\frac{2t-2}{\sqrt{5}}$$
$$=0 \quad \text{when } t=1$$
$$\frac{d^2D}{dt^2}=\frac{2}{\sqrt{5}}$$
$$>0$$

$\therefore$ it is a minimum when $t=1$.

OR

Method 2:

$D(t)$ is minimised

when t^2-2t+4 is minimised.

This is a concave up quadratic

$\therefore$ minimum occurs when

$$t=\frac{-b}{2a}$$
$$=\frac{-(-2)}{2(1)}$$
$$=1$$

$\therefore$ it is a minimum when $t=1$.

(iii) When $t=1$, $P=(1,4)$.

Gradient of tangent:

$$m=\frac{dy}{dx}$$
$$=2x$$
$$=2(1)$$
$$=2$$

Gradient of line $y=2x-1$ is 2.

$\therefore$ the tangent is parallel to ℓ.

(c) $v^2+9x^2=k$

$$v^2=k-9x^2$$

Using $\ddot{x}=\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$, we have

$$\ddot{x}=\frac{1}{2}\frac{d}{dx}(k-9x^2)$$
$$=\frac{1}{2}(-18x)$$
$$\ddot{x}=-9x$$

This is of the form $\ddot{x}=-n^2x$.

$\therefore$ this is SHM with $n=3$.

$$\text{Period}=\frac{2\pi}{n}$$
$$=\frac{2\pi}{3}.$$

Question 13

(a) (i) $V=\dfrac{4}{3}\pi r^3$

$$\frac{dV}{dr}=4\pi r^2$$
$$=A$$
$$\frac{dr}{dt}=\frac{dr}{dV}\times\frac{dV}{dt} \quad \text{but } \frac{dV}{dt}=-10^{-4}A$$
$$=\frac{1}{A}\times(-10^{-4}A)$$
$$=-10^{-4}$$

Thus $\dfrac{dr}{dt}$ is constant.

(ii) $\dfrac{dr}{dt}=-10^{-4}$

$$r=-10^{-4}t+C \quad ①$$

when $t=0$ $V=10^{-6}$

$$V=\frac{4}{3}\pi r^3$$
$$10^{-6}=\frac{4}{3}\pi r^3$$
$$r^3=\frac{3\times10^{-6}}{4\pi}$$
$$r=\sqrt[3]{\frac{3\times10^{-6}}{4\pi}}$$

Substitute into ① to find C:

$$C=\sqrt[3]{\frac{3\times10^{-6}}{4\pi}}$$
$$\therefore\ r=-10^{-4}t+\sqrt[3]{\frac{3\times10^{-6}}{4\pi}}$$

When $r=0$

$$0=-10^{-4}t+\sqrt[3]{\frac{3\times10^{-6}}{4\pi}}$$
$$10^{-4}t=\sqrt[3]{\frac{3\times10^{-6}}{4\pi}}$$
$$t=\frac{\sqrt[3]{\frac{3\times10^{-6}}{4\pi}}}{10^{-4}}$$
$$=62.03504909\ldots$$
$$=62 \text{ seconds} \quad (\text{nearest second}).$$

(b) (i) For the point G:

$$x=\frac{nx_1+mx_2}{m+n},\ y=\frac{ny_1+my_2}{m+n}$$

where $m=2$, $n=-1$

$$x=\frac{-1\times0+2\times ap}{-1+2}$$
$$=\frac{2ap}{1}$$
$$x=2ap$$
$$y=\frac{-1\times(2a+ap^2)+2\times0}{-1+2}$$
$$=\frac{-2a-ap^2}{1}$$
$$y=-2a-ap^2$$

$\therefore G$ is $(2ap,-2a-ap^2)$.

