BOSTES | BOARD OF STUDIES TEACHING & EDUCATIONAL STANDARDS NSW

2014 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I Pages 2–5

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 6–13

60 marks

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

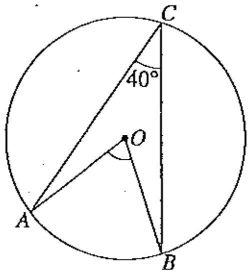
Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

---

1 The points A, B and C lie on a circle with centre O, as shown in the diagram. The size of $\angle ACB$ is 40°.



NOT TO SCALE

What is the size of $\angle AOB$?

(A) 20°

(B) 40°

(C) 70°

(D) 80°

2 Which expression is equal to $\cos x - \sin x$?

(A) $\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)$

(B) $\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$

(C) $2\cos\left(x+\frac{\pi}{4}\right)$

(D) $2\cos\left(x-\frac{\pi}{4}\right)$

3 What is the constant term in the binomial expansion of $\left(2x-\frac{5}{x^3}\right)^{12}$?

(A) $\binom{12}{3}2^9 5^3$

(B) $\binom{12}{9}2^3 5^9$

(C) $-\binom{12}{3}2^9 5^3$

(D) $-\binom{12}{9}2^3 5^9$

4 The acute angle between the lines $2x+2y=5$ and $y=3x+1$ is θ.

What is the value of $\tan\theta$?

(A) $\frac{1}{7}$

(B) $\frac{1}{2}$

(C) 1

(D) 2

5 Which group of three numbers could be the roots of the polynomial equation $x^3+ax^2-41x+42=0$?

(A) 2, 3, 7

(B) 1, –6, 7

(C) –1, –2, 21

(D) –1, –3, –14

6 What is the derivative of $3\sin^{-1}\frac{x}{2}$?

(A) $\frac{6}{\sqrt{4-x^2}}$

(B) $\frac{3}{\sqrt{4-x^2}}$

(C) $\frac{3}{2\sqrt{4-x^2}}$

(D) $\frac{3}{4\sqrt{4-x^2}}$

7 A particle is moving in simple harmonic motion with period 6 and amplitude 5.

Which is a possible expression for the velocity, v, of the particle?

(A) $v=\frac{5\pi}{3}\cos\left(\frac{\pi}{3}t\right)$

(B) $v=5\cos\left(\frac{\pi}{3}t\right)$

(C) $v=\frac{5\pi}{6}\cos\left(\frac{\pi}{6}t\right)$

(D) $v=5\cos\left(\frac{\pi}{6}t\right)$

8 In how many ways can 6 people from a group of 15 people be chosen and then arranged in a circle?

(A) $\frac{14!}{8!}$

(B) $\frac{14!}{8!6}$

(C) $\frac{15!}{9!}$

(D) $\frac{15!}{9!6}$

9 The remainder when the polynomial $P(x)=x^4-8x^3-7x^2+3$ is divided by x^2+x is $ax+3$.

What is the value of a?

(A) –14

(B) –11

(C) –2

(D) 5

10 Which equation describes the locus of points (x, y) which are equidistant from the distinct points $(a+b,\ b-a)$ and $(a-b,\ b+a)$?

(A) $bx+ay=0$

(B) $bx+ay=2ab$

(C) $bx-ay=0$

(D) $bx-ay=2ab$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Solve $\left(x+\frac{2}{x}\right)^2-6\left(x+\frac{2}{x}\right)+9=0$. **3**

(b) The probability that it rains on any particular day during the 30 days of November is 0.1. **2**

Write an expression for the probability that it rains on fewer than 3 days in November.

(c) Sketch the graph $y=6\tan^{-1}x$, clearly indicating the range. **2**

(d) Evaluate $\int_2^5 \frac{x}{\sqrt{x-1}}\,dx$ using the substitution $x=u^2+1$. **3**

(e) Solve $\frac{x^2+5}{x}>6$. **3**

(f) Differentiate $\frac{e^x \ln x}{x}$. **2**

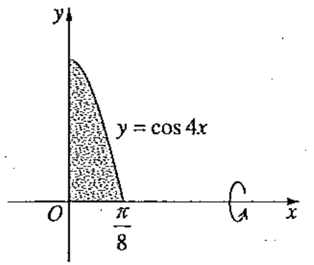
Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) A particle is moving in simple harmonic motion about the origin, with displacement x metres. The displacement is given by $x=2\sin 3t$, where t is time in seconds. The motion starts when $t=0$.

