BOARD OF STUDIES
NEW SOUTH WALES

2013

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2–6

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 7–17

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which expression is equal to $\int \tan x \, dx$?

(A) $\sec^2 x + c$

(B) $-\ln(\cos x) + c$

(C) $\dfrac{\tan^2 x}{2} + c$

(D) $\ln(\sec x + \tan x) + c$

2 Which pair of equations gives the directrices of $4x^2 - 25y^2 = 100$?

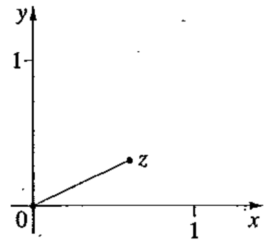
(A) $x = \pm \dfrac{25}{\sqrt{29}}$

(B) $x = \pm \dfrac{1}{\sqrt{29}}$

(C) $x = \pm \sqrt{29}$

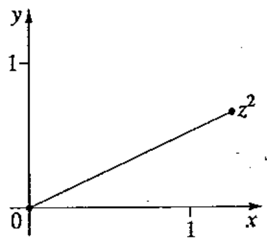
(D) $x = \pm \dfrac{\sqrt{29}}{25}$

3 The Argand diagram below shows the complex number z.

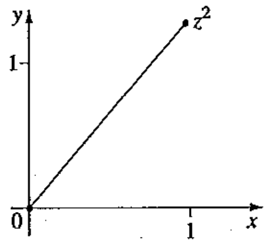


Which diagram best represents z^2?

(A)



(B)



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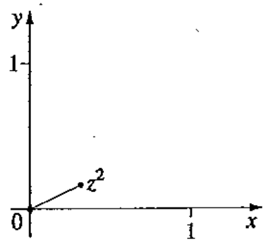

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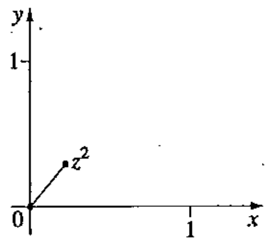


4 The polynomial equation $4x^3 + x^2 - 3x + 5 = 0$ has roots α, β and γ.

Which polynomial equation has roots $\alpha + 1$, $\beta + 1$ and $\gamma + 1$?

(A) $4x^3 - 11x^2 + 7x + 5 = 0$

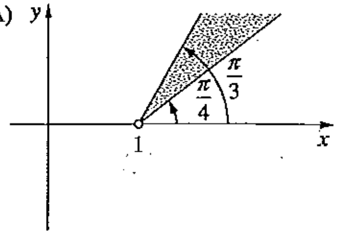
(B) $4x^3 + x^2 - 3x + 6 = 0$

(C) $4x^3 + 13x^2 + 11x + 7 = 0$

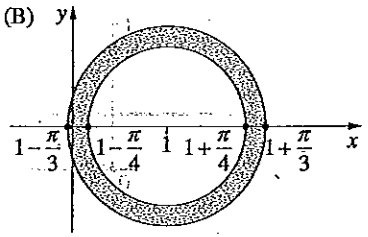
(D) $4x^3 - 2x^2 - 2x + 8 = 0$

5 Which region on the Argand diagram is defined by $\frac{\pi}{4} \le |z-1| \le \frac{\pi}{3}$?

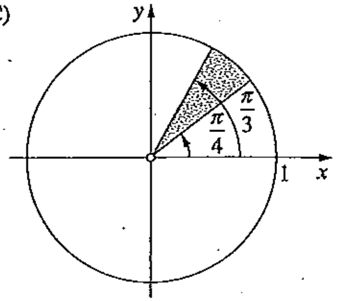
(A)



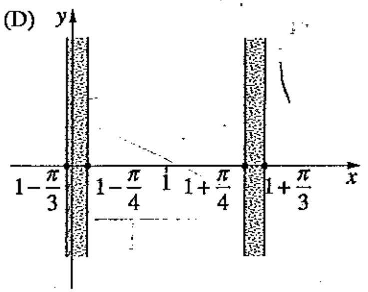
(B)



(C)



(D)



6 Which expression is equal to $\int \frac{1}{\sqrt{x^2-6x+5}}\,dx$?

(A) $\sin^{-1}\left(\frac{x-3}{2}\right)+C$

(B) $\cos^{-1}\left(\frac{x-3}{2}\right)+C$

(C) $\ln\left(x-3+\sqrt{(x-3)^2+4}\right)+C$

(D) $\ln\left(x-3+\sqrt{(x-3)^2-4}\right)+C$

7 The angular speed of a disc of radius 5 cm is 10 revolutions per minute.

What is the speed of a mark on the circumference of the disc?

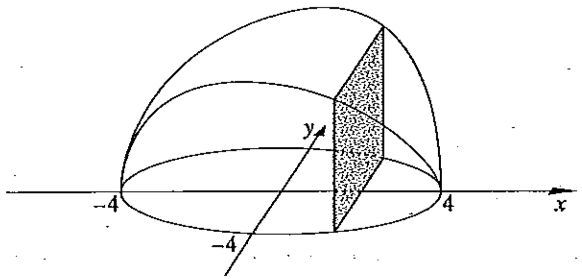
(A) 50 cm min^{-1}

(B) $\frac{1}{2}$ cm min^{-1}

(C) 100π cm min^{-1}

(D) $\frac{1}{4\pi}$ cm min^{-1}

8 The base of a solid is the region bounded by the circle $x^2+y^2=16$. Vertical cross-sections are squares perpendicular to the x-axis as shown in the diagram.



Which integral represents the volume of the solid?

(A) $\int_{-4}^{4} 4x^2\,dx$

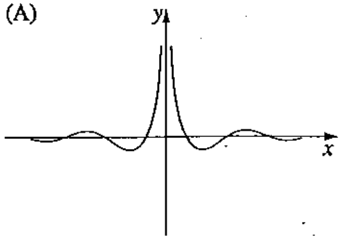
(B) $\int_{-4}^{4} 4\pi x^2\,dx$

(C) $\int_{-4}^{4} 4\left(16-x^2\right)dx$

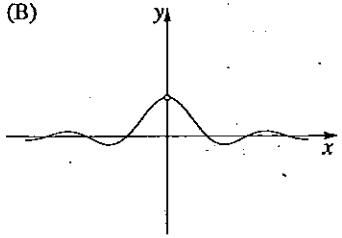
(D) $\int_{-4}^{4} 4\pi\left(16-x^2\right)dx$

9 Which diagram best represents the graph $y = \dfrac{\sin x}{x}$?

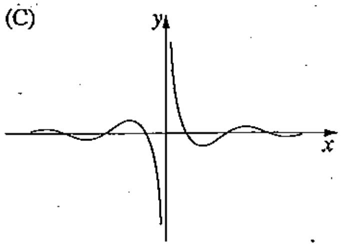
(A)



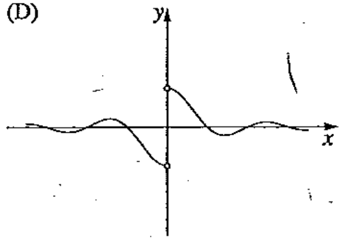
(B)



(C)



(D)



10 A hostel has four vacant rooms. Each room can accommodate a maximum of four people.

In how many different ways can six people be accommodated in the four rooms?

