BOARD OF STUDIES
NEW SOUTH WALES

2013

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2–6

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 7–15

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

---

1 What are the solutions of $2x^2 - 5x - 1 = 0$?

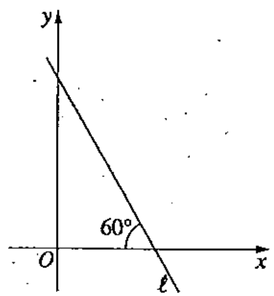
(A) $x = \dfrac{-5 \pm \sqrt{17}}{4}$

(B) $x = \dfrac{5 \pm \sqrt{17}}{4}$

(C) $x = \dfrac{-5 \pm \sqrt{33}}{4}$

(D) $x = \dfrac{5 \pm \sqrt{33}}{4}$

2 The diagram shows the line ℓ.



What is the slope of the line ℓ?

(A) $\sqrt{3}$

(B) $-\sqrt{3}$

(C) $\dfrac{1}{\sqrt{3}}$

(D) $-\dfrac{1}{\sqrt{3}}$

3 Which inequality defines the domain of the function $f(x) = \dfrac{1}{\sqrt{x+3}}$?

(A) $x > -3$

(B) $x \geq -3$

(C) $x < -3$

(D) $x \leq -3$

4 What is the derivative of $\dfrac{x}{\cos x}$?

(A) $\dfrac{\cos x + x \sin x}{\cos^2 x}$

(B) $\dfrac{\cos x - x \sin x}{\cos^2 x}$

(C) $\dfrac{x \sin x - \cos x}{\cos^2 x}$

(D) $\dfrac{-x \sin x - \cos x}{\cos^2 x}$

5 A bag contains 4 red marbles and 6 blue marbles. Three marbles are selected at random without replacement.

What is the probability that at least one of the marbles selected is red?

(A) $\dfrac{1}{6}$

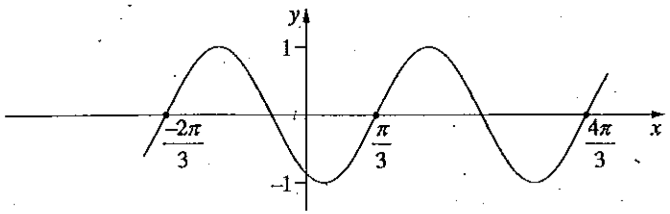
(B) $\dfrac{1}{2}$

(C) $\dfrac{5}{6}$

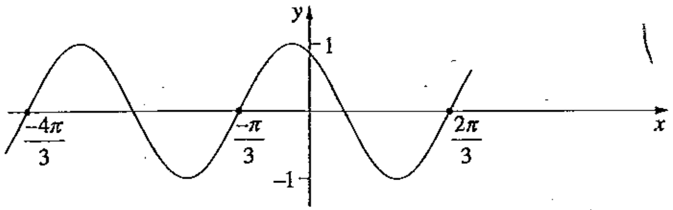
(D) $\dfrac{29}{30}$

6 Which diagram shows the graph $y = \sin\left(2x + \frac{\pi}{3}\right)$?

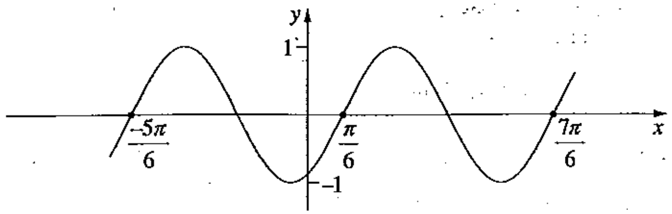
(A)



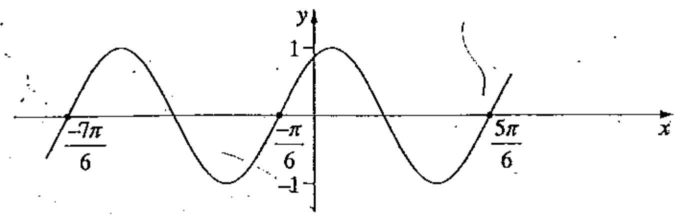
(B)



(C)



(D)



7 A parabola has focus $(5, 0)$ and directrix $x = 1$.

What is the equation of the parabola?

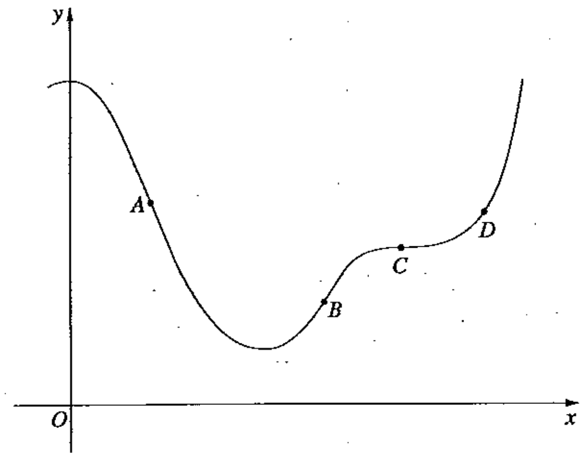
(A) $y^2 = 16(x - 5)$

(B) $y^2 = 8(x - 3)$

(C) $y^2 = -16(x - 5)$

(D) $y^2 = -8(x - 3)$

8 The diagram shows points A, B, C and D on the graph $y = f(x)$.



At which point is $f'(x) > 0$ and $f''(x) = 0$?

(A) A

(B) B

(C) C

(D) D

9 What is the solution of $5^x = 4$?

(A) $x = \dfrac{\log_e 4}{5}$

(B) $x = \dfrac{4}{\log_e 5}$

(C) $x = \dfrac{\log_e 4}{\log_e 5}$

(D) $x = \log_e\left(\dfrac{4}{5}\right)$

10 A particle is moving along the x-axis. The displacement of the particle at time t seconds is x metres.

At a certain time, $\dot{x} = -3\text{ m s}^{-1}$ and $\ddot{x} = 2\text{ m s}^{-2}$.