(i) What is the total distance travelled by the particle when it first returns to the origin? **1**

(ii) What is the acceleration of the particle when it is first at rest? **2**

(b) The region bounded by $y=\cos 4x$ and the x-axis, between $x=0$ and $x=\frac{\pi}{8}$, is rotated about the x-axis to form a solid. **3**



NOT TO SCALE

Find the volume of the solid.

(c) A particle moves along a straight line with displacement x m and velocity v m s^{-1}. The acceleration of the particle is given by **3**

$$\ddot{x}=2-e^{-\frac{x}{2}}.$$

Given that $v=4$ when $x=0$, express v^2 in terms of x.

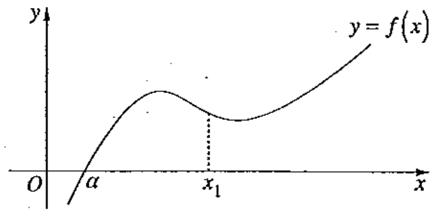
Question 12 continues on page 8

(d) Use the binomial theorem to show that **2**

$$0=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^n\binom{n}{n}.$$

(e) The diagram shows the graph of a function $f(x)$. **1**

The equation $f(x)=0$ has a root at $x=\alpha$. The value x_1, as shown in the diagram, is chosen as a first approximation of α.



A second approximation, x_2, of α is obtained by applying Newton's method once, using x_1 as the first approximation.

Using a diagram, or otherwise, explain why x_1 is a closer approximation of α than x_2.

(f) Milk taken out of a refrigerator has a temperature of 2°C. It is placed in a room of constant temperature 23°C. After t minutes the temperature, T°C, of the milk is given by **3**

$$T=A-Be^{-0.03t},$$

where A and B are positive constants.

How long does it take for the milk to reach a temperature of 10°C?

End of Question 12

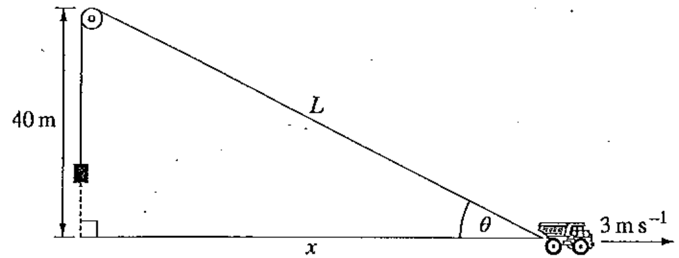
Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Use mathematical induction to prove that $2^n+(-1)^{n+1}$ is divisible by 3 for all integers $n \geq 1$. **3**

(b) One end of a rope is attached to a truck and the other end to a weight. The rope passes over a small wheel located at a vertical distance of 40 m above the point where the rope is attached to the truck.

The distance from the truck to the small wheel is L m, and the horizontal distance between them is x m. The rope makes an angle θ with the horizontal at the point where it is attached to the truck.

The truck moves to the right at a constant speed of 3 m s^{-1}, as shown in the diagram.



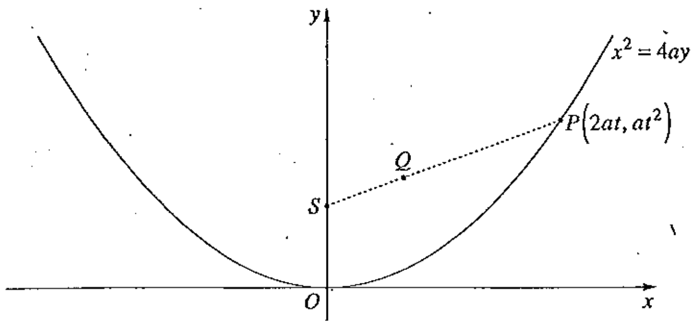
(i) Using Pythagoras' Theorem, or otherwise, show that $\dfrac{dL}{dx}=\cos\theta$. **2**

(ii) Show that $\dfrac{dL}{dt}=3\cos\theta$. **1**

Question 13 continues on page 10

(c) The point $P\left(2at, at^2\right)$ lies on the parabola $x^2 = 4ay$ with focus S.