(A) 4020
(B) 4068
(C) 4080
(D) 4096

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 2 - i\sqrt{3}$ and $w = 1 + i\sqrt{3}$.

(i) Find $z + \overline{w}$. **1**

(ii) Express w in modulus–argument form. **2**

(iii) Write w^{24} in its simplest form. **2**

(b) Find numbers A, B and C such that **2**

$$\frac{x^2 + 8x + 11}{(x-3)(x^2+2)} = \frac{A}{x-3} + \frac{Bx + C}{x^2 + 2}.$$

(c) Factorise $z^2 + 4iz + 5$. **2**

(d) Evaluate $\displaystyle\int_0^1 x^3\sqrt{1 - x^2}\,dx$. **3**

(e) Sketch the region on the Argand diagram defined by $z^2 + \overline{z}^2 \le 8$. **3**

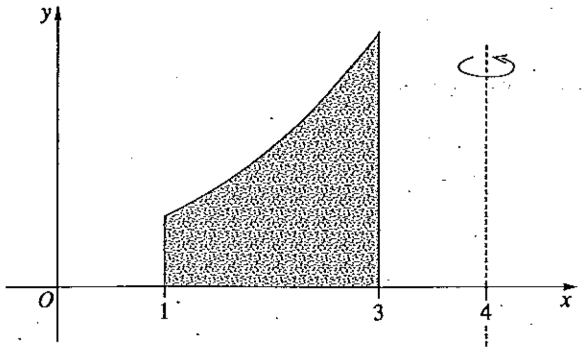
Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{4+5\cos x}\,dx$. **4**

(b) The equation $\log_e y - \log_e(1000 - y) = \frac{x}{50} - \log_e 3$ implicitly defines y as a function of x. **2**

Show that y satisfies the differential equation $\frac{dy}{dx} = \frac{y}{50}\left(1 - \frac{y}{1000}\right)$.

(c) The diagram shows the region bounded by the graph $y = e^x$, the x-axis and the lines $x = 1$ and $x = 3$. The region is rotated about the line $x = 4$ to form a solid. **4**

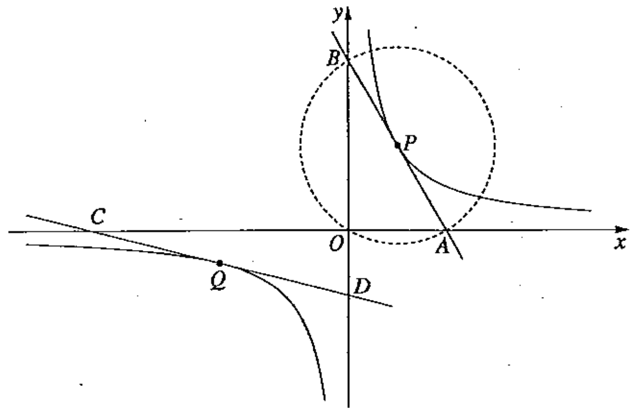


Find the volume of the solid.

Question 12 continues on page 9

(d) The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$, where $|p| \neq |q|$, lie on the rectangular hyperbola with equation $xy = c^2$.

The tangent to the hyperbola at P intersects the x-axis at A and the y-axis at B. Similarly, the tangent to the hyperbola at Q intersects the x-axis at C and the y-axis at D.



(i) Show that the equation of the tangent at P is $x + p^2y = 2cp$. **2**

(ii) Show that A, B and O are on a circle with centre P. **2**

(iii) Prove that BC is parallel to PQ. **1**

End of Question 12

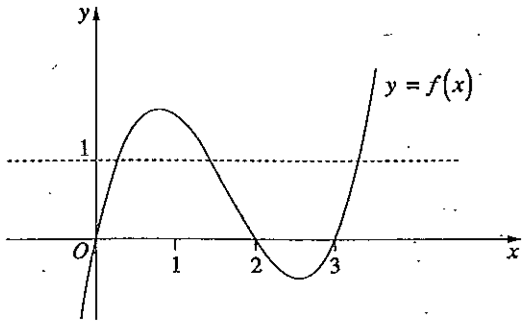
Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Let $I_n = \int_0^1 \left(1 - x^2\right)^{\frac{n}{2}} dx$, where $n \geq 0$ is an integer.

(i) Show that $I_n = \frac{n}{n+1} I_{n-2}$ for every integer $n \geq 2$. **3**

(ii) Evaluate I_5. **2**

(b) The diagram shows the graph of a function $f(x)$.



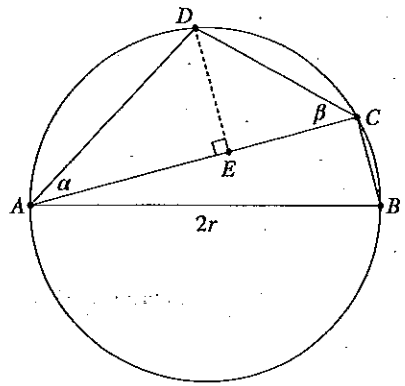
Sketch the following curves on separate half-page diagrams.

(i) $y^2 = f(x)$ **2**

(ii) $y = \frac{1}{1 - f(x)}$ **3**

Question 13 continues on page 11

(c) The points A, B, C and D lie on a circle of radius r, forming a cyclic quadrilateral. The side AB is a diameter of the circle. The point E is chosen on the diagonal AC so that $DE \perp AC$. Let $\alpha = \angle DAC$ and $\beta = \angle ACD$.



(i) Show that $AC = 2r\sin(\alpha + \beta)$. **2**

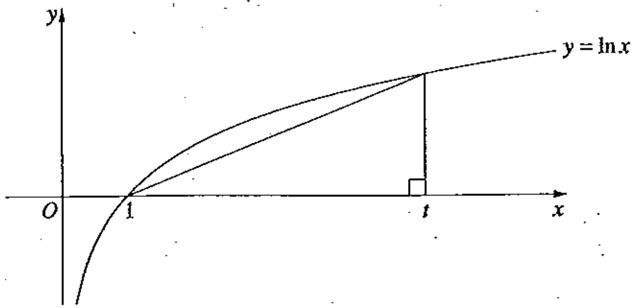
(ii) By considering $\triangle ABD$, or otherwise, show that $AE = 2r\cos\alpha\sin\beta$. **2**

(iii) Hence, show that $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$. **1**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the graph $y = \ln x$. **3**



By comparing relevant areas in the diagram, or otherwise, show that

$$\ln t > 2\left(\frac{t-1}{t+1}\right), \text{ for } t > 1.$$

(b) Let $z_2 = 1 + i$ and, for $n > 2$, let $z_n = z_{n-1}\left(1 + \frac{i}{|z_{n-1}|}\right)$. **3**

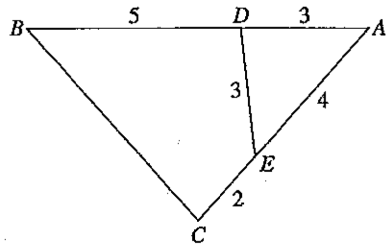
Use mathematical induction to prove that $|z_n| = \sqrt{n}$ for all integers $n \geq 2$.