Which statement describes the motion of the particle at that time?

(A) The particle is moving to the right with increasing speed.

(B) The particle is moving to the left with increasing speed.

(C) The particle is moving to the right with decreasing speed.

(D) The particle is moving to the left with decreasing speed.

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

(a) Evaluate $\ln 3$ correct to three significant figures. **1**

(b) Evaluate $\displaystyle\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4}$. **2**

(c) Differentiate $(\sin x - 1)^8$. **2**

(d) Differentiate $x^2 e^x$. **2**

(e) Find $\displaystyle\int e^{4x+1}\,dx$. **2**

(f) Evaluate $\displaystyle\int_0^1 \frac{x^2}{x^3+1}\,dx$. **3**

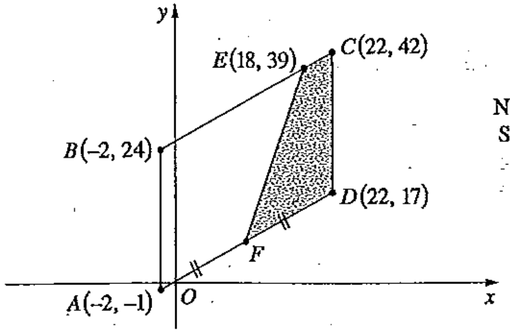
(g) Sketch the region defined by $(x-2)^2 + (y-3)^2 \geq 4$. **3**

Question 12 (15 marks) Use the Question 12 Writing Booklet.

(a) The cubic $y = ax^3 + bx^2 + cx + d$ has a point of inflexion at $x = p$. **2**

Show that $p = -\dfrac{b}{3a}$.

(b) The points $A(-2, -1)$, $B(-2, 24)$, $C(22, 42)$ and $D(22, 17)$ form a parallelogram as shown. The point $E(18, 39)$ lies on BC. The point F is the midpoint of AD.



(i) Show that the equation of the line through A and D is $3x - 4y + 2 = 0$. **2**

(ii) Show that the perpendicular distance from B to the line through A and D is 20 units. **1**

(iii) Find the length of EC. **1**

(iv) Find the area of the trapezium $EFDC$. **2**

(c) Kim and Alex start jobs at the beginning of the same year. Kim's annual salary in the first year is \$30 000, and increases by 5% at the beginning of each subsequent year. Alex's annual salary in the first year is \$33 000, and increases by \$1500 at the beginning of each subsequent year.

(i) Show that in the 10th year Kim's annual salary is higher than Alex's annual salary. **2**

(ii) In the first 10 years how much, in total, does Kim earn? **2**

(iii) Every year, Alex saves $\dfrac{1}{3}$ of her annual salary. How many years does it take her to save \$87 500? **3**

Question 13 (15 marks) Use the Question 13 Writing Booklet.

(a) The population of a herd of wild horses is given by

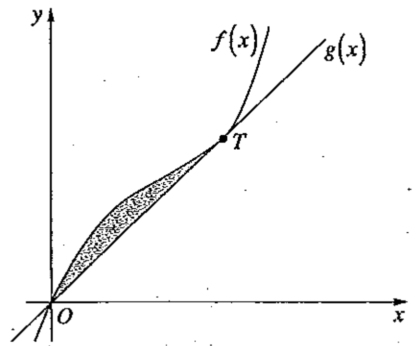
$$P(t) = 400 + 50\cos\left(\frac{\pi}{6}t\right),$$

where t is time in months.

(i) Find all times during the first 12 months when the population equals 375 horses. **2**

(ii) Sketch the graph of $P(t)$ for $0 \le t \le 12$. **2**

(b) The diagram shows the graphs of the functions $f(x) = 4x^3 - 4x^2 + 3x$ and $g(x) = 2x$. The graphs meet at O and at T.



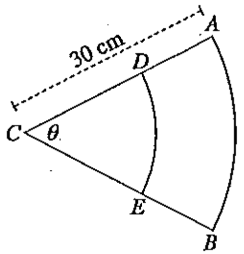
(i) Find the x-coordinate of T. **1**

(ii) Find the area of the shaded region between the graphs of the functions $f(x)$ and $g(x)$. **3**

Question 13 continues on page 10

Question 13 (continued)

(c) The region ABC is a sector of a circle with radius 30 cm, centred at C. The angle of the sector is θ. The arc DE lies on a circle also centred at C, as shown in the diagram. **2**



The arc DE divides the sector ABC into two regions of equal area.

Find the exact length of the interval CD.

(d) A family borrows \$500 000 to buy a house. The loan is to be repaid in equal monthly instalments. The interest, which is charged at 6% per annum, is reducible and calculated monthly. The amount owing after n months, $\$A_n$, is given by

$$A_n = Pr^n - M\left(1 + r + r^2 + \cdots + r^{n-1}\right), \qquad \text{(Do NOT prove this)}$$

where $\$P$ is the amount borrowed, $r = 1.005$ and $\$M$ is the monthly repayment.

(i) The loan is to be repaid over 30 years. Show that the monthly repayment is \$2998 to the nearest dollar. **2**

(ii) Show that the balance owing after 20 years is \$270 000 to the nearest thousand dollars. **1**

(iii) After 20 years the family borrows an extra amount, so that the family then owes a total of \$370 000. The monthly repayment remains \$2998, and the interest rate remains the same. **2**

How long will it take to repay the \$370 000?

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet.

(a) The velocity of a particle moving along the x-axis is given by $\dot{x} = 10 - 2t$, where x is the displacement from the origin in metres and t is the time in seconds. Initially the particle is 5 metres to the right of the origin.