The point Q divides the interval PS internally in the ratio $t^2 : 1$.



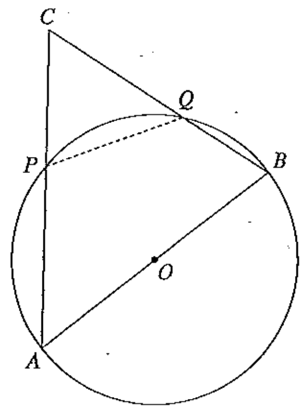
(i) Show that the coordinates of Q are $x = \dfrac{2at}{1+t^2}$ and $y = \dfrac{2at^2}{1+t^2}$. **2**

(ii) Express the slope of OQ in terms of t. **1**

(iii) Using the result from part (ii), or otherwise, show that Q lies on a fixed circle of radius a. **3**

Question 13 continues on page 11

(d) In the diagram, AB is a diameter of a circle with centre O. The point C is chosen such that $\triangle ABC$ is acute-angled. The circle intersects AC and BC at P and Q respectively.



NOT TO SCALE

Copy or trace the diagram into your writing booklet.

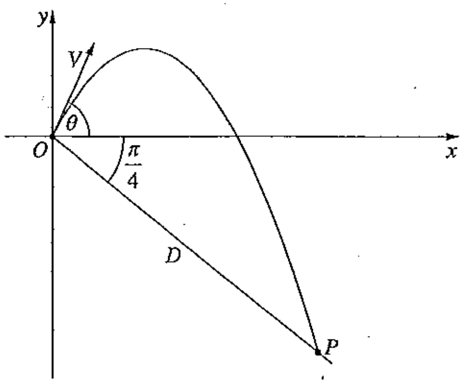
(i) Why is $\angle BAC = \angle CQP$? **1**

(ii) Show that the line OP is a tangent to the circle through P, Q and C. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) The take-off point O on a ski jump is located at the top of a downslope. The angle between the downslope and the horizontal is $\frac{\pi}{4}$. A skier takes off from O with velocity $V\ \text{m s}^{-1}$ at an angle θ to the horizontal, where $0 \le \theta < \frac{\pi}{2}$. The skier lands on the downslope at some point P, a distance D metres from O.



The flight path of the skier is given by

$$x = Vt\cos\theta, \quad y = -\frac{1}{2}gt^2 + Vt\sin\theta, \qquad \text{(Do NOT prove this.)}$$

where t is the time in seconds after take-off.

(i) Show that the cartesian equation of the flight path of the skier is given by **2**

$$y = x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta.$$

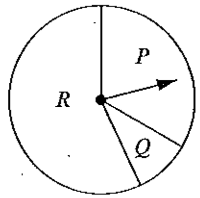
(ii) Show that $D = 2\sqrt{2}\frac{V^2}{g}\cos\theta(\cos\theta + \sin\theta)$. **3**

(iii) Show that $\frac{dD}{d\theta} = 2\sqrt{2}\frac{V^2}{g}(\cos 2\theta - \sin 2\theta)$. **2**

(iv) Show that D has a maximum value and find the value of θ for which this occurs. **3**

Question 14 continues on page 13

(b) Two players A and B play a game that consists of taking turns until a winner is determined. Each turn consists of spinning the arrow on a spinner once. The spinner has three sectors P, Q and R. The probabilities that the arrow stops in sectors P, Q and R are p, q and r respectively.



The rules of the game are as follows:

- If the arrow stops in sector P, then the player having the turn wins.
- If the arrow stops in sector Q, then the player having the turn loses and the other player wins.
- If the arrow stops in sector R, then the other player takes a turn.

Player A takes the first turn.

(i) Show that the probability of player A winning on the first or the second turn of the game is $(1-r)(p+r)$. **2**

(ii) Show that the probability that player A eventually wins the game is **3**

$$\frac{p+r}{1+r}.$$

End of paper

2014 Higher School Certificate Solutions Mathematics Extension 1

SECTION I

Summary

1	D	3	C	5	B	7	A	9	C
2	A	4	D	6	B	8	D	10	C

SECTION I

1 (D) Angle at the centre is twice the angle at circumference standing on the same arc $\therefore \angle AOB = 80°$.

