BOSTES | BOARD OF STUDIES TEACHING & EDUCATIONAL STANDARDS NSW

2014 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen. Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2–6

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 7–17

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

2630

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

1 What are the values of a, b and c for which the following identity is true?

$$\frac{5x^2-x+1}{x\left(x^2+1\right)}=\frac{a}{x}+\frac{bx+c}{x^2+1}$$

(A) $a=1,\ b=6,\ c=1$

(B) $a=1,\ b=4,\ c=1$

(C) $a=1,\ b=6,\ c=-1$

(D) $a=1,\ b=4,\ c=-1$

2 The polynomial $P(z)$ has real coefficients, and $z=2-i$ is a root of $P(z)$.

Which quadratic polynomial must be a factor of $P(z)$?

(A) z^2-4z+5

(B) z^2+4z+5

(C) z^2-4z+3

(D) z^2+4z+3

3 What is the eccentricity of the ellipse $9x^2+16y^2=25$?

(A) $\frac{7}{16}$

(B) $\frac{\sqrt{7}}{4}$

(C) $\frac{\sqrt{15}}{4}$

(D) $\frac{5}{4}$

4 Given $z=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$, which expression is equal to $(\bar{z})^{-1}$?

(A) $\frac{1}{2}\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)$

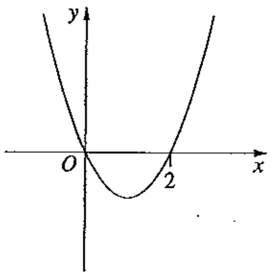
(B) $2\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)$

(C) $\frac{1}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

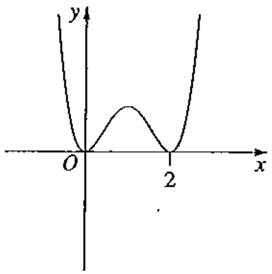
(D) $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

5 Which graph best represents the curve $y^2=x^2-2x$?

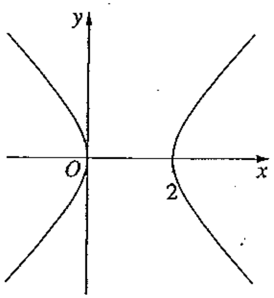
(A)



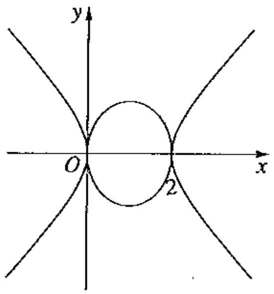
(B)



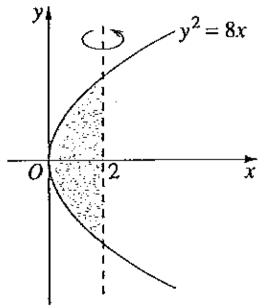
(C)



(D)



6 The region bounded by the curve $y^2=8x$ and the line $x=2$ is rotated about the line $x=2$ to form a solid.



Which expression represents the volume of the solid?

(A) $\pi\int_0^4 2^2-\left(\frac{y^2}{8}\right)^2 dy$

(B) $2\pi\int_0^4 2^2-\left(\frac{y^2}{8}\right)^2 dy$

(C) $\pi\int_0^4 \left(2-\frac{y^2}{8}\right)^2 dy$

(D) $2\pi\int_0^4 \left(2-\frac{y^2}{8}\right)^2 dy$

7 Which expression is equal to $\int \frac{1}{1-\sin x} dx$?

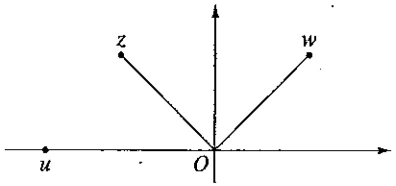
(A) $\tan x - \sec x + c$

(B) $\tan x + \sec x + c$

(C) $\log_e(1-\sin x) + c$

(D) $\frac{\log_e(1-\sin x)}{-\cos x} + c$

8 The Argand diagram shows the complex numbers w, z and u, where w lies in the first quadrant, z lies in the second quadrant and u lies on the negative real axis.



Which statement could be true?

(A) $u = zw$ and $u = z + w$

(B) $u = zw$ and $u = z - w$

(C) $z = uw$ and $u = z + w$

(D) $z = uw$ and $u = z - w$

9 A particle is moving along a straight line so that initially its displacement is $x = 1$, its velocity is $v = 2$, and its acceleration is $a = 4$.

Which is a possible equation describing the motion of the particle?

(A) $v = 2\sin(x-1) + 2$

(B) $v = 2 + 4\log_e x$

(C) $v^2 = 4(x^2 - 2)$

(D) $v = x^2 + 2x + 4$

10 Which integral is necessarily equal to $\int_{-a}^{a} f(x)\,dx$?

(A) $\int_0^a f(x) - f(-x)\,dx$

(B) $\int_0^a f(x) - f(a-x)\,dx$

(C) $\int_0^a f(x-a) + f(-x)\,dx$

(D) $\int_0^a f(x-a) + f(a-x)\,dx$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the complex numbers $z = -2 - 2i$ and $w = 3 + i$.

(i) Express $z + w$ in modulus–argument form. **2**

(ii) Express $\frac{z}{w}$ in the form $x + iy$, where x and y are real numbers. **2**

(b) Evaluate $\int_0^{\frac{1}{2}} (3x-1)\cos(\pi x)\, dx$. **3**

(c) Sketch the region in the Argand diagram where $|z| \le |z-2|$ and $-\frac{\pi}{4} \le \arg z \le \frac{\pi}{4}$. **3**

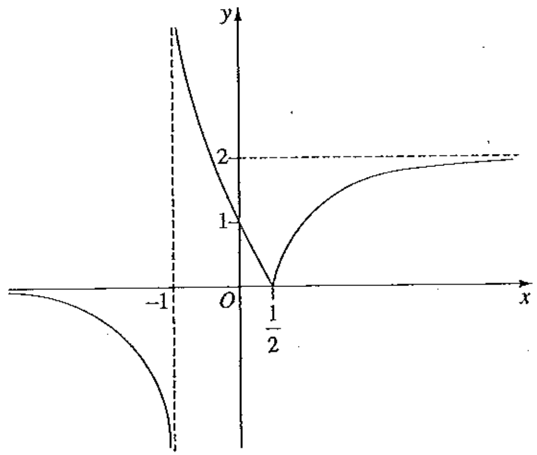
(d) Without the use of calculus, sketch the graph $y = x^2 - \frac{1}{x^2}$, showing all intercepts. **2**

(e) The region enclosed by the curve $x = y(6-y)$ and the y-axis is rotated about the x-axis to form a solid. **3**

Using the method of cylindrical shells, or otherwise, find the volume of the solid.