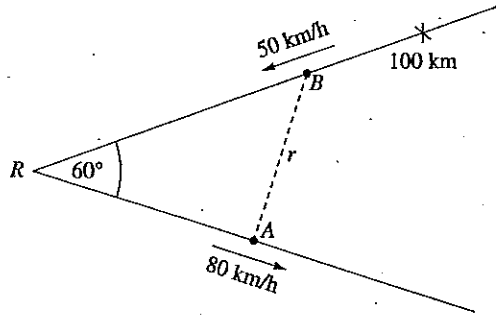
(i) Show that the acceleration of the particle is constant. **1**

(ii) Find the time when the particle is at rest. **1**

(iii) Show that the position of the particle after 7 seconds is 26 metres to the right of the origin. **2**

(iv) Find the distance travelled by the particle during the first 7 seconds. **2**

(b) Two straight roads meet at R at an angle of 60°. At time $t = 0$ car A leaves R on one road, and car B is 100 km from R on the other road. Car A travels away from R at a speed of 80 km/h, and car B travels towards R at a speed of 50 km/h.



The distance between the cars at time t hours is r km.

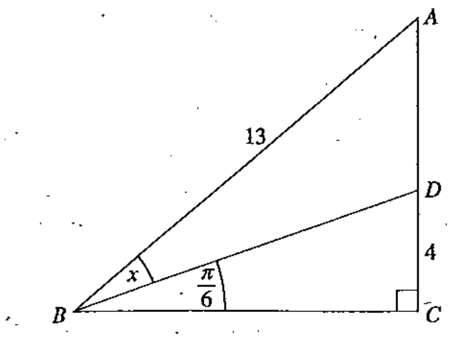
(i) Show that $r^2 = 12\,900t^2 - 18\,000t + 10\,000$. **2**

(ii) Find the minimum distance between the cars. **3**

Question 14 continues on page 12

Question 14 (continued)

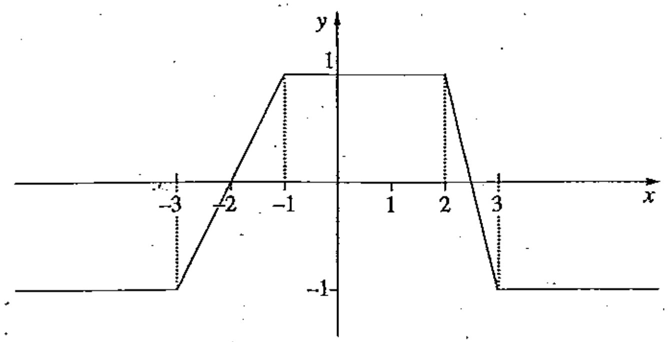
(c) The right-angled triangle ABC has hypotenuse $AB = 13$. The point D is on AC such that $DC = 4$, $\angle DBC = \frac{\pi}{6}$ and $\angle ABD = x$. **3**



NOT TO SCALE

Using the sine rule, or otherwise, find the exact value of $\sin x$.

(d) The diagram shows the graph $y = f(x)$. **1**

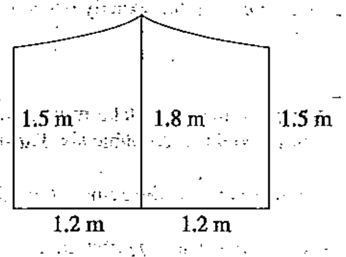


What is the value of a, where $a > 0$, so that $\int_{-a}^{a} f(x)\,dx = 0$?

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet.

(a) The diagram shows the front of a tent supported by three vertical poles. The poles are 1.2 m apart. The height of each outer pole is 1.5 m, and the height of the middle pole is 1.8 m. The roof hangs between the poles.



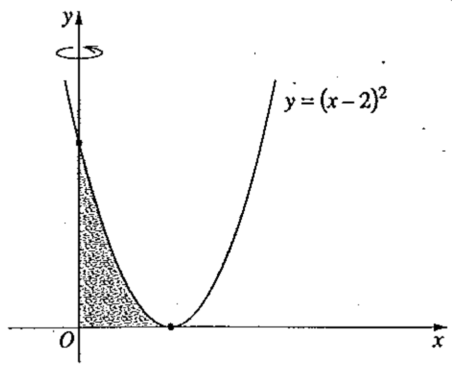
The front of the tent has area $A\,\text{m}^2$.

(i) Use the trapezoidal rule to estimate A. **1**

(ii) Use Simpson's rule to estimate A. **1**

(iii) Explain why the trapezoidal rule gives the better estimate of A. **1**

(b) The region bounded by the x-axis, the y-axis and the parabola $y = (x-2)^2$ is rotated about the y-axis to form a solid. **4**



Find the volume of the solid.

Question 15 continues on page 14

(c) (i) Sketch the graph $y = |2x - 3|$. **1**

(ii) Using the graph from part (i), or otherwise, find all values of m for which the equation $|2x - 3| = mx + 1$ has exactly one solution. **2**

(d) Pat and Chandra are playing a game. They take turns throwing two dice. The game is won by the first player to throw a double six. Pat starts the game.

(i) Find the probability that Pat wins the game on the first throw. **1**

(ii) What is the probability that Pat wins the game on the first or on the second throw? **2**

(iii) Find the probability that Pat eventually wins the game. **2**

End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet.

(a) The derivative of a function $f(x)$ is $f'(x) = 4x - 3$. The line $y = 5x - 7$ is tangent to the graph of $f(x)$. **3**

Find the function $f(x)$.

(b) Trout and carp are types of fish. A lake contains a number of trout. At a certain time 10 carp are introduced into the lake and start eating the trout. As a consequence, the number of trout, N, decreases according to

$$N = 375 - e^{0.04t},$$

where t is the time in months after the carp are introduced.