2 (A) $\cos x - \sin x \equiv R\cos(x+\alpha)$

$= R\cos x\cos\alpha - R\sin x \sin\alpha$

Equating co-efficients

$R\cos\alpha = 1$...(1)

$R\sin\alpha = 1$...(2)

(2) ÷ (1)

$\tan\alpha = 1 \Rightarrow \alpha = \frac{\pi}{4}$

$(1)^2 + (2)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2 + 1^2$

$R^2(\cos^2\alpha + \sin^2\alpha) = 2$

$R = \sqrt{2}$

$\therefore \cos x - \sin x = \sqrt{2}\cos(x + \frac{\pi}{4})$.

3 (C) For $\left(2x - \frac{5}{x^3}\right)^{12}$.

$T_{r+1} = {}^{12}C_r\,(2x)^{12-r}\left(-5x^{-3}\right)^r$

Any term is of the form

$T_{r+1} = {}^{12}C_r\,(2x)^{12-r}(-5x^{-3})^r$

$= Ax^{12-4r}$

For the constant term:

$12 - 4r = 0$

$r = 3$

The constant term is

$T_{3+1} = \binom{12}{3}(2x)^{12-3}(-5x^{-3})^3$

$T_4 = \binom{12}{3}2^9x^9(-5)^3x^{-9}$

$= -\binom{12}{3}2^9 5^3$.

4 (D) $2x + 2y = 5 \quad \Rightarrow m_1 = -1$

$y = 3x + 1 \Rightarrow m_2 = 3$

$\tan\theta = \left|\frac{3 - -1}{1 + 3\times -1}\right|$

$= 2$ (for the acute angle).

5 (B) $\alpha\beta\gamma = -42$ (1)

$\alpha\beta + \alpha\gamma + \beta\gamma = -41$ (2)

Answers (A) and (C) do not satisfy (1).
Answer (D) does not satisfy (2).

For answer (B):

$\alpha\beta\gamma = 1\times -6\times 7 = -42$

$\alpha\beta + \alpha\gamma + \beta\gamma = -6 + 7 - 42 = -41$.

6 (B) Using the standard integrals with $a = 2$:

$\frac{d}{dx}\left(3\sin^{-1}\frac{x}{2}\right) = 3\times\frac{1}{\sqrt{2^2 - x^2}}$

$= \frac{3}{\sqrt{4 - x^2}}$.

7 (A) Amplitude: $a = 5$

For a period of 6: $6 = \frac{2\pi}{n}$

$n = \frac{\pi}{3}$

$\therefore x = 5\sin\left(\frac{\pi}{3}t\right)$

$v = \frac{dx}{dt} = \frac{5\pi}{3}\cos\left(\frac{\pi}{3}t\right)$.

8 (D) 6 people can be chosen from 15 in $\binom{15}{6}$ ways. Also, 6 people can be arranged in a ring in 5! ways.

Total number is:

$\binom{15}{6}5! = \frac{15!}{(15-6)!6!}5!$

$= \frac{15!}{9!6}$.

9 (C) $P(x) = x^4 - 8x^3 - 7x^2 + 3$

$= x(x+1)Q(x) + ax + 3$

$P(-1) = 1 + 8 - 7 + 3$

$= (-1)(-1+1)Q(-1) + a(-1) + 3$

$5 = -a + 3$

$a = -2$

10 (C) The locus is the perpendicular bisector of the two points.

The midpoint is:

$M = \left(\frac{a + b + (a-b)}{2}, \frac{b - a + (b+a)}{2}\right)$

$= (a, b)$

The gradient is:

$m = \frac{b + a - (b-a)}{a - b - (a+b)}$

$= \frac{2a}{-2b}$

$= -\frac{a}{b}$

The perpendicular gradient is $\frac{b}{a}$

The equation is:

$y - b = \frac{b}{a}(x - a)$

$ay - ab = bx - ab$

$bx - ay = 0$

SECTION II

Question 11

(a) $\left(x + \frac{2}{x}\right)^2 - 6\left(x + \frac{2}{x}\right) + 9 = 0$

Let $u = x + \frac{2}{x}$

$u^2 - 6u + 9 = 0$

$(u-3)^2 = 0$

$u = 3$

$x + \frac{2}{x} = 3$

$x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

$x = 2, \quad x = 1$

$\therefore x = 1, 2$

(b) $P(\text{rain}) = 0.1$

$P(\text{no rain}) = 0.9$

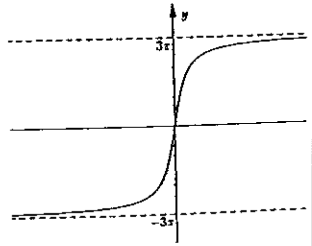
$P(<3 \text{ days rain}) = P(0 \text{ days}) + P(1 \text{ day}) + P(2 \text{ days})$