(ii) $x=2ap$

$$p=\frac{x}{2a}$$
$$y=-2a-ap^2$$
$$=-2a-a\left(\frac{x}{2a}\right)^2$$
$$=-2a-a\times\frac{x^2}{4a^2}$$
$$=-2a-\frac{x^2}{4a}$$
$$4ay=-8a^2-x^2$$
$$x^2=-4ay-8a^2$$
$$x^2=-4a(y+2a)$$

By inspection:

It is an inverted parabola.

focal length $=a$

vertex is $(0,-2a)$

directrix is $y=-2a+a$

i.e. $y=-a$

Thus the directrix and focal length are the same as the original parabola.

(c) (i) The equations of motion from A are:

$x = ut\cos\alpha$

$y = ut\sin\alpha - \frac{g}{2}t^2$

Maximum height is when $\dot{y} = 0$

$y = ut\sin\alpha - \frac{g}{2}t^2$

$\dot{y} = u\sin\alpha - gt$

$0 = u\sin\alpha - gt$

$gt = u\sin\alpha$

$t = \frac{u\sin\alpha}{g}.$

(ii) The equations of motion from B are:

$x = wt\cos\beta$

$y = wt\sin\beta - \frac{g}{2}t^2$

Its maximum height would be reached when

$t = \frac{w\sin\beta}{g}.$

When both projectiles are at their maximum height:

$\frac{u\sin\alpha}{g} = \frac{w\sin\beta}{g}$

$u\sin\alpha = w\sin\beta.$

(iii) The combined distance would be:

$d = ut\cos\alpha + wt\cos\beta$

$= u\left(\frac{u\sin\alpha}{g}\right)\cos\alpha + w\left(\frac{w\sin\beta}{g}\right)\cos\beta$

$= \frac{u\cos\alpha . u\sin\alpha}{g} + \frac{w\cos\beta . w\sin\beta}{g}$

$d = \frac{u^2\sin\alpha\cos\alpha}{g} + \frac{w^2\sin\beta\cos\beta}{g}$

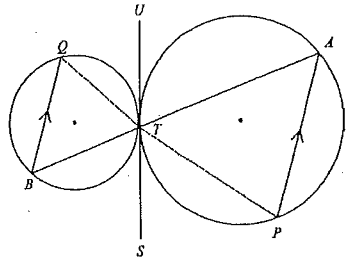
From part (ii), we have $u\sin\alpha = w\sin\beta$

$\therefore d = \frac{u\cos\alpha . w\sin\beta}{g} + \frac{w\cos\beta . u\sin\alpha}{g}$

$= \frac{uw}{g}(\cos\alpha\sin\beta + \cos\beta\sin\alpha)$

$= \frac{uw}{g}\sin(\alpha+\beta).$

(d) (i)



Draw the tangent UTS through T.
It will be a tangent to both circles.
Join TQ and TP.
Aim: To prove QTP is a straight angle.
$\angle TAP = \angle STP$ (alt. segment theorem)
$\angle QBT = \angle UTQ$ (alt. segment theorem)
$\angle TAP = \angle QBT$ (alt. $\angle$s in $\parallel$ lines)
$\therefore \angle STP = \angle UTQ$
Because $\angle STP$ and $\angle PTU$ are supplementary adjacent angles,
$\therefore \angle UTQ$ and $\angle PTU$ are supplementary adjacent angles.
$\therefore Q, T, P$ are collinear.

Question 14

(a) (i) $\frac{1}{(k+1)^2} - \frac{1}{k} + \frac{1}{k+1}$

$= \frac{k-(k+1)^2+k(k+1)}{k(k+1)^2}$

$= \frac{k-(k^2+2k+1)+k^2+k}{k(k+1)^2}$

$= \frac{-1}{k(k+1)^2}$

< 0 (since $k > 0$)

(ii) For n=2:

LHS $= \frac{1}{1^2} + \frac{1}{2^2} = 1\frac{1}{4}$

RHS $= 2 - \frac{1}{2} = 1\frac{1}{2}$

$\therefore$ True for $n = 2$, since $1\frac{1}{4} < 1\frac{1}{2}$.

Let $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{k^2} < 2 - \frac{1}{k}$ ①

be true for some integer, k.

Then for n=k+1 we need to prove that:

$\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$.