The population of carp, P, increases according to

$$\frac{dP}{dt} = 0.02P.$$

(i) How many trout were in the lake when the carp were introduced? **1**

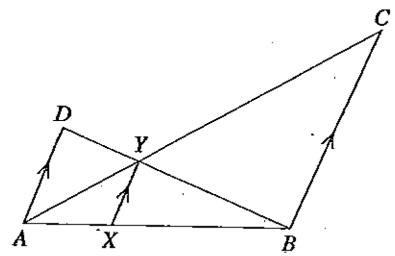
(ii) When will the population of trout be zero? **1**

(iii) Sketch the number of trout as a function of time. **1**

(iv) When is the rate of increase of carp equal to the rate of decrease of trout? **3**

(v) When is the number of carp equal to the number of trout? **2**

(c) The diagram shows triangles ABC and ABD with AD parallel to BC. The sides AC and BD intersect at Y. The point X lies on AB such that XY is parallel to AD and BC.



(i) Prove that $\triangle ABC$ is similar to $\triangle AXY$. **2**

(ii) Hence, or otherwise, prove that $\dfrac{1}{XY} = \dfrac{1}{AD} + \dfrac{1}{BC}$. **2**

End of paper

2013 Higher School Certificate Solutions Mathematics

SECTION I

Summary

1	D	4	A	7	B	9	C
2	B	5	C	8	B	10	D
3	A	6	D				

1 (D) $2x^2-5x-1=0$

$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4(2)(-1)}}{2(2)}$$
$$=\frac{5\pm\sqrt{33}}{4}.$$

2 (B) $m=\tan\theta$ where θ is the angle with the positive x-axis

$$\theta=180°-60°$$
$$=120°$$
$$\therefore m=\tan 120°$$
$$=-\sqrt{3}.$$

3 (A) $x+3\geq 0$

$x\geq -3$

$x\neq -3$ (denominator cannot equal 0)

so $x>-3$ is correct domain.

4 (A) Using the product rule:

$$\frac{d}{dx}\left[\frac{x}{\cos x}\right]=\frac{\cos x(1)-x(-\sin x)}{(\cos x)^2}$$
$$=\frac{\cos x+x\sin x}{\cos^2 x}.$$

5 (C) $P(\text{at least 1 red})=1-P(\text{all blue})$

$$=1-\left(\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}\right)$$
$$=1-\frac{1}{6}$$
$$=\frac{5}{6}.$$

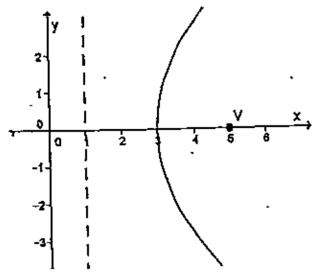
6 (D) $y=\sin\left(2x+\frac{\pi}{3}\right)$

x-intercept occurs when $y=0$

$$\therefore\ 2x+\frac{\pi}{3}=0$$
$$2x=-\frac{\pi}{3}$$
$$x=-\frac{\pi}{6}.$$

This means that the graph of $y=\sin 2x$ has been shifted $\frac{\pi}{6}$ units to the left.

7 (B)



From the diagram, $a=2$ and $V=(3,0)$.

$$(y-k)^2=4a(x-h)$$
$$\therefore\quad y^2=4(2)(x-3)$$
$$y^2=8(x-3).$$

8 (B) $f'(x)>0$ means that the gradient is positive and the curve is then increasing. $f''(x)=0$ means that the curve has a possible point of inflexion.

9 (C) $5^x=4$

Taking logs of both sides:

$$\log_e 5^x=\log_e 4$$
$$x\log_e 5=\log_e 4$$
$$x=\frac{\log_e 4}{\log_e 5}.$$

10 (D) Since $\dot{x}=-3<0$ the particle is moving to the left. Since $\ddot{x}=2>0$ the acceleration is to the right and thus is opposing the motion.

Therefore the particle is moving to the left with decreasing speed.

SECTION II

Question 11

(a) $\ln 3 \doteqdot 1.098612289\ldots$

$$\approx 1.10\quad (3\text{ sig. fig.}).$$

(b)
$$\lim_{x\to 2}\frac{x^3-8}{x^2-4}=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}$$
$$=\lim_{x\to 2}\frac{(x^2+2x+4)}{(x+2)}$$
$$=\frac{2^2+2(2)+4}{2+2}$$
$$=3.$$

(c) Using the chain rule:

$$\frac{d}{dx}(\sin x-1)^8=8(\sin x-1)^7(\cos x)$$
$$=8\cos x(\sin x-1)^7.$$

(d) Using the product rule:

$$\frac{d}{dx}(x^2e^x)=e^x(2x)+x^2(e^x)$$
$$=xe^x(2+x).$$

(e)
$$\int e^{4x+1}\,dx=\frac{1}{4}\int 4e^{4x+1}\,dx$$
$$=\frac{1}{4}e^{4x+1}+C.$$

(f) Note that $\frac{d}{dx}(x^3+1)=3x^2$ and so the integrand is of the form:

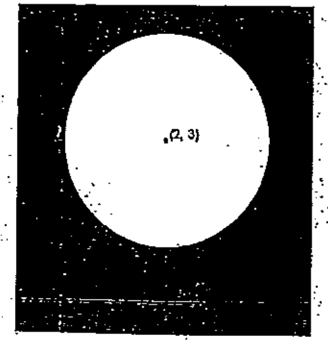
$$\int\frac{f'(x)}{f(x)}dx=\ln[f(x)]+C$$
$$\int_0^1\frac{x^2}{x^3+1}\,dx=\frac{1}{3}\int_0^1\frac{3x^2}{x^3+1}\,dx$$
$$=\frac{1}{3}\left[\ln(x^3+1)\right]_0^1$$
$$=\frac{1}{3}[\ln 2-\ln 1]$$
$$=\frac{1}{3}\ln 2.$$

(g) $(x-2)^2+(y-3)^2=4$ represents a circle with centre $(2,3)$ and radius 2.

Test $(0,0)$:

$$(0-2)^2+(0-3)^2=4+9$$
$$=13$$
$$\geq 4$$

$\therefore$ $(0,0)$ lies in the required region.