Question 14 continues on page 13

(c) (i) Given a positive integer n, show that $\sec^{2n}\theta = \sum_{k=0}^{n}\binom{n}{k}\tan^{2k}\theta$. **1**

(ii) Hence, by writing $\sec^8\theta$ as $\sec^6\theta\sec^2\theta$, find $\int \sec^8\theta \, d\theta$. **2**

(d) A triangle has vertices A, B and C. The point D lies on the interval AB such that $AD = 3$ and $DB = 5$. The point E lies on the interval AC such that $AE = 4$, $DE = 3$ and $EC = 2$.



NOT TO SCALE

(i) Prove that $\triangle ABC$ and $\triangle AED$ are similar. **1**

(ii) Prove that $BCED$ is a cyclic quadrilateral. **1**

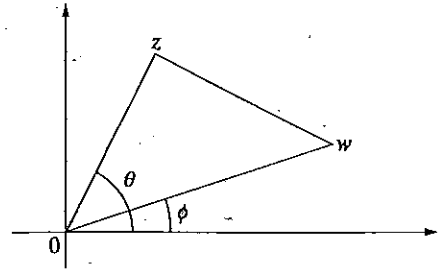
(iii) Show that $CD = \sqrt{21}$. **2**

(iv) Find the exact value of the radius of the circle passing through the points B, C, E and D. **2**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) The Argand diagram shows complex numbers w and z with arguments ϕ and θ respectively, where $\phi < \theta$. The area of the triangle formed by 0, w and z is A. **3**



Show that $z\overline{w} - w\overline{z} = 4iA$.

(b) The polynomial $P(x) = ax^4 + bx^3 + cx^2 + e$ has remainder -3 when divided by $x - 1$. The polynomial has a double root at $x = -1$.

(i) Show that $4a + 2c = -\frac{9}{2}$. **2**

(ii) Hence, or otherwise, find the slope of the tangent to the graph $y = P(x)$ when $x = 1$. **1**

(c) Eight cars participate in a competition that lasts for four days. The probability that a car completes a day is 0.7. Cars that do not complete a day are eliminated.

(i) Find the probability that a car completes all four days of the competition. **1**

(ii) Find an expression for the probability that at least three cars complete all four days of the competition. **2**

Question 15 continues on page 15

(d) A ball of mass m is projected vertically into the air from the ground with initial velocity u. After reaching the maximum height H it falls back to the ground. While in the air, the ball experiences a resistive force kv^2, where v is the velocity of the ball and k is a constant.

The equation of motion when the ball falls can be written as

$$m\dot{v} = mg - kv^2. \qquad \text{(Do NOT prove this.)}$$

(i) Show that the terminal velocity v_T of the ball when it falls is $\sqrt{\frac{mg}{k}}$. **1**

(ii) Show that when the ball goes up, the maximum height H is **3**

$$H = \frac{v_T^2}{2g}\ln\left(1 + \frac{u^2}{v_T^2}\right).$$

(iii) When the ball falls from height H it hits the ground with velocity w. **2**

Show that $\frac{1}{w^2} = \frac{1}{u^2} + \frac{1}{v_T^2}$.

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Find the minimum value of $P(x) = 2x^3 - 15x^2 + 24x + 16$, for $x \geq 0$. **2**

(ii) Hence, or otherwise, show that for $x \geq 0$, **1**

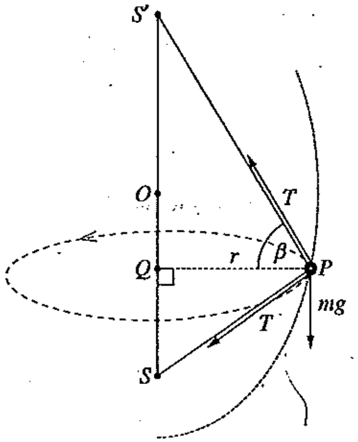
$$(x+1)\left(x^2 + (x+4)^2\right) \geq 25x^2.$$

(iii) Hence, or otherwise, show that for $m \geq 0$ and $n \geq 0$, **2**

$$(m+n)^2 + (m+n+4)^2 \geq \frac{100mn}{m+n+1}.$$

(b) A small bead P of mass m can freely move along a string. The ends of the string are attached to fixed points S and S', where S' lies vertically above S. The bead undergoes uniform circular motion with radius r and constant angular velocity ω in a horizontal plane.

The forces acting on the bead are the gravitational force and the tension forces along the string. The tension forces along PS and PS' have the same magnitude T.



The length of the string is $2a$ and $SS' = 2ae$, where $0 < e < 1$. The horizontal plane through P meets SS' at Q. The midpoint of SS' is O and $\beta = \angle S'PQ$. The parameter θ is chosen so that $OQ = a\cos\theta$.

Question 16 continues on page 17

Question 16 (continued)

(i) What information indicates that P lies on an ellipse with foci S and S', and with eccentricity e? **1**

(ii) Using the focus–directrix definition of an ellipse, or otherwise, show that $SP = a(1 - e\cos\theta)$. **1**

(iii) Show that $\sin\beta = \dfrac{e + \cos\theta}{1 + e\cos\theta}$. **2**

(iv) By considering the forces acting on P in the vertical direction, show that **2**

$$mg = \frac{2T\left(1-e^2\right)\cos\theta}{1 - e^2\cos^2\theta}.$$

(v) Show that the force acting on P in the horizontal direction is **3**

$$mr\omega^2 = \frac{2T\sqrt{1-e^2}\sin\theta}{1 - e^2\cos^2\theta}.$$

(vi) Show that $\tan\theta = \dfrac{r\omega^2}{g}\sqrt{1-e^2}$. **1**

End of paper

2013 Higher School Certificate Solutions Mathematics Extension 2

SECTION I

Summary

1	B	4	A	7	C	9	B
2	A	5	B	8	C	10	A
3	D	6	D				

1 (B) $\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$

$= -\ln(\cos x) + C$

2 (A) The equation can be written as:

$\frac{x^2}{5^2} - \frac{y^2}{2^2} = 1$, $\therefore$ $a = 5$ and $b = 2$.

Since $b^2 = a^2(e^2 - 1)$ then

$e = \frac{\sqrt{a^2+b^2}}{a} = \frac{\sqrt{5^2+2^2}}{5} = \frac{\sqrt{29}}{5}$

Equations of the directrices:

$x = \pm\frac{a}{e}$

$= \pm 5 \div \frac{\sqrt{29}}{5}$

$= \pm\frac{25}{\sqrt{29}}$.

3 (D) Since $|z| < 1$

$|z^2| < |z|$

and $\arg(z^2) = 2\arg(z)$.

4 (A) Let $x = \alpha + 1$, $\therefore$ $\alpha = x - 1$.

$4(x-1)^3 + (x-1)^2 - 3(x-1) + 5 = 0$

$4(x^3 - 3x^2 + 3x - 1) + (x^2 - 2x + 1) - 3x + 3 + 8 = 0$

$4x^3 - 11x^2 + 7x + 5 = 0$

5 (B) z lies between 2 circles, centre $(1,0)$, radii $\frac{\pi}{4}$ and $\frac{\pi}{3}$.