$LHS = \frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{k^2} + \frac{1}{(k+1)^2}$

$< 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$ from ①

but $\frac{1}{(k+1)^2} - \frac{1}{k} < -\frac{1}{k+1}$ from part (i)

So $2 - \frac{1}{k} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$

$< RHS$

$\therefore$ by mathematical induction

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n}$

is true for all integers $n \geq 2$.

(b) (i) $(1+x)^{4n} = \sum_{r=0}^{4n}\binom{4n}{r}1^{4n-r}x^r$

$= \sum_{r=0}^{4n}\binom{4n}{r}x^r$

Hence coefficient of x^{2n} is $\binom{4n}{2n}$.

(ii) $(1+x^2+2x)^{2n} = \left(1+\left(x^2+2x\right)\right)^{2n}$

$= \sum_{k=0}^{2n}\binom{2n}{k}1^k\left(x^2+2x\right)^{2n-k}$

$= \sum_{k=0}^{2n}\binom{2n}{k}\left(x(x+2)\right)^{2n-k}$

$= \sum_{k=0}^{2n}\binom{2n}{k}x^{2n-k}(x+2)^{2n-k}$

(iii) $(1+x^2+2x)^{2n} = \left((1+x)^2\right)^{2n}$

$= (1+x)^{4n}$

Thus, from part (ii):

$(1+x)^{4n} = \sum_{k=0}^{2n}\binom{2n}{k}x^{2n-k}(x+2)^{2n-k}$ ①

From part (i), $\binom{4n}{2n}$ is the coefficient of x^{2n} in the expansion of

$(1+x)^{4n} = \sum_{k=0}^{2n}\binom{2n}{k}x^{2n-k}(x+2)^{2n-k}$.

Also, it is given that:

$x^{2n-k}(x+2)^{2n-k} = \sum_{r=0}^{2n-k}\binom{2n-k}{r}2^{2n-k-r}x^{2n-k+r}$

and the term in x^{2n} occurs when $r = k$, with coefficient $\binom{2n-k}{k}2^{2n-2k}$

Therefore, the coefficient of x^{2n} in the expansion of:

$\sum_{k=0}^{2n}\binom{2n}{k}x^{2n-k}(x+2)^{2n-k}$ is:

$\sum_{k=0}^{2n}\binom{2n}{k}\binom{2n-k}{k}2^{2n-2k}$

And from ① above, this equals $\binom{4n}{2n}$.

$\therefore \binom{4n}{2n} = \sum_{k=0}^{n}\binom{2n}{k}\binom{2n-k}{k}2^{2n-2k}$

$= \sum_{k=0}^{n}2^{2n-2k}\binom{2n}{k}\binom{2n-k}{k}.$

(c) (i) $e^t = \frac{1}{t}$

Let $f(t) = e^t - \frac{1}{t}$

$= e^t - t^{-1}$

$f'(t) = e^t + t^{-2}$

$= e^t - \frac{1}{t^2}$

$t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}$

$t_1 = 0.5 - \frac{f(0.5)}{f'(0.5)}$

$= 0.5 - \frac{e^{0.5}-2}{e^{0.5}+4}$

$= 0.5621873\ldots$

$= 0.56$ (2 dp).

(ii) Let $f(x) = e^{rx}$ and $g(x) = \ln x$

$f'(x) = re^{rx}$ $\quad g'(x) = \frac{1}{x}$

Since the tangents are equal:

$re^{rx} = \frac{1}{x}$

$e^{rx} = \frac{1}{rx}$ let $t = rx$

$e^{t} = \frac{1}{t}$

From part (i), this has a solution when $t \approx 0.56$.

At the intersection:

$e^{rx} = \ln x$ but $t = rx \approx 0.56$

$e^{0.56} = \ln x$

$x = e^{e^{0.56}}$

$= 5.75847395\ldots$

$rx = 0.56$

$r = \frac{0.56}{x}$

$= 0.097247987\ldots$

≈ 0.1 (1 dec. pl.).

End of Mathematics Extension 1 solutions