Question 12

(a) $y = ax^3 + bx^2 + cx + d$

$y' = 3ax^2 + 2bx + c$

$y'' = 6ax + 2b$

If an inflexion exists at $x = p$, then $y'' = 0$

$6ax + 2b = 0$

$6ax = -2b$

$x = -\frac{b}{3a}$

$\therefore\ p = -\frac{b}{3a}$ as required.

(b) (i) $A = (-2,-1)$ and $D = (22,17)$

$m_{AD} = \frac{17-(-1)}{22-(-2)}$

$= \frac{3}{4}$

Using point A, the equation of AD is:

$y-(-1) = \frac{3}{4}(x-(-2))$

$4(y+1) = 3(x+2)$

$4y+4 = 3x+6$

$\therefore\ 3x-4y+2=0.$

(ii) $B(-2,24)$ and line $3x-4y+2=0$

Using the perpendicular distance formula:

$d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

$= \frac{|3(-2)-4(24)+2|}{\sqrt{3^2+(-4)^2}}$

$= \frac{|-100|}{5}$

$= 20$ units.

(iii) Using $E(18,39)$ and $C(22,42)$:

$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$d = \sqrt{(22-18)^2+(42-39)^2}$

$= \sqrt{16+9}$

$= 5$ units.

(iv) $d_{AD} = \sqrt{(22-(-2))^2+(17-(-1))^2}$

$= \sqrt{(22+2)^2+(17+1)^2}$

$= \sqrt{576+324}$

$= \sqrt{900}$

$AD = 30$ units

$FD = \frac{1}{2}AD$

$= 15$ units

Using $h = 20$ from (ii)

$A_{trapezium} = \frac{1}{2}h[EC+FD]$

$= \frac{1}{2}(20)[5+15]$

$= 200$

$\therefore$ the area is 200 unit^2.

(c) (i) Kim's salary is a geometric series:

$a = 30\,000,\ r = 1.05,\ n = 10$

$T_n = ar^{n-1}$

$T_{10} = 30000(1.05)^{10-1}$

$= \$46539.85$

Alex's salary is an arithmetic series:

$a = 30\,000;\ d = 1500;\ n = 10$

$T_n = a+(n-1)d$

$T_{10} = 30\,000+(10-1)1500$

$= \$46\,500.00$

$\therefore$ Kim's salary is higher.

(ii) Using $S_n = \frac{a(r^n-1)}{r-1}$

$S_{10} = \frac{30\,000(1.05^{10}-1)}{1.05-1}$

$= 377336.776\ldots$

$= 377\,336.776\ldots$

Kim's earnings are $377 337 (to the nearest $).

(iii) Let the number of years be n.

Alex needs to save $87 500.

Her total salary after n years is:

$3\times 87\,500 = 262\,500$

Using $S_n = \frac{n}{2}[2a+(n-1)d]$

$\frac{n}{2}[2(33\,000)+(n-1)1500] = 262\,500$

$n(66\,000+1500n-1500) = 525\,000$

$n(64\,500+1500n) = 525\,000$

$1500n^2+64\,500n-525\,000 = 0$

$n^2+43n-350 = 0$

$(n+50)(n-7) = 0$

$n = -50, 7$

but $n > 0\ \therefore n = 7$

$\therefore$ it takes 7 years.

Question 13

(a) (i) $P(t) = 400+50\cos\left(\frac{\pi}{6}t\right)$

$P(t) = 375$ and $0 \le t \le 12$

$375 = 400+50\cos\left(\frac{\pi}{6}t\right)$

$50\cos\left(\frac{\pi}{6}t\right) = -50 \quad 0 \le t \le 12$

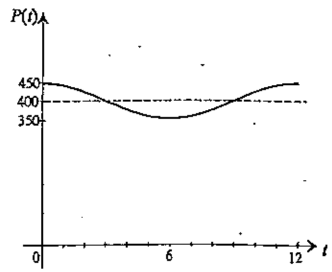
$\cos\left(\frac{\pi}{6}t\right) = -1 \quad 0 \le \frac{\pi}{6}t \le 2\pi$

$\frac{\pi}{6}t = \frac{2\pi}{3}, \frac{4\pi}{3}$

$t = 4, 8$

$\therefore$ after 4 months and 8 months.

(ii) $P(0) = 400+50\cos 0 = 450$



(b) (i) At T, $f(x) = g(x)$:

$\therefore 4x^3-4x^2+3x = 2x$

$4x^3-4x^2+x = 0$

$x(4x^2-4x+1) = 0$

$x(2x-1)(2x-1) = 0$

$x = 0, \frac{1}{2}$

$\therefore$ at T, $x = \frac{1}{2}$

(ii) $A = \int_0^{\frac{1}{2}}[f(x)-g(x)]\,dx$

$= \int_0^{\frac{1}{2}}(4x^3-4x^2+x)\,dx$

$= \left[x^4-\frac{4}{3}x^3+\frac{1}{2}x^2\right]_0^{\frac{1}{2}}$

$= \left(\frac{1}{2}\right)^4-\frac{4}{3}\left(\frac{1}{2}\right)^3+\frac{1}{2}\left(\frac{1}{2}\right)^2-(0)$

$= \frac{1}{48}$

$\therefore$ the area is $\frac{1}{48}$ unit^2.

(c) Let $CD = r$

Area sector $CDE = \frac{1}{2}$Area sector CAB

$\frac{1}{2}r^2\theta = \frac{1}{2}\left(\frac{1}{2}(30)^2\theta\right)$

$r^2\theta = \frac{1}{2}(30)^2\theta$

$r^2 = \frac{1}{2}(30)^2$

$= 450$

$r = \sqrt{450}$ as $r > 0$

$= 15\sqrt{2}$

$\therefore CD = 15\sqrt{2}$ cm.