6 (D) By completing the square

$x^2 - 6x + 5 = (x-3)^2 - 4$

$\int \frac{1}{\sqrt{x^2-6x+5}} dx = \int \frac{1}{\sqrt{(x-3)^2 - 2^2}} dx$

$= \ln\left(x - 3 + \sqrt{(x-3)^2 - 2^2}\right) + C.$

7 (C) $v = r\omega$,

where $\omega = 2\pi f$

$= 2\pi \times 10$

$= 20\pi$

$\therefore$ $v = 5 \times 20\pi$

$= 100\pi$.

8 (C) Area $= (2y)^2$

$= 4y^2$

$= 4(16 - x^2)$

$\therefore$ $V = 4\int_{-4}^{4} 16 - x^2\, dx$.

9 (B) Let $f(x) = \frac{\sin x}{x}$

$f(-x) = \frac{\sin(-x)}{-x}$

$= \frac{-\sin(x)}{-x}$

$= \frac{\sin(x)}{x}$

$= f(x)$

$\therefore$ $f(x)$ is even

and $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$

So $f(x)$ is symmetrical about the y-axis and curve approaches the point (0,1) with $x = 0$ excluded.

10 (A) Each man has 4 rooms to choose from, hence $4^6 = 4096$ with no restrictions.

6 in 1 room can occur 4 ways.

5 in 1 room can occur ${}^6C_5\,{}^4C_1\,{}^3C_1$ ways.

Thus the number of ways is:

$P(X) = \text{total} - (6 \text{ in room}) - (5 \text{ in room})$

$= 4^6 - {}^4C_1 - {}^4C_1\,{}^6C_5\,{}^3C_1$

$= 4020$.

SECTION II

Question 11

(a) (i) $z + \overline{w} = 2 - i\sqrt{3} + (1 - i\sqrt{3})$

$= 3 - i2\sqrt{3}$.

(ii) *Method 1:*

$|w| = \sqrt{1^2 + (\sqrt{3})^2} = 2$

$\arg(w) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$w = 2cis\frac{\pi}{3}$.

OR

Method 2:

$|w| = \sqrt{1+3} = 2$

$w = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= 2cis\frac{\pi}{3}$.

(iii) $w^{24} = 2^{24} cis\left(24 \times \frac{\pi}{3}\right)$

$= 2^{24} cis 8\pi$

$= 2^{24}$.

(b) $\frac{x^2+8x+11}{(x-3)(x^2+2)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+2}$

$x^2 + 8x + 11 = A(x^2+2) + (Bx+C)(x-3)$

$= (A+B)x^2 + (C-3B)x + (2A-3C)$

$A + B = 1 \Rightarrow B = 1 - A$

$C - 3B = 8 \Rightarrow C = 11 - 3A$

$2A - 3C = 11$

$2A - 33 + 9A = 11$

$11A = 44$

$A = 4$

$$B=-3$$
$$C=-1$$
$$\frac{x^2+8x+11}{(x-3)(x^2+2)}=\frac{4}{x-3}+\frac{-3x-1}{x^2+2}.$$

(c) *Method 1:*

By inspection:

$z^2+4iz+5=(z-i)(z+5i)$.

OR

Method 2:

$$z=\frac{-4i\pm\sqrt{16i^2-4(5)}}{2}$$
$$=\frac{-4i\pm\sqrt{-36}}{2}$$
$$=\frac{-4i\pm 6i}{2}$$
$$=i \text{ or } -5i$$
$$z^2+4iz+5=(z-i)(z+5i).$$

(d) Let $u^2=1-x^2$

$$2u.\frac{du}{dx}=-2x$$
$$u\,du=-x\,dx$$

When $x=0,\ u=1$

$x=1,\ u=0$

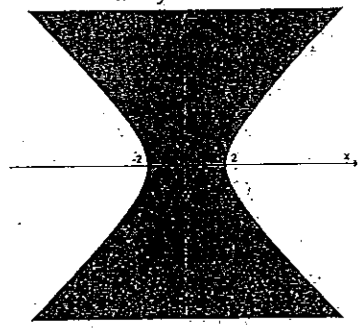
$$\int_0^1 x^3\sqrt{1-x^2}\,dx=\int_0^1 x^2\sqrt{1-x^2}\,x\,dx$$
$$=\int_1^0(1-u^2)u.-u\,du$$
$$=-\int_1^0(1-u^2)u^2\,du$$
$$=\int_0^1(u^2-u^4)\,du$$
$$=\left[\frac{u^3}{3}-\frac{u^5}{5}\right]_0^1$$
$$=\frac{2}{15}.$$

(e) Let $z=x+iy$

$$z^2+\bar{z}^2=(x+iy)^2+(x-iy)^2$$
$$=x^2+2ixy-y^2+x^2-2ixy-y^2$$
$$=2(x^2-y^2)$$

Thus

$$z^2+\bar{z}^2\le 8$$
$$2(x^2-y^2)\le 8$$
$$x^2-y^2\le 4$$



Question 12

(a) Let $t=\tan\frac{x}{2}$.

$$\cos x=\frac{1-t^2}{1+t^2}\qquad dx=\frac{2}{1+t^2}dt$$

When $x=0,\ t=0$

$x=\frac{\pi}{2},\ t=1$

$$I=\int_0^{\frac{\pi}{2}}\frac{1}{4+5\cos x}dx$$
$$=\int_0^1\frac{1}{4+5\left(\frac{1-t^2}{1+t^2}\right)}\cdot\frac{2}{1+t^2}dt$$
$$=\int_0^1\frac{2}{4(1+t^2)+5(1-t^2)}dt$$
$$=\int_0^1\frac{2}{9-t^2}dt$$
$$=2\int_0^1\frac{1}{(3+t)(3-t)}dt$$
$$=\frac{1}{3}\int_0^1\left(\frac{1}{(3+t)}+\frac{1}{(3-t)}\right)dt$$
$$=\frac{1}{3}\left[\ln\left(\frac{(3+t)}{(3-t)}\right)\right]_0^1$$
$$=\frac{1}{3}(\ln 2-\ln 1)$$
$$=\frac{1}{3}\ln 2.$$

(b) $\ln y-\ln(1000-y)=\frac{x}{50}-\ln 3$

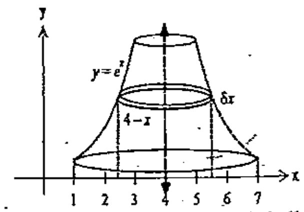
By implicit differentiation:

$$\frac{1}{y}\times\frac{dy}{dx}+\frac{1}{1000-y}\times\frac{dy}{dx}=\frac{1}{50}$$
$$\frac{dy}{dx}\left(\frac{1}{y}+\frac{1}{1000-y}\right)=\frac{1}{50}$$
$$\frac{dy}{dx}\left(\frac{1000-y+y}{y(1000-y)}\right)=\frac{1}{50}$$
$$\frac{dy}{dx}\left(\frac{1000}{y(1000-y)}\right)=\frac{1}{50}$$
$$\frac{dy}{dx}\left(\frac{1000}{1000-y}\right)=\frac{y}{50}$$