$= 0.9^{30} + \binom{30}{1}0.1^1\times 0.9^{29} + \binom{30}{2}0.1^2$

(c) $y = 6\tan^{-1}x$

Range: $-\frac{\pi}{2} < \frac{y}{6} < \frac{\pi}{2}$

$-3\pi < y < 3\pi$



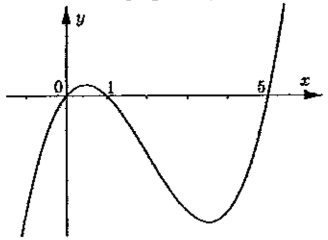
(d) $x = u^2 + 1 \qquad x = 2 \Rightarrow u = 1$

$dx = 2u\,du \qquad x = 5 \Rightarrow u = 2$

$$\int_2^5 \frac{x}{\sqrt{x-1}}\,dx = \int_1^2 \frac{u^2+1}{u} \times 2u\,du$$
$$= 2\left[\frac{u^3}{3} + u\right]_1^2$$
$$= 2\left(\frac{8}{3} + 2 - \frac{1}{3} - 1\right)$$
$$= 2 \times \frac{10}{3}$$
$$= 6\frac{2}{3}.$$

(e)
$$\frac{x^2+5}{x} > 6$$
$$\frac{x^2\left(x^2+5\right)}{x} > 6 \times x^2$$
$$x\left(x^2+5\right) > 6x^2$$
$$x\left(x^2+5\right) - 6x^2 > 0$$
$$x\left(x^2-6x+5\right) > 0$$
$$x(x-1)(x-5) > 0$$

Consider the graph of $y = x(x-1)(x-5)$



Require x values that give positive y values.
These occur when $0 < x < 1$ or $x > 5$.

(f)
$$\frac{d}{dx}\left(\frac{e^x \ln x}{x}\right) = \frac{\left(e^x \ln x + \frac{1}{x}e^x\right)x - e^x \ln x \times 1}{x^2}$$
$$= \frac{xe^x \ln x + e^x - e^x \ln x}{x^2}$$
$$= \frac{e^x(x\ln x - \ln x + 1)}{x^2}$$

Question 12

(a) (i) The displacement function $x = 2\sin 3t$ has an amplitude of 2 and when $t=0, x=0$. The particle travels 2 metres out then 2 metres back. So the total distance travelled is 4 metres.

(ii) $x = 2\sin 3t$

$v = 6\cos 3t$

$a = -18\sin 3t$

When first at rest:

$0 = 6\cos 3t$

$\cos 3t = 0$

$3t = \frac{\pi}{2}$

$t = \frac{\pi}{6}$ (first at rest)

$a = -18\sin 3\left(\frac{\pi}{6}\right)$

$= -18\text{ms}^{-2}$.

(b) Consider: $\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

Hence $\cos^2 4x = \frac{1}{2}(1 + \cos 8x)$

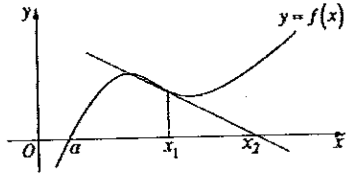
$$V = \pi \int_0^{\frac{\pi}{8}} \cos^2 4x\,dx$$
$$= \frac{\pi}{2}\int_0^{\frac{\pi}{8}} (1 + \cos 8x)\,dx$$
$$= \frac{\pi}{2}\left[x + \frac{1}{8}\sin 8x\right]_0^{\frac{\pi}{8}}$$
$$= \frac{\pi}{2}\left[\left(\frac{\pi}{8} + \frac{1}{8}\sin\pi\right) - (0)\right]$$
$$= \frac{\pi^2}{16} \text{ units}^3.$$