(d) (i) $A_n = Pr^n - M(1+r+r^2+\ldots+r^{n-1})$

$1+r+r^2+\ldots+r^{n-1}$ is a geometric series with n terms

Using $S_n = \frac{a(r^n-1)}{r-1}$

$1+r+r^2+\ldots+r^{n-1} = \frac{1(r^n-1)}{r-1}$

$$\therefore A_n = Pr^n - \frac{M(r^n - 1)}{r-1}$$

Now $n = 30 \times 12 = 360$,

$r = 1.005, P = 500\,000, A_n = 0$

$$0 = 500\,000(1.005)^{360} - \frac{M(1.005^{360}-1)}{1.005-1}$$

$$\frac{M(1.005^{360}-1)}{0.005} = 500\,000(1.005)^{360}$$

$$M = \frac{500\,000(1.005)^{360}(0.005)}{(1.005^{360}-1)}$$

$$= 2997.754\ldots$$

$\therefore$ the monthly payment is \$2 998.

(ii) After 20 years, $n = 240$

$$A_{240} = 500\,000(1.005)^{240} - \frac{2\,998(1.005^{240}-1)}{1.005-1}$$

$$= 269903.634\ldots$$

$$\doteqdot 270\,000 \quad \text{(nearest thousand)}$$

$\therefore$ approximately \$270 000 remains.

(iii) Now $P = \$370\,000$

$$A_n = 370\,000(1.005)^n - \frac{2\,998(1.005^n-1)}{1.005-1}$$

and $A_n = 0$

$$\therefore 370\,000(1.005)^n - \frac{2\,998(1.005^n-1)}{1.005-1} = 0$$

$$3\,700(1.005)^n - 5\,996(1.005^n - 1) = 0$$

$$3\,700(1.005)^n - 5\,996(1.005^n) + 5\,996 = 0$$

$$-2\,296(1.005)^n + 5\,996 = 0$$

$$2\,296(1.005)^n = 5\,996$$

$$(1.005)^n = \frac{5\,996}{2\,296}$$

$$n\log 1.005 = \log\left(\frac{5\,996}{2\,296}\right)$$

$$n = \frac{\log\left(\frac{5\,996}{2\,296}\right)}{\log 1.005}$$

$$= 192.464\ldots$$

$\therefore$ Time remaining is about 193 months.

Question 14

(a) (i) $\dot{x} = 10 - 2t$

$\ddot{x} = -2$

$\therefore$ the acceleration is constant.

(ii) At rest, $\dot{x} = 0$

$10 - 2t = 0$

$2t = 10$

$t = 5$ seconds.

(iii) $x = \int \dot{x}\, dt$

$= \int (10 - 2t)\, dt$

$= 10t - t^2 + C$

When $t = 0, x = 5$:

$5 = 10(0) - 0^2 + C$

$\therefore C = 5$

$\therefore x = 10t - t^2 + 5$

When $t = 7$:

$x = 10(7) - (7)^2 + 5$

$= 26$

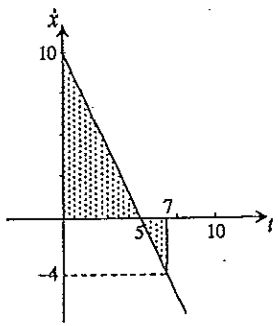
$\therefore$ it will be 26 m to the right of the origin.

(iv) *Method 1*

The distance travelled is the area under the $\dot{x} - t$ graph.

When $t = 7$: $\dot{x} = 10 - 2(7)$

$= -4$



$A = \frac{1}{2}(5)(10) + \frac{1}{2}(2)(4)$

$= 29$

$\therefore$ the distance travelled is 29 m.

Method 2

The particle stops when $\dot{x} = 0$.

$\therefore 10 - 2t = 0$

$10 = 2t$

$t = 5$

Then $x = 10(5) - (5)^2 + 5$

$= 30$

t	0	5	7
x	5	30	26

The particle travels 25 m in the positive direction in the first 5 seconds and then 4 m in the negative direction in the next 2 seconds.

$\therefore$ the total distance travelled is 25 m + 4 m = 29 m

(b) (i) Car *A* travels at 80 km/h for *t* hours:

$\therefore AR = 80t$

Car B travels at 50 km/h for t hours:

$\therefore BR = 100 - 50t$

By the cosine rule in ΔABR:

$AB^2 = AR^2 + BR^2 - 2AR.BR\cos 60°$

$r^2 = (80t)^2 + (100 - 50t)^2 - 2(80t)(100 - 50t)(\frac{1}{2})$

$= 6\,400 + 10\,000 - 10\,000t + 2\,500t^2 - 8\,000t + 4\,000t^2$

$= 12\,900t^2 - 18\,000t + 10\,000$ as required.

(ii) *Method 1:*

The minimum distance occurs at the vertex of this parabolic function

ie when $t = -\frac{b}{2a}$

$= \frac{-(-18000)}{2(12900)}$

$= \frac{30}{43}$

$r^2 = 12\,900\left(\frac{30}{43}\right)^2 - 18\,000\left(\frac{30}{43}\right) + 10\,000$

$= 3720.930\ldots$

$r = 60.999\ldots$

$\doteqdot 61$ (nearest whole)

$\therefore$ the minimum distance between cars is 61 km (nearest km).

Method 2:

$r^2 = 12900t^2 - 18000t + 10000$

$\frac{d(r^2)}{dt} = 25800t - 18000$

For a minimum, $\frac{d(r^2)}{dt} = 0$

$25800t - 18000 = 0$

$t = \frac{30}{43}$

$r^2 = 12900\left(\frac{30}{43}\right)^2 - 18000\left(\frac{30}{43}\right) + 10000$

$= \frac{160000}{43}$

$r = \frac{400}{\sqrt{43}} = \frac{400\sqrt{43}}{43} \approx 61$ m

$\frac{d^2(r^2)}{dt^2} = 25800 > 0$

$\therefore$ the minimum distance between cars is 61km (nearest km).