Thus

$$\frac{dy}{dx}=\frac{y}{50}\left(\frac{1000-y}{1000}\right)$$
$$\frac{dy}{dx}=\frac{y}{50}\left(1-\frac{y}{1000}\right).$$

(c)



The volume of the cylindrical shell ∂V is given by

$$\partial V\approx 2\pi rh\,\partial x$$
$$=2\pi(4-x)e^x\,\partial x$$
$$V=2\pi\int_1^3(4-x)e^x\,dx$$

Let $u=4-x,\quad du=-dx$

$dv=e^x,\qquad v=e^x$

$$I=\int_1^3(4-x)e^x\,dx$$
$$=\left[(4-x)e^x\right]_1^3+\int_1^3 e^x\,dx$$
$$=\left[(4-x)e^x\right]_1^3+\left[e^x\right]_1^3$$
$$=\left[(4-x)e^x+e^x\right]_1^3$$
$$=\left[(5-x)e^x\right]_1^3$$
$$=2e^3-4e$$
$$\therefore V=2\pi(2e^3-4e)$$
$$=4\pi e(e^2-2).$$

(d) (i) $xy=c^2$

$$y=c^2x^{-1}$$
$$\frac{dy}{dx}=-c^2x^{-2}$$
$$=-\frac{c^2}{x^2}$$

When $x=cp$

$$\frac{dy}{dx}=-\frac{c^2}{c^2p^2}$$
$$=-\frac{1}{p^2}$$

Equation of the tangent is:

$$y-\frac{c}{p}=-\frac{1}{p^2}(x-cp)$$
$$p^2y-cp=-x+cp$$
$$x+p^2y=2cp.$$

(ii) At A, $y=0\ \therefore x=2cp$

and A is $(2cp,0)$.

At B, $x=0\ \therefore y=\frac{2c}{p}$

and B is $\left(0,\frac{2c}{p}\right)$.

$$OP=\sqrt{(cp)^2+\left(\frac{c}{p}\right)^2}$$

$$AP=\sqrt{(2cp-cp)^2+\left(0-\frac{c}{p}\right)^2}$$
$$=\sqrt{(cp)^2+\left(\frac{c}{p}\right)^2}$$
$$=OP$$
$$BP=\sqrt{(cp-0)^2+\left(\frac{c}{p}-\frac{2c}{p}\right)^2}$$
$$=\sqrt{(cp)^2+\left(\frac{c}{p}\right)^2}$$
$$=OP$$
$\therefore$ A, B and O lie on a circle with centre at P as required.

(iii) Using similar methods, the equation of the tangent at Q is $x+q^2y=2cq$, and that $C(2cq,0)$ and $D\left(0,\frac{2c}{q}\right)$.

$$m_{BC}=\frac{\frac{2c}{p}-0}{0-2cq}$$
$$=\frac{2c}{p}\times\frac{1}{-2cq}$$
$$=-\frac{1}{pq}$$
$$m_{PQ}=\frac{\frac{c}{p}-\frac{c}{q}}{cp-cq}$$
$$=\left(\frac{1}{p}-\frac{1}{q}\right)\times\frac{1}{p-q}$$
$$=\left(\frac{q-p}{pq}\right)\times\frac{1}{p-q}$$
$$=-\frac{1}{pq}$$
$\therefore BC \parallel PQ$ as required.

Question 13

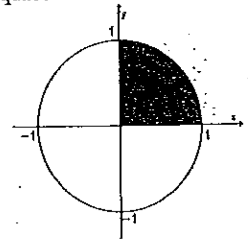
(a) (i) $I_n=\int_0^1(1-x^2)^{\frac{n}{2}}\,dx$

Let $u=(1-x^2)^{\frac{n}{2}}$, $du=-nx(1-x^2)^{\frac{n}{2}-1}\,dx$

Let $dv=dx$, $v=x$

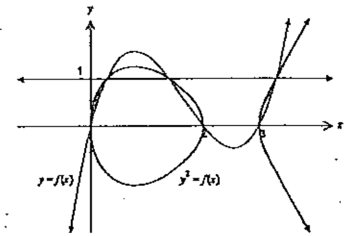
$$I_n=\left[x(1-x^2)^{\frac{n}{2}}\right]_0^1+n\int_0^1x^2(1-x^2)^{\frac{n}{2}-1}\,dx$$
$$=0+n\int_0^1x^2(1-x^2)^{\frac{n}{2}-1}\,dx$$
$$=n\int_0^1(x^2-1+1)(1-x^2)^{\frac{n}{2}-1}\,dx$$
$$=n\int_0^1(x^2-1)(1-x^2)^{\frac{n}{2}-1}\,dx+n\int_0^1 1(1-x^2)^{\frac{n}{2}-1}\,dx$$
$$=-n\int_0^1(1-x^2)(1-x^2)^{\frac{n}{2}-1}\,dx+n\int_0^1(1-x^2)^{\frac{n}{2}-1}\,dx$$
$$=-nI_n+nI_{n-2}\quad\text{since }\frac{n-2}{2}=\frac{n}{2}-1$$
$$\therefore (n+1)I_n=nI_{n-2}$$
$$I_n=\frac{n}{n+1}I_{n-2}.$$

(ii) $I_5=\frac{5}{6}I_3$
$$=\frac{5}{6}\left(\frac{3}{4}I_1\right)$$
$$=\frac{5}{8}I_1$$
and $I_1=\int_0^1(1-x^2)^{\frac{1}{2}}\,dx$
$$=\int_0^1\sqrt{1-x^2}\,dx$$
$$=\frac{\pi}{4}$$
since $\int_0^1\sqrt{1-x^2}\,dx$ is the area of the first quadrant of a unit circle.

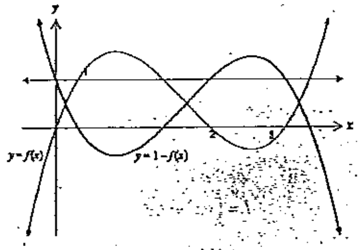


$$\therefore I_5=\frac{5}{8}I_1$$
$$=\frac{5}{8}\times\frac{\pi}{4}$$
$$=\frac{5\pi}{32}.$$

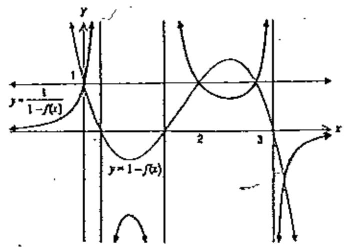
(b) (i)



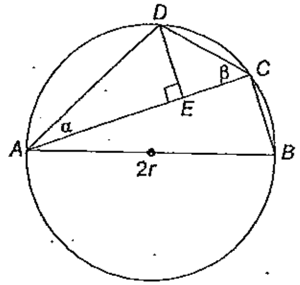
(ii) Reflect $f(x)$ about the x axis then shift the entire graph up 1 unit.