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the graph of a function $f(x)$.



Draw a separate half-page graph for each of the following functions, showing all asymptotes and intercepts.

(i) $y = f(|x|)$ **2**

(ii) $y = \frac{1}{f(x)}$ **2**

(b) It can be shown that $4\cos^3\theta - 3\cos\theta = \cos 3\theta$. (Do NOT prove this.)

Assume that $x = 2\cos\theta$ is a solution of $x^3 - 3x = \sqrt{3}$.

(i) Show that $\cos 3\theta = \frac{\sqrt{3}}{2}$. **1**

(ii) Hence, or otherwise, find the three real solutions of $x^3 - 3x = \sqrt{3}$. **2**

Question 12 continues on page 9

(c) The point $P(x_0, y_0)$ lies on the curves $x^2 - y^2 = 5$ and $xy = 6$. **3**

Prove that the tangents to these curves at P are perpendicular to one another.

(d) Let $I_n = \displaystyle\int_0^1 \frac{x^{2n}}{x^2+1}\,dx$, where n is an integer and $n \geq 0$.

(i) Show that $I_0 = \dfrac{\pi}{4}$. **1**

(ii) Show that $I_n + I_{n-1} = \dfrac{1}{2n-1}$. **2**

(iii) Hence, or otherwise, find $\displaystyle\int_0^1 \frac{x^4}{x^2+1}\,dx$. **2**

End of Question 12

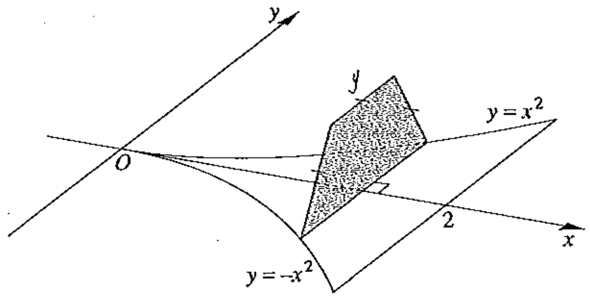
Please turn over

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Using the substitution $t = \tan\dfrac{x}{2}$, or otherwise, evaluate **3**

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{3\sin x - 4\cos x + 5}\,dx.$$

(b) The base of a solid is the region bounded by $y = x^2$, $y = -x^2$ and $x = 2$. Each cross-section perpendicular to the x-axis is a trapezium, as shown in the diagram. The trapezium has three equal sides and its base is twice the length of any one of the equal sides. **4**



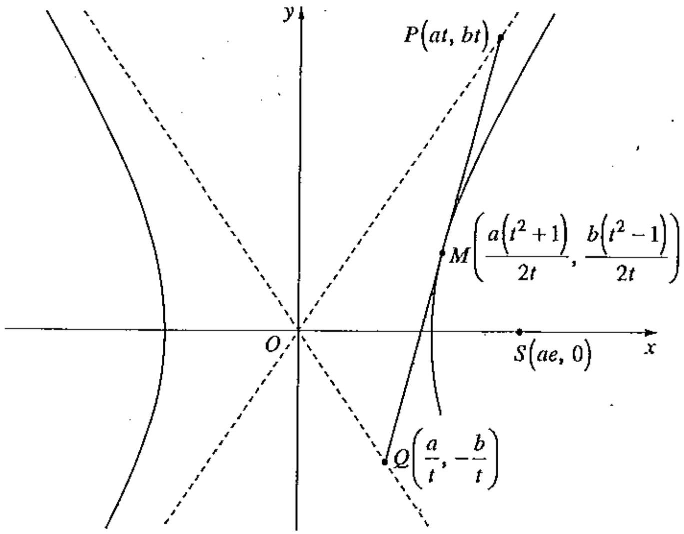
Find the volume of the solid.

Question 13 continues on page 11

(c) The point $S(ae, 0)$ is the focus of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ on the positive x-axis.

The points $P(at, bt)$ and $Q\left(\dfrac{a}{t}, -\dfrac{b}{t}\right)$ lie on the asymptotes of the hyperbola, where $t > 0$.

The point $M\left(\dfrac{a(t^2+1)}{2t}, \dfrac{b(t^2-1)}{2t}\right)$ is the midpoint of PQ.



(i) Show that M lies on the hyperbola. **1**

(ii) Prove that the line through P and Q is a tangent to the hyperbola at M. **3**

(iii) Show that $OP \times OQ = OS^2$. **2**

(iv) If P and S have the same x-coordinate, show that MS is parallel to one of the asymptotes of the hyperbola. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

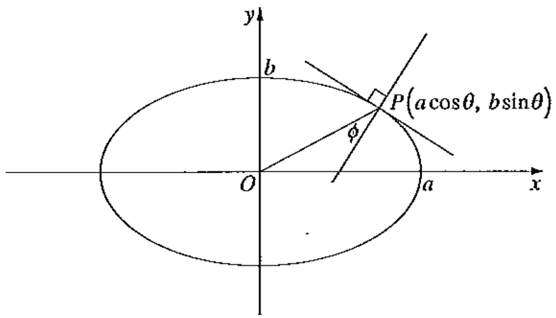
(a) Let $P(x) = x^5 - 10x^2 + 15x - 6$.

(i) Show that $x = 1$ is a root of $P(x)$ of multiplicity three. **2**

(ii) Hence, or otherwise, find the two complex roots of $P(x)$. **2**

(b) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a > b$.

The acute angle between OP and the normal to the ellipse at P is ϕ.



(i) Show that $\tan\phi = \left(\dfrac{a^2 - b^2}{ab}\right)\sin\theta\cos\theta$. **3**

(ii) Find a value of θ for which ϕ is a maximum. **2**

Question 14 continues on page 13

(c) A high speed train of mass m starts from rest and moves along a straight track. At time t hours, the distance travelled by the train from its starting point is x km, and its velocity is v km/h.

The train is driven by a constant force F in the forward direction. The resistive force in the opposite direction is Kv^2, where K is a positive constant. The terminal velocity of the train is 300 km/h.