(c)
$$\ddot{x} = 2 - e^{-\frac{x}{2}}$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2 - e^{-\frac{x}{2}}$$
$$\frac{1}{2}v^2 = 2x + 2e^{-\frac{x}{2}}$$
$$v^2 = 4x + 4e^{-\frac{x}{2}} + C$$

When $x = 0, \quad v = 4$
$$4^2 = 4(0) + 4e^{-\frac{0}{2}} + C$$
$$C = 12$$
$$v^2 = 4x + 4e^{-\frac{x}{2}} + 12.$$

(d) From the Binomial Theorem:
$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$$
Let $x = -1$:
$$0 = \binom{n}{0} + \binom{n}{1}.-1 + \binom{n}{2}.(-1)^2 + \dots + \binom{n}{n}(-1)^n$$
$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - + \dots + (-1)^n\binom{n}{n}$$
As required.

(e) *Method 1:*



When the tangent at $x = x_1$ is drawn, it intersects the x-axis at the point x_2 which is further away from $x = \alpha$ than x_1.

OR

Method 2:
From the diagram, it can be seen that
$$f'(x_1) < 0 \text{ and } f(x_1) > 0$$
$$\therefore \frac{f(x_1)}{f'(x_1)} < 0 \text{ and } \therefore -\frac{f(x_1)}{f'(x_1)} > 0$$
Hence $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} > x_1$ since x_2 is further away from $x = \alpha$ than x_1.

(f) $T = A - Be^{-0.03t}$

As $t \to \infty, \quad T \to A \quad \therefore A = 23°C$ which is the constant temperature.

$\therefore T = 23 - Be^{-0.03t}$

When $t = 0, \quad T = 2$

$\therefore 2 = 23 - Be^{-0.03(0)} \quad \Rightarrow \quad B = 21$

$\therefore T = 23 - 21e^{-0.03t}$

When $T = 10$:
$$10 = 23 - 21e^{-0.03t}$$
$$e^{-0.03t} = \frac{13}{21}$$
$$-0.03t = \ln\left(\frac{13}{21}\right)$$
$$t = \frac{\ln\left(\frac{13}{21}\right)}{-0.03}$$
$$= 15.98576934\ldots$$
$$\approx 16 \text{ minutes.}$$

Question 13

(a) Let $P(n)$ be the proposition that $2^n + (-1)^{n+1}$ is divisible by 3 for all integers $n \geq 1$.

$P(1)$: $2^1 + (-1)^{1+1} = 3.P \qquad (P \in \mathbb{Z})$
$$LHS = 2^1 + (-1)^{1+1}$$
$$= 2 + 1$$
$$= 3.P$$
$$= RHS$$
This is divisible by 3 $\therefore P(1)$ is true.

Assume $P(k)$ is true for integer $k \geq$

i.e. $2^k + (-1)^{k+1} = 3Q \qquad (Q \in \mathbb{Z})$

or $2^k = 3Q - (-1)^{k+1}$

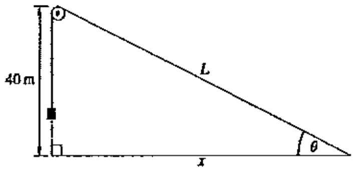
$P(k+1)$: $2^{k+1}+(-1)^{k+2}=3R \quad (R\in\mathbb{Z})$

$$LHS=2^{k+1}+(-1)^{k+2}$$
$$=2.2^k+(-1)^{k+2}$$
$$=2\left(3Q-(-1)^{k+1}\right)+(-1)^{k+2}$$
$$=6Q-2(-1)^{k+1}+(-1)(-1)^{k+1}$$
$$=6Q-3(-1)^{k+1}$$
$$=3\left(2Q-(-1)^{k+1}\right)$$
$$=3R$$
$$=RHS$$

$\therefore P(k+1)$ is true assuming $P(k)$ is true.

$\therefore P(n)$ is true by Mathematical Induction.

(b) (i)



Using Pythagoras: $x^2+40^2=L^2$

$$L=\sqrt{x^2+40^2}$$
$$\frac{dL}{dx}=\frac{1}{2}\left(x^2+40^2\right)^{\frac{-1}{2}}\times 2x$$
$$=\frac{x}{\sqrt{x^2+40^2}}$$
$$=\frac{x}{L}$$

$\cos\theta=\frac{x}{L} \Rightarrow \frac{dL}{dx}=\cos\theta$ as required.