Now sketch $y=\frac{1}{1-f(x)}$



(c) (i)



$\angle ADC=180°-(\alpha+\beta)$ ($\angle$ sum of Δ)
$\angle ABC=\alpha+\beta$ (opp. $\angle$s of cyclic quad.)
$\angle ACB=90°$ ($\angle$ in a semi-circle)
$\therefore \Delta ABC$ is right angled.
$$\sin\angle ABC=\frac{AC}{AB}$$
$$\sin(\alpha+\beta)=\frac{AC}{2r}$$
$$AC=2r\sin(\alpha+\beta).$$

(ii) $\angle DBC=\angle DAC=\alpha$ ($\angle$s on same arc)
$\angle ABC=\alpha+\beta$ from part (i)
$\therefore \angle ABD=\beta$

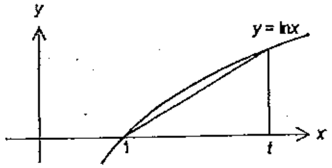
In ΔABD,
$\angle ADB=90°$ ($\angle$ in a semi-circle)
$$\sin\beta=\frac{AD}{AB}$$
$$AD=2r\sin\beta$$

In ΔADE,
$$\cos\alpha=\frac{AE}{AD}$$
$$=\frac{AE}{2r\sin\beta}$$
$$AE=2r\cos\alpha\sin\beta.$$

(iii) Similarly $EC=2r\cos\beta\sin\alpha$
$$AC=AE+EC$$
$$2r\sin(\alpha+\beta)=2r\cos\alpha\sin\beta+2r\sin\alpha\cos\beta$$
$$\sin(\alpha+\beta)=\cos\alpha\sin\beta+\sin\alpha\cos\beta.$$

Question 14

(a)



Consider the area under the curve and the area of the triangle:

$$\int_1^t \ln x\,dx > \frac{1}{2}(t-1)\ln t$$

$$[x\ln x]_1^t - \int_1^t x\cdot\frac{1}{x}\,dx > \frac{1}{2}(t-1)\ln t$$

$$[x\ln x]_1^t - \int_1^t dx > \frac{1}{2}(t-1)\ln t$$

$$[x\ln x]_1^t - [x]_1^t > \frac{1}{2}(t-1)\ln t$$

$$(t\ln t - 0) - (t-1) > \frac{1}{2}(t-1)\ln t$$

$$2t\ln t - 2(t-1) > t\ln t - \ln t$$

$$t\ln t + \ln t > 2(t-1)$$

$$\ln t(t+1) > 2(t-1)$$

$$\ln t > 2\left(\frac{t-1}{t+1}\right) \text{ for } t > 1.$$

(b)

$z_2 = 1 + i$

$|z_2| = \sqrt{1^2 + 1^2}$

$= \sqrt{2}$

Assume it is true for some $n = k$ then $|z_k| = \sqrt{k}$.

For $n = k+1$,

$$z_{k+1} = z_k\left(1 + \frac{i}{|z_k|}\right)$$

$$= z_k\left(1 + \frac{i}{\sqrt{k}}\right) \quad \text{from } n = k$$

$$|z_{k+1}| = |z_k|\left|1 + \frac{i}{\sqrt{k}}\right|$$

$$= \sqrt{k}\cdot\sqrt{1^2 + \left(\frac{1}{\sqrt{k}}\right)^2}$$

$$= \sqrt{k}\cdot\sqrt{1 + \frac{1}{k}}$$

$$= \sqrt{k}\cdot\sqrt{\frac{k+1}{k}}$$

$$= \sqrt{k}\cdot\frac{\sqrt{k+1}}{\sqrt{k}}$$

$$= \sqrt{k+1}$$

$\therefore$ true for $n = k+1$.

$\therefore$ By the principle of Mathematical Induction it is true for all $n \geq 2$.

(c) (i)

$$\sec^{2n}\theta = \left(\sec^2\theta\right)^n$$

$$= \left(1 + \tan^2\theta\right)^n$$

$$= \sum_{k=0}^{n}\binom{n}{k}1^{n-k}\left(\tan^2\theta\right)^k$$

$$= \sum_{k=0}^{n}\binom{n}{k}\tan^{2k}\theta.$$

(ii) *Method 1:*

$$I = \int \sec^8\theta\,d\theta$$

$$= \int \sec^6\theta\sec^2\theta\,d\theta$$

$$= \int\left[\binom{3}{0} + \binom{3}{1}\tan^2\theta + \binom{3}{2}\tan^4\theta + \binom{3}{3}\tan^6\theta\right]\sec^2\theta\,d\theta$$

$$= \int\left[1 + 3\tan^2\theta + 3\tan^4\theta + \tan^6\theta\right]\sec^2\theta\,d\theta$$

$$= \tan\theta + \tan^3\theta + \frac{3}{5}\tan^5\theta + \frac{1}{7}\tan^5\theta + C.$$

OR

Method 2:

$$\sec^8\theta = \sec^6\theta\sec^2\theta$$

$$= \left(\sec^2\theta\right)^3\sec^2\theta$$

$$= \left(1 + \tan^2\theta\right)^3\sec^2\theta$$

$$\therefore \int \sec^8\theta\,d\theta = \int\left(1 + \tan^2\theta\right)^3\sec^2\theta\,d\theta$$

Let $u = \tan\theta$

$\therefore du = \sec^2\theta\,d\theta$

$$I = \int\left(1 + u^2\right)^3 du$$

$$= \int\left(1 + 3u^2 + 3u^4 + u^6\right)du$$

$$= u + u^3 + \frac{3}{5}u^5 + \frac{1}{7}u^7 + C$$

$$= \tan\theta + \tan^3\theta + \frac{3}{5}\tan^5\theta + \frac{1}{7}\tan^7\theta + C$$

(d) (i) $\angle BAC = \angle EAD$ (common angle)

$\frac{AE}{AB} = \frac{4}{8}$ (given)

$= \frac{1}{2}$

$\frac{AD}{AC} = \frac{3}{6}$ (given)

$= \frac{1}{2}$

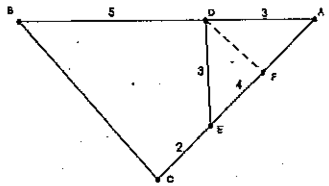
$\therefore \frac{AE}{AB} = \frac{AD}{AC}$

Thus $\Delta ABC \,|||\, \Delta AED$ (2 sides proportional and included angle)

(ii) $\angle ADE = \angle ACB$ (corr. $\angle$s in similar Δs)

$BCED$ is cyclic (ext $\angle$ equal int opp $\angle$)

(iii)



ΔDEA is isosceles.

Let F be the midpoint of AE.