(i) Show that the equation of motion for the train is **2**

$$m\ddot{x} = F\left[1 - \left(\frac{v}{300}\right)^2\right].$$

(ii) Find, in terms of F and m, the time it takes the train to reach a velocity of 200 km/h. **4**

End of Question 14

Please turn over

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) Three positive real numbers a, b and c are such that $a + b + c = 1$ and $a \le b \le c$. **2**

By considering the expansion of $(a + b + c)^2$, or otherwise, show that

$$5a^2 + 3b^2 + c^2 \le 1.$$

(b) (i) Using de Moivre's theorem, or otherwise, show that for every positive integer n, **2**

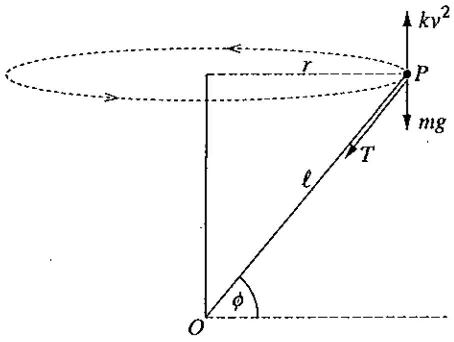
$$(1+i)^n + (1-i)^n = 2\left(\sqrt{2}\right)^n \cos\frac{n\pi}{4}.$$

(ii) Hence, or otherwise, show that for every positive integer n divisible by 4, **3**

$$\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots + \binom{n}{n} = (-1)^{\frac{n}{4}}\left(\sqrt{2}\right)^n.$$

Question 15 continues on page 15

(c) A toy aeroplane P of mass m is attached to a fixed point O by a string of length ℓ. The string makes an angle ϕ with the horizontal. The aeroplane moves in uniform circular motion with velocity v in a circle of radius r in a horizontal plane.



The forces acting on the aeroplane are the gravitational force mg, the tension force T in the string and a vertical lifting force kv^2, where k is a positive constant.

(i) By resolving the forces on the aeroplane in the horizontal and the vertical directions, show that $\dfrac{\sin\phi}{\cos^2\phi} = \dfrac{\ell k}{m} - \dfrac{\ell g}{v^2}$. **3**

(ii) Part (i) implies that $\dfrac{\sin\phi}{\cos^2\phi} < \dfrac{\ell k}{m}$. (Do NOT prove this.) **2**

Use this to show that

$$\sin\phi < \frac{\sqrt{m^2 + 4\ell^2 k^2} - m}{2\ell k}.$$

(iii) Show that $\dfrac{\sin\phi}{\cos^2\phi}$ is an increasing function of ϕ for $-\dfrac{\pi}{2} < \phi < \dfrac{\pi}{2}$. **2**

(iv) Explain why ϕ increases as v increases. **1**

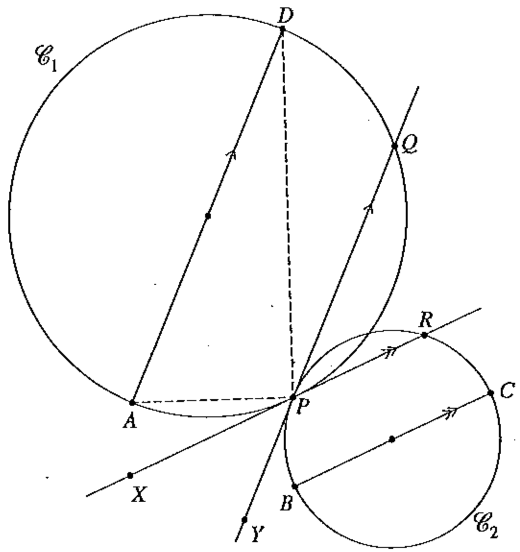
End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows two circles $\mathscr{C}_1$ and $\mathscr{C}_2$. The point P is one of their points of intersection. The tangent to $\mathscr{C}_2$ at P meets $\mathscr{C}_1$ at Q, and the tangent to $\mathscr{C}_1$ at P meets $\mathscr{C}_2$ at R.

The points A and D are chosen on $\mathscr{C}_1$ so that AD is a diameter of $\mathscr{C}_1$ and parallel to PQ. Likewise, points B and C are chosen on $\mathscr{C}_2$ so that BC is a diameter of $\mathscr{C}_2$ and parallel to PR.

The points X and Y lie on the tangents PR and PQ, respectively, as shown in the diagram.



Copy or trace the diagram into your writing booklet.

(i) Show that $\angle APX = \angle DPQ$. **2**

(ii) Show that A, P and C are collinear. **3**

(iii) Show that $ABCD$ is a cyclic quadrilateral. **1**

Question 16 continues on page 17

Question 16 (continued)

(b) Suppose n is a positive integer.

(i) Show that **3**

$$-x^{2n} \le \frac{1}{1+x^2} - \left(1 - x^2 + x^4 - x^6 + \cdots + (-1)^{n-1} x^{2n-2}\right) \le x^{2n}.$$

(ii) Use integration to deduce that **2**

$$-\frac{1}{2n+1} \le \frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{n-1}\frac{1}{2n-1}\right) \le \frac{1}{2n+1}.$$

(iii) Explain why $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$. **1**

(c) Find $\int \frac{\ln x}{(1+\ln x)^2}\,dx$. **3**

End of paper

2014 Higher School Certificate Solutions Mathematics Extension 2

SECTION I

Summary

1	D	3	B	5	C	7	B	9	A
2	A	4	C	6	D	8	B	10	D

SECTION I

1 (D) Using partial fractions:

$$\frac{5x^2-x+1}{x(x^2+1)}\equiv\frac{a}{x}+\frac{bx+c}{x^2+1}$$

$$5x^2-x+1\equiv a(x^2+1)+(bx+c)x$$
$$\equiv(a+b)x^2+cx+a$$

Equating coefficients:

$c=-1,\quad a=1$

$a+b=5$

$1+b=5$

$b=4$

$\therefore a=1, b=4, c=-1.$

2 (A) If $z=2-i$ is a root of $P(z)$,
then $\overline{z}=2+i$ is also a root.
Sum of the roots is $z+\overline{z}=4$.
Product of the roots is $z\overline{z}=2^2-i^2=5$.
The quadratic equation with these roots is z^2-4z+5.