(c) (i) In ΔBCD:

$\tan\frac{\pi}{6} = \frac{4}{BC}$

$\frac{1}{\sqrt{3}} = \frac{4}{BC}$

$BC = 4\sqrt{3}$

In ΔABC:

$13^2 = AC^2 + (4\sqrt{3})^2$

$AC^2 = 13^2 - (4\sqrt{3})^2$

$= 121$

$AC = 11, \quad AC > 0$

$\therefore AD = 11 - 4 = 7$

Also:

$\angle BDA = \frac{\pi}{2} + \frac{\pi}{6}$ (exterior angle of ΔBCD)

$= \frac{2\pi}{3}$

In ΔABD, using the sine rule:

$\frac{\sin x}{7} = \frac{\sin\frac{2\pi}{3}}{13}$

$\sin x = \frac{7}{13} \cdot \frac{\sqrt{3}}{2}$

$= \frac{7\sqrt{3}}{26}$.

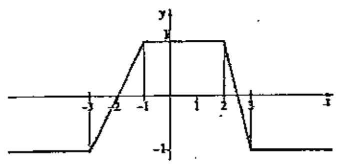
(d) We need the area above the x-axis to equal the area below.
By observation that happens for:

$\int_{-3}^{-1} f(x)dx = 0$ and $\int_{2}^{3} f(x)dx = 0$

Then this area needs to be balanced:

$$\int_{-1}^{2} f(x)dx = 3$$

Since the 3 previous areas cover the domain $-3 \le x \le 3$. The balancing area needs to be divided equally on each side, under the x-axis



$\therefore \int_{-a}^{-3} f(x)\,dx = \int_{3}^{a} f(x)\,dx = -1.5$

The height of the respective rectangles is 1 unit. So to balance the area of 3 units^2, a distance of 1.5 is needed on either side.
$\therefore\ a = 3 + 1.5 = 4.5$.

Question 15

(a) (i) Let the ground be the x-axis and the origin at the bottom left of the tent.

x	0	1.2	2.4
h	1.5	1.8	1.5

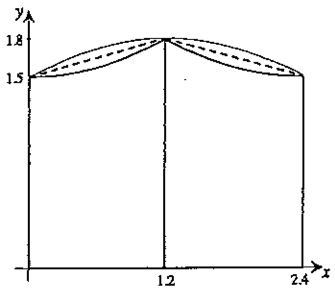
$A \doteq \frac{1}{2}(1.2)[1.5+1.8] + \frac{1}{2}(1.2)[1.8+1.5]$
$= 3.96\ \text{m}^2$.

(ii) $A \doteq \frac{1}{6}(2.4)[1.5+4(1.8)+1.5]$
$= 4.08\ \text{m}^2$.

(iii) Simpson's rule approximates the curve with a parabola. In this case the parabola will pass through the three points on the roof of the tent and will so be *concave* down as illustrated in the diagram below.

The trapezoidal rule approximates the curve with two line segments joining the same three points.

In this case, the line segments are a better approximation to the curve than the parabolic arc and so the trapezoidal rule i the better approximation.



(b) $V = \pi \int x^2\,dy$

At $x = 0$: $y = (0-2)^2 = 4$

Also:

$y = (x-2)^2$

$x - 2 = \pm\sqrt{y}$

$x = 2 \pm \sqrt{y}$

But we need the part of the parabola that lies on the left of the vertex.
This is $x = 2 - \sqrt{y}$.

$V = \pi \int_0^4 \left(2-\sqrt{y}\right)^2 dy$

$= \pi \int_0^4 \left(4 - 4y^{\frac{1}{2}} + y\right) dy$

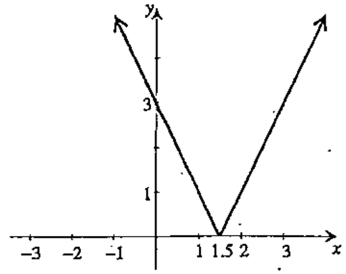
$V = \pi\left[4y - \frac{2}{3}\cdot 4y^{\frac{3}{2}} + \frac{1}{2}y^2\right]_0^4$

$= \pi\left[4(4) - \frac{2}{3}\cdot 4(4)^{\frac{3}{2}} + \frac{1}{2}(4)^2 - 0\right]$

$= \frac{8\pi}{3}$

$\therefore$ the area is $\frac{8\pi}{3}$ unit^2.

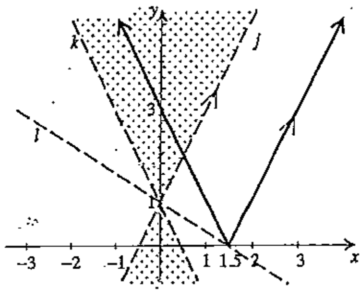
(c) (i) $y = |2x-3|$
If $x = 0$: $y = |-3| = 3$
If $y = 0$: $2x - 3 = 0$
$2x = 3$
$x = 1.5$



(ii) For $|2x-3| = mx + 1$ to have only one solution we need to consider the relationship between the graphs of $y = |2x-3|$ and $y = mx + 1$.

Now $y = mx + 1$ represents a family of lines through (0, 1) and with variable gradient m. For there to be one solution, this variable line can only intersect the absolute value graph at one point.

There are two critical positions we need to consider – the lines labelled j and k in the diagram below. Any line which lies in the shaded region between these lines will meet the condition of only one point of intersection. There is also a line, labelled ℓ, which intersects the graph once, at the vertex.



Now j is parallel to the right branch of the absolute value graph $\therefore$ gradient of $j = 2$ and k is parallel to the right branch of the absolute value graph $\therefore$ gradient of $k = -2$.
Also ℓ passes through the vertex $(1.5, 0)$ and $(0,1)$ $\therefore$ gradient of $\ell = -\frac{2}{3}$.
This means we need lines steeper than j, lines with negative gradient 'steeper' than k or lines with the gradient of ℓ.
Furthermore if the lines are parallel to k, there will be no point of intersection.
$\therefore\ m \ge 2,\ m < -2$ or $m = -\frac{2}{3}$.

(d) (i) $\text{P(double 6)} = \frac{1}{6} \times \frac{1}{6}$
$= \frac{1}{36}$.