(ii) Note that $\frac{dx}{dt}=3$

$$\frac{dL}{dt}=\frac{dL}{dx}\times\frac{dx}{dt}$$
$$=\cos\theta\times 3$$
$$=3\cos\theta.$$

(c) (i) $Q(x,y)=\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n}\right)$

$$=\left(\frac{t^2.0+1.2at}{t^2+1},\frac{t^2.a+1.at^2}{t^2+1}\right)$$
$$=\left(\frac{2at}{t^2+1},\frac{2at^2}{t^2+1}\right).$$

(ii) $m_{OQ}=\frac{y_2-y_1}{x_2-x_1}$

$$=\frac{\frac{2at^2}{t^2+1}-0}{\frac{2at}{t^2+1}-0}$$
$$=\frac{2at^2}{t^2+1}\times\frac{t^2+1}{2at}$$
$$=t$$

(iii) If Q lies on a circle then it should be a fixed distance from a fixed point. The fixed point in its definition is S.

$$QS^2=\left(\frac{2at}{t^2+1}-0\right)^2+\left(\frac{2at^2}{t^2+1}-\frac{a(t^2+1)}{t^2+1}\right)^2$$
$$=\frac{4a^2t^2}{(t^2+1)^2}+\frac{(at^2-a)^2}{(t^2+1)^2}$$
$$=\frac{4a^2t^2+a^2t^4-2a^2t^2+a^2}{(t^2+1)^2}$$
$$=\frac{a^2\left(t^4+2t^2+1\right)}{(t^2+1)^2}$$
$$=\frac{a^2(t^2+1)^2}{(t^2+1)^2}$$
$$=a^2$$
$$QS=a$$

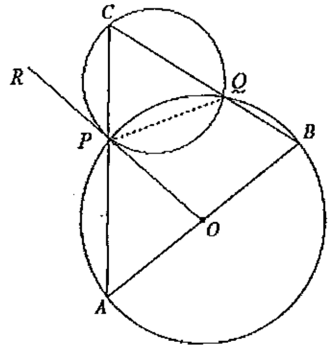
Thus Q lies on a fixed circle of radius a.

(d) (i) $\angle CQP=\angle BAP$

The exterior angle of the cyclic quadrilateral $PQBA$ is equal to the opposite interior angle.

$\therefore \angle CQP=\angle BAC$.

(ii)



Produce OP to R.

$\angle CQP=\angle BAC$ from (i)

$OA=OP$ (equal radii, $\triangle APO$ isosceles)

$\angle APO=\angle BAC$ (isosceles $\triangle APO$)

$\angle RPC=\angle APO$ (vert. opp. angles)

$\therefore \angle RPC=\angle CQP$

These 2 angles occupy the positions for the Alternate Segment Theorem.

$\therefore OP$ is tangent to circle.

Question 14

(a) (i) $x=Vt\cos\theta$

$$t=\frac{x}{V\cos\theta} \qquad (1)$$
$$y=-\frac{1}{2}gt^2+Vt\sin\theta \quad \ldots (2)$$

Sub (1) in (2)

$$y=-\frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2+V\left(\frac{x}{V\cos\theta}\right)\sin\theta$$
$$=-\frac{gx^2}{2V^2\cos^2\theta}+x\frac{\sin\theta}{\cos\theta}$$
$$=x\tan\theta-\frac{gx^2}{2V^2}\sec^2\theta$$

As required.

(ii) *Method 1:*

P lies on the line $y=-x$.

Substitute this into the result from (i).

$$x\tan\theta-\frac{gx^2}{2V^2}\sec^2\theta=-x$$
$$\frac{gx^2}{2V^2}\sec^2\theta-x-x\tan\theta=0$$
$$x\left(\frac{gx}{2V^2\cos^2\theta}-1-1\tan\theta\right)=0$$

$x=0$ is the result for the origin O.