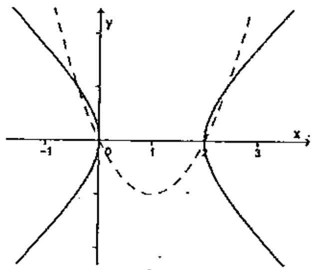
3 (B) $b^2=a^2\left(1-e^2\right)$

$$e^2=1-\frac{b^2}{a^2}$$
$$=1-\frac{\frac{25}{16}}{\frac{25}{9}}$$
$$=1-\frac{9}{16}$$
$$e^2=\frac{7}{16}$$
$$e=\frac{\sqrt{7}}{4}.$$

4 (C) $z=2cis\left(\frac{\pi}{3}\right)$

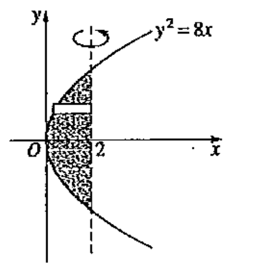
$$\overline{z}=2cis\left(-\frac{\pi}{3}\right)$$
$$\left(\overline{z}\right)^{-1}=\left(2cis\left(-\frac{\pi}{3}\right)\right)^{-1}$$
$$=2^{-1}cis\left(\frac{\pi}{3}\right)$$
$$=\frac{1}{2}cis\left(\frac{\pi}{3}\right)$$
$$=\frac{1}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right).$$

5 (C)



Consider $y=x^2-2x$
It is the dashed curve above.
For $y^2=x^2-2x$
$$y=\pm\sqrt{x^2-2x}$$
This curve is undefined for $0<x<2$.

6 (D)

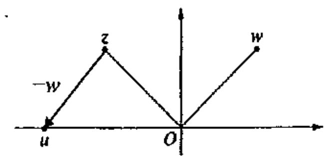


$$\delta V=\pi r^2\delta y$$
$$V=\int_{-4}^{4}\pi(2-x)^2\,dy$$
$$=2\pi\int_0^4\left(2-\frac{y}{8}\right)^2\,dy$$

7 (B) $\displaystyle\int\frac{dx}{1-\sin x}=\int\frac{(1+\sin x)}{(1-\sin x)(1+\sin x)}\,dx$

$$=\int\frac{(1+\sin x)}{\cos^2 x}\,dx$$
$$=\int\sec^2 x+\tan x\sec x\,dx$$
$$=\tan x+\sec x+c.$$

8 (B)



$u=z-w$ (add the negative of w)
$u=zw$ (add the arguments)
Thus it is answer B.

9 (A) For $x=1$:
$v=2$ for (A) and (B)
(C) is undefined and (D) is 7.

For answer (A):
$$a=v\frac{dv}{dx}$$
$$=(2\sin(x-1)+2)\times2\cos(x-1).1$$
$$=4\cos(1-1)[\sin(1-1)+1]$$
$$=4$$

For answer (B):
$$a=v\frac{dv}{dx}$$
$$=(2+4\log_e x)\times\frac{4}{x}$$
$$=(2+4\times0)\times\frac{4}{1}$$
$$=8$$
Thus it is answer (A).

10 (D) $\displaystyle\int_{-a}^{a}f(x)\,dx=\int_{-a}^{0}f(x)\,dx+\int_0^a f(x)$

$$=I_1+I_2$$

For I_1:
let $u=x+a\Rightarrow x=u-a$ and $dx=d$
$$I_1=\int_{-a}^{0}f(x)\,dx$$
$$=\int_0^a f(u-a)\,du$$
$$=\int_0^a f(x-a)\,dx$$

For I_2:
let $u=a-x\Rightarrow x=a-u$ and $dx=-$
$$I_2=\int_0^a f(x)\,dx$$
$$=\int_a^0 f(a-u)-du$$
$$=\int_0^a f(a-u)\,du$$
$$=\int_0^a f(a-x)\,dx$$
$$I_1+I_2=\int_0^a f(x-a)\,dx+\int_0^a f(a-$$
$$=\int_0^a f(x-a)+f(a-x)\,dx.$$

SECTION II

Question 11

(a) (i)
$$z+w=-2-2i+3+i$$
$$=1-i$$
$$|z+w|=\sqrt{2}$$
$$\arg(z+w)=-\frac{\pi}{4}$$
Hence $z+w=\sqrt{2}\,cis\left(-\frac{\pi}{4}\right)$.

(ii) $\frac{z}{w}=\frac{-2-2i}{3+i}\times\frac{3-i}{3-i}$

$=\frac{-8-4i}{9+1}$

$=-\frac{4}{5}-\frac{2}{5}i$

(b) Let $I=\int_0^{\frac{1}{2}}(3x-1)\cos(\pi x)\,dx$

$u=3x-1 \qquad du=3dx$

$dv=\cos(\pi x) \qquad v=\frac{1}{\pi}\sin(\pi x)$

$I=\frac{1}{\pi}\left[(3x-1)\sin(\pi x)\right]_0^{\frac{1}{2}}-\frac{3}{\pi}\int_0^{\frac{1}{2}}\sin(\pi x)\,dx$

$=\frac{1}{\pi}\left[\frac{1}{2}\right]-\frac{3}{\pi}\cdot\frac{1}{\pi}\left[-\cos(\pi x)\right]_0^{\frac{1}{2}}$

$=\frac{1}{2\pi}-\frac{3}{\pi^2}$

(c) Let $z=x+iy$

$|z|=\sqrt{x^2+y^2}$

$z-2=x+iy-2$

$=x-2+iy$

$|z-2|=\sqrt{(x-2)^2+y^2}$

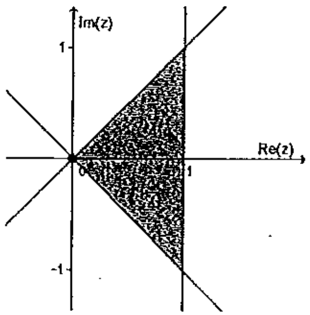
$|z|\le|z-2|$

$\sqrt{x^2+y^2}\le\sqrt{(x-2)^2+y^2}$

$x^2+y^2\le x^2-4x+4+y^2$

$x\le 1$

$-\frac{\pi}{4}\le\arg z\le\frac{\pi}{4}$ is the region between $y=x$ and $y=-x$

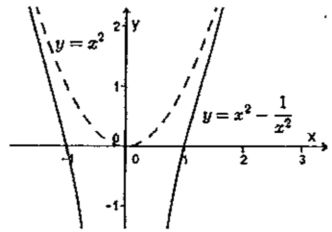


(d) $y=x^2-\frac{1}{x^2}$ is an even function.

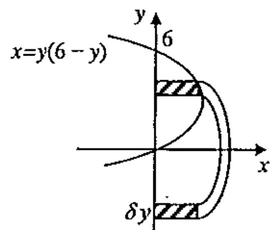
Its domain is all $x, x\neq 0$.

x-intercepts: $x=\pm 1$.