(ii) P(Pat wins on her first or second throw)
$=$ P(Pat win) + P(Pat miss, Chandra miss, Pat win)
$= \frac{1}{36} + \frac{35}{36} \times \frac{35}{36} \times \frac{1}{36}$
$= \frac{2521}{46656}$.

(iii) P(Pat wins at some stage)
$= \text{P(W)} + \text{P(LLW)} + \text{P(LLLLW)} + \ldots$
$= \frac{1}{36} + \left(\frac{35}{36}\right)^2 \times \frac{1}{36} + \left(\frac{35}{36}\right)^4 \times \frac{1}{36} + \ldots$
$= \frac{a}{1-r}$ where $a = \frac{1}{36}$ and $r = \left(\frac{35}{36}\right)^2$

$$P(\text{Pat wins}) = \frac{\frac{1}{36}}{1-\left(\frac{35}{36}\right)^2}$$

$$= \frac{36}{71}$$

Question 16

(a) $y = 5x - 7$ is a tangent, the gradient is 5

$4x - 3 = 5$

$4x = 8$

$x = 2$

When $x = 2$, $y = 5(2) - 7 = 3$

$(2,3)$ is the common point between the curve and the tangent.

$f'(x) = 4x - 3$

$\therefore\ f(x) = 2x^2 - 3x + C$

But $(2,3)$ should satisfy this.

$3 = 2(2)^2 - 3(2) + C$

$3 = 8 - 6 + C$

$C = 1$

$\therefore\ f(x) = 2x^2 - 3x + 1.$

(b) (i) When $t = 0$:

$N = 375 - e^{0.04(0)}$

$= 375 - 1$

$= 374$

$\therefore$ there were 374 trout to begin with.

(ii) When $N = 0$:

$0 = 375 - e^{0.04t}$

$e^{0.04t} = 375$

$\ln\left(e^{0.04t}\right) = \ln 375$

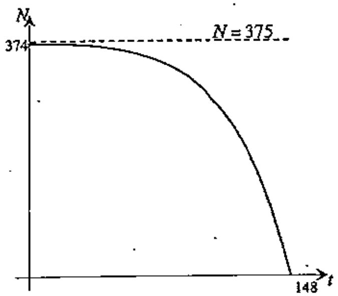
$0.04t = \ln 375$

$t = \frac{\ln 375}{0.04}$

$= 148.173\ldots$

$\therefore$ after about 148 months.

(iii)



(iv) $N = 375 - e^{0.04t}$

$\frac{dN}{dt} = -0.04e^{0.04t}$

$\therefore$ the rate of decrease is $0.04e^{0.04t}$

But $\frac{dP}{dt} = 0.02P$

$\therefore P = P_0 e^{0.02t}$

When $t = 0: P = 10$

$\therefore P_0 = 10$

i.e. $P = 10e^{0.02t}$

and $\frac{dP}{dT} = 0.02\left(10e^{0.02t}\right)$

$= 0.2e^{0.02t}$

Now the rates are equal:

$\therefore 0.04e^{0.04t} = 0.2e^{0.02t}$

$e^{0.04t} = 5e^{0.02t}$ but $e^{0.02t} > 0$ for all t

$\frac{e^{0.04t}}{e^{0.02t}} = 5$

$\therefore e^{0.02t} = 5$

$0.02t = \ln 5$

$t = \frac{\ln 5}{0.02}$

$= 80.471\ldots$

$\therefore$ after about 80 months

(v) For $N = P$:

$375 - e^{0.04t} = 10e^{0.02t}$

$375 - \left(e^{0.02t}\right)^2 = 10e^{0.02t}$

$\left(e^{0.02t}\right)^2 + 10e^{0.02t} - 375 = 0$

Let $k = e^{0.02t}$

$\therefore\ k^2 + 10k - 375 = 0$

$(k+25)(k-15) = 0$

$k = -25, 15$

But as $e^{0.02t} > 0 \Rightarrow k > 0$

$\therefore\ k = 15$ is the only solution

$\therefore e^{0.02t} = 15$

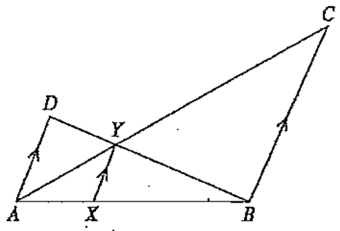
$0.02t = \ln 15$

$t = \frac{\ln 15}{0.02}$

$= 135.40\ldots$

$\therefore$ after about 135 months.

(c) (i)



In ΔABC and ΔAXY:

$\angle BAC = \angle XAY$ (common)

$\angle ABC = \angle AXY$ (corr., $XY \parallel BC$)

$\therefore\ \Delta ABC \equiv \Delta AXY$ (equi-angular)

(ii) $\frac{AB}{AX} = \frac{BC}{XY} = \frac{AC}{AY}$ ①

(corresponding. sides of similar Δs)

Similarly,

$\Delta ADB \equiv \Delta XYB$ (equiangular)

$\frac{AD}{XY} = \frac{DB}{YB} = \frac{AB}{XB}$ ②

(corresponding. sides of similar Δs)

Using parts of ① and ②:

$\frac{XY}{BC} + \frac{XY}{AD} = \frac{AX}{AB} + \frac{XB}{AB}$

$\frac{XY}{BC} + \frac{XY}{AD} = \frac{AX}{AB} + \frac{XB}{AB}$

$XY\left(\frac{1}{BC} + \frac{1}{AD}\right) = \frac{AX + XB}{AB}$

But $AX + XB = AB$

$\therefore XY\left(\frac{1}{BC} + \frac{1}{AD}\right) = \frac{AB}{AB}$

$= 1$

$\frac{1}{BC} + \frac{1}{AD} = \frac{1}{XY}$

$\therefore\ \frac{1}{XY} = \frac{1}{AD} + \frac{1}{BC}$ as required.

End of Mathematics solutions