The other result is needed.

$$0=\frac{gx}{2V^2\cos^2\theta}-1-1\tan\theta$$
$$\frac{gx}{2V^2\cos^2\theta}=\tan\theta+1$$
$$=\frac{\sin\theta}{\cos\theta}+1$$
$$=\frac{\sin\theta+\cos\theta}{\cos\theta}$$
$$x=\left(\frac{\sin\theta+\cos\theta}{\cos\theta}\right)\left(\frac{2V^2\cos^2}{g}\right.$$
$$=\frac{2V^2}{g}\left(\cos\theta\sin\theta+\cos^2\theta\right)$$
$$=\frac{2V^2}{g}\cos\theta\left(\sin\theta+\cos\theta\right)$$

Using Pythagoras theorem:

$$D^2=x^2+x^2$$
$$=2x^2$$
$$D=\sqrt{2}x$$
$$D=2\sqrt{2}\frac{V^2}{g}\cos\theta\left(\sin\theta+\cos\theta\right)$$

As required.

OR

Method 2:

At P:

$$\frac{y}{x}=\frac{-\frac{1}{2}gt^2+Vt\sin\theta}{Vt\cos\theta}$$
$$\tan\left(-\frac{\pi}{4}\right)=\frac{-gt^2+2Vt\sin\theta}{2Vt\cos\theta}$$

$$-1=\frac{-gt^2+2Vt\sin\theta}{2Vt\cos\theta}$$

$$2Vt\cos\theta=gt^2-2Vt\sin\theta$$

$$t^2=\frac{2Vt}{g}(\sin\theta+\cos\theta)$$

$$t=\frac{2V}{g}(\sin\theta+\cos\theta)$$

Using Pythagoras theorem:

$$D^2=x^2+x^2$$
$$=2x^2$$
$$D=\sqrt{2}x$$
$$=\sqrt{2}Vt\cos\theta$$
$$D=2\sqrt{2}\frac{V^2}{g}\cos\theta(\sin\theta+\cos\theta)$$

As required.

(iii) $D=2\sqrt{2}\frac{V^2}{g}\cos\theta(\sin\theta+\cos\theta)$

$$=2\sqrt{2}\frac{V^2}{g}\left[\cos\theta(\sin\theta+\cos\theta)\right]$$

$$\frac{dD}{d\theta}=2\sqrt{2}\frac{V^2}{g}\begin{bmatrix}-\sin\theta(\sin\theta+\cos\theta)+\\ \cos\theta(\cos\theta-\sin\theta)\end{bmatrix}$$

$$=2\sqrt{2}\frac{V^2}{g}\begin{bmatrix}-\sin^2\theta-2\sin\theta\cos\theta\\ +\cos^2\theta\end{bmatrix}$$

$$=2\sqrt{2}\frac{V^2}{g}\left[\cos 2\theta-\sin 2\theta\right].$$

(iv) For a possible maximum $\frac{dD}{d\theta}=0$

From part (iii), that means:

$$2\sqrt{2}\frac{V^2}{g}(\cos 2\theta-\sin 2\theta)=0$$
$$\cos 2\theta-\sin 2\theta=0$$
$$\cos 2\theta=\sin 2\theta$$
$$\tan 2\theta=1$$
$$\theta=\frac{\pi}{8}$$

$$\frac{d^2D}{d\theta^2}=2\sqrt{2}\frac{V^2}{g}(-2\sin 2\theta-2\cos 2\theta)$$

<0 when $\theta=\frac{\pi}{8}$

$\therefore D$ is a maximum when $\theta=\frac{\pi}{8}$.

(b) (i) $P(A \text{ wins 1st turn})=P(\text{lands on P})$
$=p$

$P(A \text{ wins 2nd turn})=P(A\to \text{R}, B\to \text{Q})$
$=rq$

$p+q+r=1 \Rightarrow q=1-p-r$

$$P(A \text{ wins 1st/2nd turn})=p+rq$$
$$=p+r(1-p-r)$$
$$=p+r-pr-r^2$$
$$=(p+r)-r(p+r)$$
$$=(p+r)(1-r)$$

As required.

(ii) $P(w)=P(\text{P})+P(\text{R,Q})+P(\text{R,R,P})+P(\text{R,R,R,Q})+\ldots$

$$=p+qr+r^2p+r^3q+\ldots$$
$$=p(1+r^2+\ldots)+q(r+r^3+\ldots)$$
$$=\frac{p}{1-r^2}+\frac{qr}{1-r^2}\quad \text{use } S_\infty \text{ since } 0<r^2<1$$
$$=\frac{p+qr}{1-r^2}$$
$$=\frac{(p+r)(1-r)}{1-r^2}$$
$$=\frac{p+r}{1+r}$$

End of Mathematics Extension 1 solutions