$\lim_{x\to 0^+}\left(x^2-\frac{1}{x^2}\right)=-\infty$.



(e)



$\delta V=2\pi y.y(6-y)\delta y$

$V=2\pi\int_0^6(6y^2-y^3)\,dy$

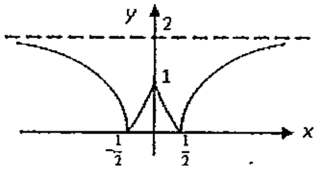
$=2\pi\left[2y^3-\frac{1}{4}y^4\right]_0^6$

$=2\pi\left[(432-324)-(0)\right]$

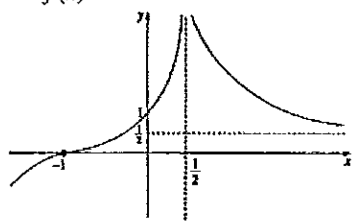
$=216\pi$ units^3.

Question 12

(a) (i) $y=f(|x|)$



(ii) $y=\frac{1}{f(x)}$



(b) (i) Substitute $x=2\cos\theta$ in

$x^3-3x=\sqrt{3}$

$8\cos^3\theta-6\cos\theta=\sqrt{3}$

$4\cos^3\theta-3\cos\theta=\frac{\sqrt{3}}{2}$

$\therefore\ \cos 3\theta=\frac{\sqrt{3}}{2}$.

(ii) $\cos 3\theta=\frac{\sqrt{3}}{2}$ (3 solutions)

$3\theta=\frac{\pi}{6},\frac{11\pi}{6},\frac{13\pi}{6}$

$\theta=\frac{\pi}{18},\frac{11\pi}{18},\frac{13\pi}{18}$

$x=2\cos\frac{\pi}{18},2\cos\frac{11\pi}{18},2\cos\frac{13\pi}{18}$.

(c) For $x^2-y^2=5$

$2x-2y\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{x}{y}$

At $P(x_0,y_0)$:

$\frac{dy}{dx}=\frac{x_0}{y_0}$

For $xy=6$

$1y+1\frac{dy}{dx}x=0$

$\frac{dy}{dx}=-\frac{y}{x}$

At $P(x_0,y_0)$:

$\frac{dy}{dx}=-\frac{y_0}{x_0}$

Now $\frac{x_0}{y_0}\times-\frac{y_0}{x_0}=-1$

$\therefore$ The tangents are perpendicular.

(d) (i) $I_0=\int_0^1\frac{x^{2(0)}}{x^2+1}\,dx$

$=\int_0^1\frac{1}{x^2+1}\,dx$

$=\left[\tan^{-1}x\right]_0^1$

$=\frac{\pi}{4}$.

(ii) $I_n+I_{n-1}=\int_0^1\left(\frac{x^{2n}}{x^2+1}+\frac{x^{2(n-1)}}{x^2+1}\right)dx$

$=\int_0^1\frac{x^{2n-2}(x^2+1)}{x^2+1}\,dx$

$=\int_0^1 x^{2n-2}\,dx$

$=\frac{1}{2n-1}\left[x^{2n-1}\right]_0^1$

$=\frac{1}{2n-1}(1-0)$

$=\frac{1}{2n-1}$.

(iii) *Method 1*

$\int_0^1\frac{x^4}{x^2+1}\,dx=I_2$

From (ii):

$I_n+I_{n-1}=\frac{1}{2n-1}$

$I_2+I_1=\frac{1}{2(2)-1}=\frac{1}{3}$

$I_2=\frac{1}{3}-I_1$

$I_1+I_0=\frac{1}{2(1)-1}=1$

$I_1=1-I_0$

But from (i) $I_0=\dfrac{\pi}{4}$

$$I_1=1-\frac{\pi}{4}$$

$$I_2=\frac{1}{3}-I_1$$

$$=\frac{1}{3}-\left(1-\frac{\pi}{4}\right)$$

$$=\frac{\pi}{4}-\frac{2}{3}$$

OR

Method 2 "otherwise solution"
By division:

$$\frac{x^4}{x^2+1}\equiv x^2-1+\frac{1}{x^2+1}$$

$$\int_0^1\frac{x^4}{x^2+1}\,dx=\int_0^1 x^2-1+\frac{1}{x^2+1}\,dx$$

$$=\left[\frac{x^3}{3}-x+\tan^{-1}x\right]_0^1$$

$$=\frac{1}{3}-1+\frac{\pi}{4}-(0-0+0)$$

$$=\frac{\pi}{4}-\frac{2}{3}.$$

Question 13

(a)

$$t=\tan\frac{x}{2},\quad \sin x=\frac{2t}{1+t^2}$$

$$dx=\frac{2dt}{1+t^2},\quad \cos x=\frac{1-t^2}{1+t^2}$$

when $x=\dfrac{\pi}{3}$, $t=\tan\dfrac{\pi}{6}=\dfrac{1}{\sqrt{3}}$

when $x=\dfrac{\pi}{2}$, $t=\tan\dfrac{\pi}{4}=1$

$$I=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{1}{3\sin x-4\cos x+5}\,dx$$

$$=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{1}{3\times\frac{2t}{1+t^2}-4\times\frac{1-t^2}{1+t^2}+5}\,\frac{2dt}{1+t^2}$$

$$I=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{1}{6t-4+4t^2+5+5t^2}\,2dt$$

$$=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{2dt}{9t^2+6t+1}$$

$$=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{2dt}{(3t+1)^2}$$

$$=-\frac{2}{3}\left[\frac{1}{3t+1}\right]_{\frac{1}{\sqrt{3}}}^{1}$$

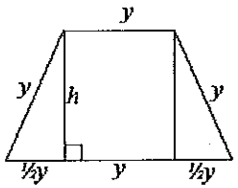
$$=-\frac{2}{3}\left(\frac{1}{4}-\frac{1}{\sqrt{3}+1}\right)$$

$$=-\frac{2}{3}\left(\frac{1}{4}-\frac{\sqrt{3}-1}{2}\right)$$

$$=-\frac{1}{6}+\frac{\sqrt{3}}{3}-\frac{1}{3}$$

$$=\frac{2\sqrt{3}-3}{6}.$$

(b)