$\angle DFE = 90°$

$AF = FE = 2$

$3^2 = DF^2 + 2^2$

$DF = \sqrt{5}$

$CF = 2 + 2 = 4$

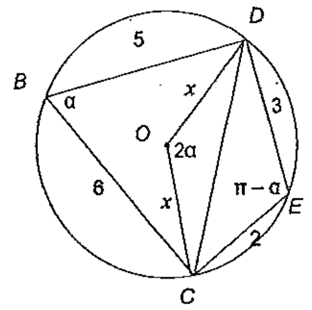
$CD^2 = DF^2 + CF^2$

$= \sqrt{5}^2 + 4^2$

$= 21$

$CD = \sqrt{21}.$

(iv)



Let $\angle DOC = 2\alpha$ (angle at centre)

Reflex $\angle DOC = 2\pi - 2\alpha$

$\angle CED = \frac{1}{2}\times\angle DOC$ ($\angle$ at circumference)

$= \pi - \alpha$

Let $OC = OD = x$

In ΔCOD:

$CD^2 = x^2 + x^2 - 2\times x\times x\times\cos 2\alpha$

$21 = 2x^2 - 2x^2\cos 2\alpha$ ①

In ΔCED:

$CD^2 = 2^2 + 3^2 - 2\times 2\times 3\times\cos(\pi - \alpha)$

$21 = 13 + 12\cos\alpha$

$\cos\alpha = \frac{8}{12} = \frac{2}{3}$

$\cos 2\alpha = 2\cos^2\alpha - 1$

$= 2\left(\frac{2}{3}\right)^2 - 1$

$= -\frac{1}{9}$

From ①:

$21 = 2x^2 - 2x^2\left(-\frac{1}{9}\right)$

$189 = 18x^2 + 2x^2$

$x = \sqrt{\frac{189}{20}} = \frac{3\sqrt{21}}{2\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}$

$= \frac{3\sqrt{105}}{10}.$

Question 15

(a) (i) Area $=\frac{1}{2}ab\sin C$

$$A=\frac{1}{2}|z||w|\sin(\theta-\phi)$$

$$\begin{aligned}
z\bar{w}-w\bar{z} &= |z|cis\theta|w|cis(-\phi) \\
&\quad -|w|cis\phi|z|cis(-\theta) \\
&= |z||w|\left(cis\theta cis(-\phi)-cis\phi cis(-\theta)\right) \\
&= |z||w|\left(cis(\theta-\phi)-cis(\phi-\theta)\right)
\end{aligned}$$

But $\cos(\phi-\theta)=\cos(\theta-\phi)$

$\sin(\phi-\theta)=-\sin(\theta-\phi)$

$$\begin{aligned}
\therefore\ z\bar{w}-w\bar{z} &= |z||w|2i\sin(\theta-\phi) \\
&= 4i\left(\frac{1}{2}|z||w|\sin(\theta-\phi)\right) \\
&= 4iA.
\end{aligned}$$

(b) (i) $P(x)=ax^4+bx^3+cx^2+e$

$P'(x)=4ax^3+3bx^2+2cx$

$P(1)=a+b+c+e=-3$

$P'(-1)=P(-1)=0$ (repeated root)

$P(-1)=a-b+c+e=0$

$P'(-1)=-4a+3b-2c=0$

$a+b+c+e=-3$ ①

$a-b+c+e=0$ ②

① − ②

$2b=-3$

$b=-\frac{3}{2}$

$-4a+3b-2c=0$ ③

$4a+2c=3b$

$4a+2c=3\times-\frac{3}{2}$

$4a+2c=-\frac{9}{2}.$

(ii)
$$\begin{aligned}
P'(1) &= 4a+3b+2c \\
&= 4a+2c+3b \\
&= -\frac{9}{2}+3\times-\frac{3}{2} \\
&= -9.
\end{aligned}$$

(c) (i) $0.7^4=0.2401$

(ii) $P(X\ge 3)=1-P(X=0)-P(X=1)-P(X=2)$

$$=1-\left(\begin{aligned}&(0.7599)^8\\ &+{}^8C_1(0.7599)^7(0.2401)\\ &+{}^8C_2(0.7599)^6(0.2401)^2\end{aligned}\right).$$

(d) (i) v_T is when $\dot{v}=0$.

$m\dot{v}=mg-kv^2$

$0=mg-kv_T^2$

$kv_T^2=mg$

$v_T^2=\frac{mg}{k}$

$v_T=\sqrt{\frac{mg}{k}}.$ \

(ii) When the ball rises: $m\dot{v}=-mg-kv^2$

$m\dot{v}=-mg-kv^2$

$mv\frac{dv}{dx}=-mg-kv^2$

$dx=-\frac{mv}{mg+kv^2}dv$

$\int_0^H dx=-\int_u^0\frac{mv}{mg+kv^2}dv$

$[x]_0^H=-\frac{m}{2k}\left[\ln(mg+kv^2)\right]_u^0$

$$\begin{aligned}
H &= -\frac{m}{2k}\left[\ln(mg)-\ln(mg+ku^2)\right] \\
&= \frac{m}{2k}\ln\left(\frac{mg+ku^2}{mg}\right) \quad \text{but } k=\frac{mg}{v_T^2} \\
&= \frac{mv_T^2}{2mg}\ln\left(\frac{mg+\frac{mg}{v_T^2}u^2}{mg}\right) \\
&= \frac{v_T^2}{2g}\ln\left(1+\frac{u^2}{v_T^2}\right)
\end{aligned}$$

(iii) When the ball falls: $m\dot{v}=mg-kv^2$

$m\dot{v}=mg-kv^2$

$mv\frac{dv}{dx}=mg-kv^2$

$dx=\frac{mv}{mg-kv^2}dv$

$\int_0^H dx=\int_0^w\frac{mv}{mg-kv^2}dv$

$[x]_0^H=-\frac{m}{2k}\left[\ln(mg-kv^2)\right]_0^w$

$$\begin{aligned}
H &= -\frac{m}{2k}\left[\ln(mg-kw^2)-\ln(mg)\right] \\
&= -\frac{m}{2k}\ln\left(\frac{mg-kw^2}{mg}\right) \quad \text{but } k=\frac{mg}{v_T^2} \\
&= -\frac{mv_T^2}{2mg}\ln\left(\frac{mg-\frac{mg}{v_T^2}w^2}{mg}\right) \\
&= -\frac{v_T^2}{2g}\ln\left(1-\frac{w^2}{v_T^2}\right) \\
&= -\frac{v_T^2}{2g}\ln\left(\frac{v_T^2-w^2}{v_T^2}\right) \\
&= \frac{v_T^2}{2g}\ln\left(\frac{v_T^2}{v_T^2-w^2}\right)
\end{aligned}$$

But $H=\frac{v_T^2}{2g}\ln\left(1+\frac{u^2}{v_T^2}\right)$

$=\frac{v_T^2}{2g}\ln\left(\frac{v_T^2+u^2}{v_T^2}\right)$

$\therefore\ \frac{v_T^2}{2g}\ln\left(\frac{v_T^2+u^2}{v_T^2}\right)=\frac{v_T^2}{2g}\ln\left(\frac{v_T^2}{v_T^2-w^2}\right)$

$\ln\left(\frac{v_T^2+u^2}{v_T^2}\right)=\ln\left(\frac{v_T^2}{v_T^2-w^2}\right)$

$\frac{v_T^2+u^2}{v_T^2}=\frac{v_T^2}{v_T^2-w^2}$

$v_T^4+v_T^2u^2-v_T^2w^2-u^2w^2=v_T^4$

$v_T^2u^2-v_T^2w^2-u^2w^2=0$

$v_T^2w^2+u^2w^2=v_T^2u^2$

$\frac{v_T^2w^2}{v_T^2u^2w^2}+\frac{u^2w^2}{v_T^2u^2w^2}=\frac{v_T^2u^2}{v_T^2u^2w^2}$

$\frac{1}{u^2}+\frac{1}{v_T^2}=\frac{1}{w^2}$

$\frac{1}{w^2}=\frac{1}{u^2}+\frac{1}{v_T^2}.$

Question 16

(a) (i) $P(x)=2x^3-15x^2+24x+16$

$P'(x)=6x^2-30x+24$

Turning points:

$6x^2-30x+24=0$

$x^2-5x+4=0$

$(x-4)(x-1)=0$

$x=1,4$

$P''(x)=12x-30$

$P''(1)=-18$

$P''(4)=18$

$P(4)=128-240+96+16$

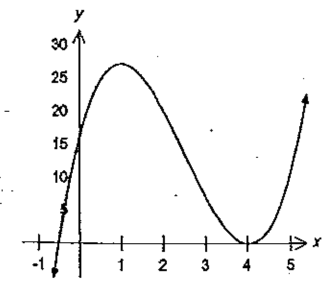
$=0$

$P(0)=16$

as $x\to\infty$, $y\to\infty$

Thus there is a local minimum at $(4,0)$ and this is less than the value at the extremities.