$$h^2=y^2-\left(\frac{y}{2}\right)^2$$

$$=y^2-\frac{y^2}{4}$$

$$=\frac{3y^2}{4}$$

$$h=\frac{\sqrt{3}}{2}y$$

$$\text{Area}=\frac{1}{2}\times\frac{\sqrt{3}}{2}y(y+2y)$$

$$=\frac{\sqrt{3}}{4}\times3y^2\quad\text{but } y=x^2$$

$$=\frac{3\sqrt{3}x^4}{4}$$

$$\delta V=\frac{3\sqrt{3}x^4}{4}\delta x$$

$$V=\int_0^2\frac{3\sqrt{3}x^4}{4}\,dx$$

$$=\frac{3\sqrt{3}}{4}\left[\frac{x^5}{5}\right]_0^2$$

$$=\frac{3\sqrt{3}}{4}\left(\frac{32}{5}-0\right)$$

$$=\frac{24\sqrt{3}}{5}\text{ units}^3.$$

(c) (i) Substitute M into the equation:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{\left(\frac{a(t^2+1)}{2t}\right)^2}{a^2}-\frac{\left(\frac{b(t^2-1)}{2t}\right)^2}{b^2}$$

$$=\frac{(t^4+2t^2+1)}{4t^2}-\frac{(t^4-2t^2+1)}{4t^2}$$

$$=\frac{4t^2}{4t^2}$$

$$=1$$

$\therefore M$ satisfies the equation of the hyperbola.

(ii)

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}-\frac{2y\frac{dy}{dx}}{b^2}=0$$

$$\frac{dy}{dx}=\frac{b^2x}{a^2y}$$

$$m_{\text{at } M}=\frac{b^2\times\frac{a(t^2+1)}{2t}}{a^2\times\frac{b(t^2-1)}{2t}}$$

$$=\frac{b(t^2+1)}{a(t^2-1)}$$

$$m_{PQ}=\frac{bt-\frac{-b}{t}}{at-\frac{a}{t}}$$

$$=\frac{\frac{bt^2+b}{t}}{\frac{at^2-a}{t}}$$

$$=\frac{b(t^2+1)}{a(t^2-1)}$$

Thus the line through PQ is a tangent To the hyperbola at M.

(iii)

$$OP\times OQ=\sqrt{b^2t^2+a^2t^2}\times\sqrt{\frac{b^2}{t^2}+\frac{a^2}{t^2}}$$

$$=t\sqrt{b^2+a^2}\times\frac{1}{t}\sqrt{b^2+a^2}$$

$$=a^2+b^2$$

$$=a^2+a^2(e^2-1)$$

$$=a^2e^2$$

$$=OS^2.$$

(iv) For P and S to have the same x-coordinate $ae=at\ \Rightarrow\ e=t$.

$$m_{MS}=\frac{\frac{b(e^2-1)}{2e}-0}{\frac{a(e^2+1)}{2e}-ae}$$

$$=\frac{b(e^2-1)}{2e}\div\left(\frac{a(e^2+1)-2ae^2}{2e}\right)$$

$$=\frac{b(e^2-1)}{2e}\div\left(\frac{a(1-e^2)}{2e}\right)$$

$$=\frac{b(e^2-1)}{2e}\times\left(\frac{-2e}{a(e^2-1)}\right)$$

$$=-\frac{b}{a}$$

For the hyperbola, the equation of the asymptotes are $y=\pm\dfrac{b}{a}x$. These have gradients $=\pm\dfrac{b}{a}$.

$\therefore MS$ is parallel to one of the asymptotes

Question 14

(a) (i) $P(x) = x^5 - 10x^2 + 15x - 6$

$P'(x) = 5x^4 - 20x + 15$

$P''(x) = 20x^3 - 20$

$P'''(x) = 60x^2$

$P(1) = (1)^5 - 10(1)^2 + 15(1) - 6$
$= 0$

$P'(1) = 5(1)^4 - 20(1) + 15$
$= 0$

$P''(1) = 20(1)^3 - 20$
$= 0$

$P'''(1) = 60(1)^2$
$\neq 0$

$\therefore x = 1$ is a root of multiplicity 3.

(ii) Let the other roots be α and β.
The sum of the roots is:

$$\alpha + \beta + 1 + 1 + 1 = \frac{-b}{a}$$
$$= 0$$
$$\alpha + \beta = -3$$

The product of the roots is:

$$\alpha\beta(1)(1)(1) = -\frac{f}{a}$$
$$= -\frac{-6}{1}$$
$$\alpha\beta = 6$$

Hence α and β satisfy the quadratic:

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0$$
$$x^2 + 3x + 6 = 0$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(6)}}{2}$$
$$= \frac{-3 \pm \sqrt{-15}}{2}$$
$$= \frac{-3 \pm i\sqrt{15}}{2}$$

These are the complex roots of $P(x)$.

(b) (i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$
$$\frac{2y}{b^2}\frac{dy}{dx} = -\frac{2x}{a^2}$$
$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

At $P(a\cos\theta, b\sin\theta)$:

$$m_T = -\frac{b^2 a\cos\theta}{a^2 b\sin\theta}$$
$$= -\frac{b\cos\theta}{a\sin\theta}$$
$$m_N = \frac{a\sin\theta}{b\cos\theta}$$
$$= \frac{a}{b}\tan\theta$$
$$m_{OP} = \frac{b\sin\theta - 0}{a\cos\theta - 0}$$
$$= \frac{b}{a}\tan\theta$$

ϕ is the acute angle between OP and the normal at P:

$$\tan\phi = \left|\frac{m_N - m_{OP}}{1 + m_N m_{OP}}\right|$$
$$= \left|\frac{\frac{a}{b}\tan\theta - \frac{b}{a}\tan\theta}{1 + \frac{a}{b}\tan\theta \times \frac{b}{a}\tan\theta}\right|$$
$$= \left|\frac{\tan\theta\left(\frac{a^2-b^2}{ba}\right)}{1+\tan^2\theta}\right|$$
$$= \left|\frac{\tan\theta\left(\frac{a^2-b^2}{ba}\right)}{\sec^2\theta}\right|$$
$$= \left|\frac{\sin\theta}{\cos\theta}\left(\frac{a^2-b^2}{ba}\right)\cos^2\theta\right|$$
$$= \left(\frac{a^2-b^2}{ba}\right)\sin\theta\cos\theta.$$

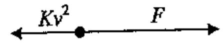
(b) (ii)

$$\tan\phi = \left(\frac{a^2-b^2}{ba}\right)\sin\theta\cos\theta$$
$$= \left(\frac{a^2-b^2}{ba}\right)\frac{1}{2}\times 2\sin\theta\cos\theta$$
$$= \frac{1}{2}\left(\frac{a^2-b^2}{ba}\right)\sin 2\theta$$

Since a and b are constant, ϕ will be a maximum when $\sin 2\theta$ is a maximum. The maximum value of $\sin 2\theta$ is 1.

This occurs when $\theta = \dfrac{\pi}{4}$.

(c) (i)



The resultant force is $m\ddot{x} = F - Kv^2$
The terminal velocity of $v = 300$ occurs when $\ddot{x} = 0$.

$$m \times 0 = F - K300^2$$
$$K = \frac{F}{300^2}$$

Thus $m\ddot{x} = F - \dfrac{F}{300^2}v^2$

$$= F\left[1 - \left(\frac{v}{300}\right)^2\right].$$

(ii)

$$m\ddot{x} = F\left[1 - \left(\frac{v}{300}\right)^2\right]$$
$$m\frac{dv}{dt} = F\left[\frac{300^2 - v^2}{300^2}\right]$$
$$\int_0^{200} \frac{300^2}{300^2 - v^2}\,dv = \frac{F}{m}\int_0^T dt$$
$$\int_0^T dt = \frac{300m}{F}\int_0^{200} \frac{300}{(300-v)(300+v)}\,dv$$

$$T = \frac{300m}{F}\int_0^{200} \frac{\frac{1}{2}}{300+v} + \frac{\frac{1}{2}}{300-v}\,dv$$
$$= \frac{150m}{F}\int_0^{200} \frac{1}{300+v} + \frac{1}{300-v}\,dv$$
$$= \frac{150m}{F}\left[\ln\left(\frac{300+v}{300-v}\right)\right]_0^{200}$$
$$= \frac{150m}{F}[\ln 5 - \ln 1]$$
$$= \frac{150m\ln 5}{F}.$$

Question 15

(a) Note: $a > 0, b > 0, c > 0$ and $a \le b \le c$

Thus: $2ab \ge 2a^2, 2ac \ge 2a^2, 2bc \ge 2b^2$

$$(a+b+c)^2 = 1$$
$$a^2 + b^2 + c^2 + 2ab + 2ac + 2bc = 1$$
$$a^2 + b^2 + c^2 + 2a^2 + 2a^2 + 2b^2 \le 1$$
$$5a^2 + 3b^2 + c^2 \le 1.$$

(b) (i)

$$1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$$
$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$
$$(1+i)^n = \left(\sqrt{2}\right)^n\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) \quad ①$$

$$1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$$
$$= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$
$$= \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$
$$(1-i)^n = \left(\sqrt{2}\right)^n\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right) \quad ②$$

①+②

$$(1+i)^n + (1-i)^n = \left(\sqrt{2}\right)^n\left(2\cos\frac{n\pi}{4}\right)$$
$$= 2\left(\sqrt{2}\right)^n\cos\frac{n\pi}{4}.$$

(ii) $(1+i)^n+(1-i)^n$

$=\sum_{k=0}^{n}\binom{n}{k}i^k+\sum_{k=0}^{n}(-1)^k\binom{n}{k}i^k$

$=\sum_{k=0}^{n}\binom{n}{k}i^k+(-1)^k\binom{n}{k}i^k$

$=\sum_{k=0}^{n}\binom{n}{k}i^k\left(1+(-1)^k\right)$

$=\sum_{k=0}^{n}2\binom{n}{k}i^k$ (k even)

The result when k is odd is 0.

Now equating parts (i) and (ii):

$\sum_{k=0}^{n}2\binom{n}{k}i^k=2\left(\sqrt{2}\right)^n\left(\cos\frac{n\pi}{4}\right)$

$\sum_{k=0}^{n}\binom{n}{k}i^k=\left(\sqrt{2}\right)^n\left(\cos\frac{n\pi}{4}\right)$

Since k is even:

$\sum_{k=0}^{n}\binom{n}{k}i^k=\binom{n}{0}i^0+\binom{n}{2}i^2+..+\binom{n}{n}i^n$

$=\binom{n}{0}-\binom{n}{2}+\binom{n}{4}-...+\binom{n}{n}$

When n is a multiple of 4:

$\binom{n}{0}-\binom{n}{2}+\binom{n}{4}-\binom{n}{6}+...+\binom{n}{n}$

$=\left(\sqrt{2}\right)^n\left(\cos\frac{n\pi}{4}\right)$

$=(-1)^{\frac{n}{4}}\left(\sqrt{2}\right)^n.$

(c) (i) Resolving the forces vertically:

$kv^2=mg+T\sin\phi$ ①

Resolving the forces horizontally:

$T\cos\phi=\frac{mv^2}{r}$

$T=\frac{mv^2}{r\cos\phi}$ but $r=\ell\cos\phi$

$T=\frac{mv^2}{\ell\cos^2\phi}$ ②

Substitute ② in ①

$kv^2=mg+\frac{mv^2}{\ell\cos^2\phi}\sin\phi$

$\ell kv^2=\ell mg+\frac{mv^2\sin\phi}{\cos^2\phi}$

$\frac{mv^2\sin\phi}{\cos^2\phi}=\ell kv^2-\ell mg$

$\frac{\sin\phi}{\cos^2\phi}=\frac{\ell k}{m}-\frac{\ell g}{v^2}.$

(ii) $\frac{\sin\phi}{\cos^2\phi}<\frac{\ell k}{m}$

$m\sin\phi<\ell k\cos^2\phi$

$\frac{m}{\ell k}\sin\phi<\cos^2\phi$

$\frac{m}{\ell k}\sin\phi<1-\sin^2\phi$

$\sin^2\phi+\frac{m}{\ell k}\sin\phi<1$

$\sin^2\phi+\frac{m}{lk}\sin\phi+\left(\frac{m}{2\ell k}\right)^2<1+\left(\frac{m}{2\ell k}\right)^2$

$\left(\sin\phi+\frac{m}{2\ell k}\right)^2<\frac{4\ell^2k^2+m^2}{4\ell^2k^2}$

Looking at all solutions:

$-\frac{\sqrt{4\ell^2k^2+m^2}}{2\ell k}<\sin\phi+\frac{m}{2\ell k}<\frac{\sqrt{4\ell^2k^2+m^2}}{2\ell k}$

Only set of solutions required as ϕ is acute:

$\sin\phi<-\frac{m}{2\ell k}+\frac{\sqrt{4\ell^2k^2+m^2}}{2\ell k}$

$\sin\phi<\frac{\sqrt{4\ell^2k^2+m^2}-m}{2\ell k}.$

(iii) $\frac{d}{d\phi}\left(\frac{\sin\phi}{\cos^2\phi}\right)$

$=\frac{\cos\phi\cos^2\phi-2\cos\phi(-\sin\phi)\sin\phi}{\cos^4\phi}$

$=\frac{\cos^3\phi+2\cos\phi\sin^2\phi}{\cos^4\phi}$

$=\frac{\cos^2\phi+2\sin^2\phi}{\cos^3\phi}$

Since $-\frac{\pi}{2}<\phi<\frac{\pi}{2}$ are in the 1st and 4th quadrants $\therefore$ $\cos\phi>0$

Hence $\cos^3\phi>0$ and $\sin^2\phi\geq 0$.