The minimum value for $x\ge 0$ of $P(x)$ is 0.



(ii) For $x \ge 0$:

$$2x^3 - 15x^2 + 24x + 16 \ge 0$$
$$2x^3 + 10x^2 + 24x + 16 \ge 25x^2$$
$$(x+1)(2x^2 + 8x + 16) \ge 25x^2$$
$$(x+1)(x^2 + x^2 + 8x + 16) \ge 25x^2$$
$$(x+1)\left(x^2 + (x+4)^2\right) \ge 25x^2.$$

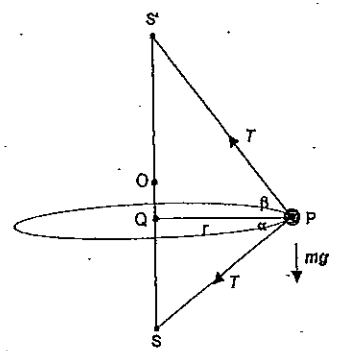
(iii) Let $x = m + n$
where $m \ge 0$ and $n \ge 0$ and so $x \ge 0$.
From part (ii):

$$(x+1)\left(x^2 + (x+4)^2\right) \ge 25x^2$$
$$(m+n+1)\left((m+n)^2 + (m+n+4)^2\right) \ge 25(m+n)^2$$
$$(m+n)^2 + (m+n+4)^2 \ge \frac{25(m+n)^2}{m+n+1}$$

But $(m-n)^2 \ge 0$

$$m^2 + n^2 \ge 2mn$$
$$(m+n)^2 \ge 4mn$$
$$(m+n)^2 + (m+n+4)^2 \ge \frac{25(m+n)^2}{m+n+1} \ge \frac{25 \times 4mn}{m+n+1} = \frac{100mn}{m+n+1}.$$

(b) (i) $PS' + PS = 2a$
$\therefore$ P lies on an ellipse with foci S' and S.



(ii) $SP = ePM$

$$= e(OM - OQ)$$
$$= e\left(\frac{a}{e} - a\cos\theta\right)$$
$$= a(1 - e\cos\theta).$$

(iii)

$$\sin\beta = \frac{QS'}{PS'} = \frac{OQ + OS'}{2a - PS} = \frac{a\cos\theta + ae}{2a - a(1 - e\cos\theta)} = \frac{\cos\theta + e}{2 - (1 - e\cos\theta)} = \frac{e + \cos\theta}{1 + e\cos\theta}.$$

(iv) Vertically: let $\angle SPQ = \alpha$
The vertical equilibrium is

$$\therefore\ T\sin\beta = T\sin\alpha + mg \qquad ①$$
$$\sin\alpha = \frac{QS}{PS} = \frac{ae - a\cos\theta}{a(1 - e\cos\theta)} = \frac{e - \cos\theta}{1 - e\cos\theta} \qquad ②$$

From ① and ②:

$$mg = T\sin\beta - T\sin\alpha$$
$$= T\left(\frac{e + \cos\theta}{1 + e\cos\theta}\right) - T\left(\frac{e - \cos\theta}{1 - e\cos\theta}\right)$$
$$= T\left(\frac{(e + \cos\theta)(1 - e\cos\theta)}{(1 + e\cos\theta)(1 - e\cos\theta)} - \frac{(e - \cos\theta)(1 + e\cos\theta)}{(1 + e\cos\theta)(1 - e\cos\theta)}\right)$$
$$= T\left(\frac{e - e^2\cos\theta + \cos\theta - e\cos^2\theta}{(1 + e\cos\theta)(1 - e\cos\theta)} - \frac{e + e^2\cos\theta - \cos\theta - e\cos^2}{(1 + e\cos\theta)(1 - e\cos\theta)}\right)$$
$$= T\left(\frac{-2e^2\cos\theta + 2\cos\theta}{1 - e^2\cos^2\theta}\right)$$

$$= \frac{2T\cos\theta(1 - e^2)}{1 - e^2\cos^2\theta}$$
$$= \frac{2T(1 - e^2)\cos\theta}{1 - e^2\cos^2\theta}.$$

(v) Horizontally $mr\omega^2 = T\cos\beta + T\cos\alpha$

$$mr\omega^2 = T\cos\beta + T\cos\alpha$$
$$= T(\cos\beta + \cos\alpha)$$
$$= T\left(\frac{PQ}{PS'} + \frac{PQ}{PS}\right)$$
$$= T \times PQ\left(\frac{1}{PS'} + \frac{1}{PS}\right)$$
$$= T \times PQ\left(\frac{PS + PS'}{PS \times PS'}\right)$$
$$= T \times PQ \times \frac{2a}{a(1 - e\cos\theta) \times a(1 + e\cos\theta)}$$
$$= T \times PQ \times \frac{2a}{a^2(1 - e^2\cos^2\theta)}$$

But $PQ = b\sin\theta$
and is the semi-minor axis.
Also $b^2 = a^2(1 - e^2)$

$$\frac{b}{a} = \sqrt{1 - e^2}$$

$$mr\omega^2 = T \times PQ \times \frac{2a}{a^2(1 - e^2\cos^2\theta)}$$
$$= T \times b\sin\theta \times \frac{2}{a(1 - e^2\cos^2\theta)}$$
$$= \frac{2Tb\sin\theta}{a(1 - e^2\cos^2\theta)}$$
$$= \frac{b}{a} \times \frac{2T\sin\theta}{(1 - e^2\cos^2\theta)}$$
$$= \frac{2T\sqrt{1 - e^2}\sin\theta}{1 - e^2\cos^2\theta}.$$

(vi) Divide the answers to (v) by (iv):

$$\frac{mr\omega^2}{mg} = \frac{2T\sqrt{1 - e^2}\sin\theta}{1 - e^2\cos^2\theta} \div \frac{2T(1 - e^2)\cos\theta}{1 - e^2\cos^2\theta}$$
$$\frac{r\omega^2}{g} = \frac{2T\sqrt{1 - e^2}\sin\theta}{2T(1 - e^2)\cos\theta}$$
$$= \frac{\tan\theta}{\sqrt{1 - e^2}}$$
$$\tan\theta = \frac{r\omega^2}{g}\sqrt{1 - e^2}.$$

End of Mathematics Extension 2 solutions