$\therefore\frac{\cos^2\phi+2\sin^2\phi}{\cos^3\phi}>0$ and hence

$\frac{\sin\phi}{\cos^2\phi}$ is increasing for $-\frac{\pi}{2}<\phi<\frac{\pi}{2}$.

(iv) Consider:

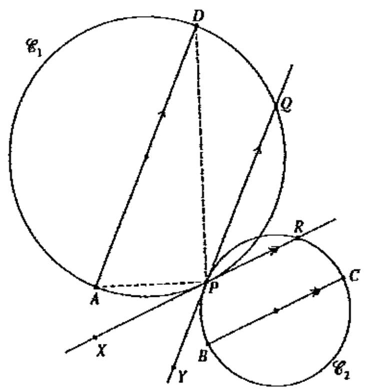
$\frac{\sin\phi}{\cos^2\phi}=\frac{lk}{m}-\frac{lg}{v^2}$

As v increases $\frac{\sin\phi}{\cos^2\phi}\to\frac{lk}{m}$

and since $\frac{\sin\phi}{\cos^2\phi}$ is increasing from part (iii), ϕ must increase.

Question 16

(a) (i)



$\angle APX=\angle ADP$

(Alternate Segment Theorem)

$\angle ADP=\angle DPQ$ (alt. $\angle$s, AD||PQ)

Thus $\angle APX=\angle DPQ$.

(ii) $\angle APX=\angle DPQ$ (from part (i))

Similarly $\angle BPY=\angle CPR$

$\angle APD=\angle BPC=90°$ ($\angle$ in semicircle)

$2\angle APX+90°=2\angle CPR+90°$

$\angle XPQ=\angle YPR$ (vert. opp. $\angle$s)

$\therefore\angle APX=\angle CPR$

These angles are in the position for vertically opposite angles, thus they are two straight lines.

$\therefore A,P,C$ are collinear.

(iii) $\angle ADP=\angle APX$ (from part (i))

$\angle CPR=\angle BCP$ (alt. $\angle$s, $XR \parallel BC$)

$\angle APX=\angle CPR$

(vert. opp. $\angle$s, APC collinear)

$\therefore\angle ADP=\angle CPR=\angle BCP$

$\therefore\angle ADB=\angle BCA$

(same angles as $\angle ADP$, $\angle BCP$)

These angles are in the positions for equal angles in same segment on chord AB).

$\therefore ABCD$ is a cyclic quadrilateral.

(b) (i) $1-x^2+x^4-x^6+...+(-1)^{n-1}x^{2n-2}$

is a G.P. with $a=1,r=-x^2,n$ terms.

$S_n=\frac{1[1-(-x^2)^n]}{1--x^2}$

$=\frac{1-(-x^2)^n}{1+x^2}$

Thus:

$\frac{1}{1+x^2}-\left(1-x^2+x^4-x^6+...+(-1)^{n-1}x^{2n-2}\right)$

$=\frac{1}{1+x^2}-\frac{1-(-x^2)^n}{1+x^2}$

$=\frac{(-x^2)^n}{1+x^2}$

$=\frac{\left(-1\right)^n x^{2n}}{1+x^2}$

Since $-x^{2n}\leq\left(-1\right)^n x^{2n}\leq x^{2n}$

and $1+x^2\geq 1$

$-x^{2n}\leq\frac{(-1)^n x^{2n}}{1+x^2}\leq x^{2n}$

Replacing the middle term gives the desired result:

$-x^{2n}\leq\frac{1}{1+x^2}-\left(\begin{matrix}1-x^2+x^4-x^6+...\\+(-1)^{n-1}x^{2n-2}\end{matrix}\right)\leq x$

(ii) Integrating all parts with respect to x between 0 and 1.

$\int_0^1-x^{2n}\,dx=\left[\frac{1}{2n+1}(-x^{2n+1})\right]_0^1$

$=-\frac{1}{2n+1}$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_0^1$$
$$= \frac{\pi}{4}$$
$$\int_0^1 \left(1 - x^2 + x^4 - \ldots\right) dx = \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right]_0^1$$
$$= 1 - \frac{1}{3} + \frac{1}{5} - \ldots$$
$$\int_0^1 x^{2n}\,dx = \left[\frac{1}{2n+1}x^{2n+1}\right]_0^1$$
$$= \frac{1}{2n+1}$$

Thus
$$-\frac{1}{2n+1} \le \frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \ldots + \frac{(-1)^{n-1}}{2n-1}\right) \le \frac{1}{2n+1}$$

(iii) As $n \to \infty$, $\pm\dfrac{1}{2n+1} \to 0$

So:
$$0 \le \frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) \le 0$$
$$\therefore \frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) = 0$$

Thus $\dfrac{\pi}{4} = \left(1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots\right)$.

(c) *Method 1:*

For $I = \displaystyle\int \frac{\ln x}{(1+\ln x)^2}\,dx$

Let $u = \ln x$
$$\frac{du}{dx} = \frac{1}{x} = \frac{1}{e^u}$$
$$du = \frac{1}{e^u}dx$$
$$dx = e^u\,du$$
$$I = \int \frac{ue^u}{(1+u)^2}\,du$$
$$= \int \frac{(1+u)e^u}{(1+u)^2}\,du - \int \frac{e^u}{(1+u)^2}\,du$$

use integration by parts
$$= \int \frac{e^u}{1+u}\,du - \left[\frac{-e^u}{1+u} + \int \frac{e^u}{1+u}\,du\right]$$
$$= \frac{e^u}{1+u} + c$$
$$= \frac{x}{1+\ln x} + c.$$

OR

Method 2:
$$\int \frac{\ln x}{(1+\ln x)^2}\,dx = \int \frac{(1+\ln x) - \frac{x}{x}}{(1+\ln x)^2}\,dx$$
$$= \frac{x}{1+\ln x} + c.$$

This can be verified by using the quotient rule on the result.

End of Mathematics Extension 2 solutions