BOSTES | BOARD OF STUDIES TEACHING & EDUCATIONAL STANDARDS NSW

2014 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2–4

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 5–15

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

TABLE OF STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a>0, \ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x>0$

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

---

1 What is the value of $\frac{\pi^2}{6}$, correct to 3 significant figures?

(A) 1.64

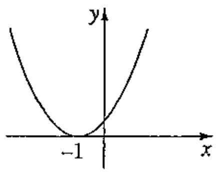
(B) 1.65

(C) 1.644

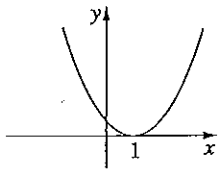
(D) 1.645

2 Which graph best represents $y=(x-1)^2$?

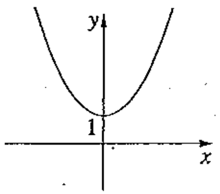
(A)



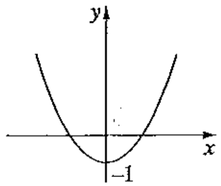
(B)



(C)



(D)



3 What is the solution to the equation $\log_2(x-1)=8$?

(A) 4

(B) 17

(C) 65

(D) 257

4 Which expression is equal to $\int e^{2x}\,dx$?

(A) $e^{2x}+c$

(B) $2e^{2x}+c$

(C) $\frac{e^{2x}}{2}+c$

(D) $\frac{e^{2x+1}}{2x+1}+c$

5 Which equation represents the line perpendicular to $2x-3y=8$, passing through the point $(2,0)$?

(A) $3x+2y=4$

(B) $3x+2y=6$

(C) $3x-2y=-4$

(D) $3x-2y=6$

6 Which expression is a factorisation of $8x^3+27$?

(A) $(2x-3)(4x^2+12x-9)$

(B) $(2x+3)(4x^2-12x+9)$

(C) $(2x-3)(4x^2+6x-9)$

(D) $(2x+3)(4x^2-6x+9)$

7 How many solutions of the equation $(\sin x-1)(\tan x+2)=0$ lie between 0 and 2π?

(A) 1

(B) 2

(C) 3

(D) 4

8 Which expression is a term of the geometric series $3x - 6x^2 + 12x^3 - \cdots$?

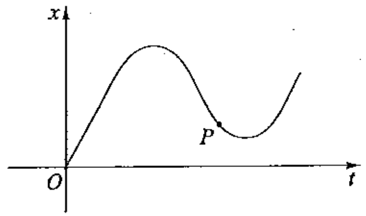
(A) $3072x^{10}$

(B) $-3072x^{10}$

(C) $3072x^{11}$

(D) $-3072x^{11}$

9 The graph shows the displacement x of a particle moving along a straight line as a function of time t.



Which statement describes the motion of the particle at the point P?

(A) The velocity is negative and the acceleration is positive.

(B) The velocity is negative and the acceleration is negative.

(C) The velocity is positive and the acceleration is positive.

(D) The velocity is positive and the acceleration is negative.

10 Three runners compete in a race. The probabilities that the three runners finish the race in under 10 seconds are $\frac{1}{4}$, $\frac{1}{6}$ and $\frac{2}{5}$ respectively.

What is the probability that at least one of the three runners will finish the race in under 10 seconds?

(A) $\frac{1}{60}$

(B) $\frac{37}{60}$

(C) $\frac{3}{8}$

(D) $\frac{5}{8}$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

(a) Rationalise the denominator of $\frac{1}{\sqrt{5}-2}$. **2**

(b) Factorise $3x^2 + x - 2$. **2**

(c) Differentiate $\frac{x^3}{x+1}$. **2**

(d) Find $\int \frac{1}{(x+3)^2}\,dx$. **2**

(e) Evaluate $\int_0^{\frac{\pi}{2}} \sin\frac{x}{2}\,dx$. **3**

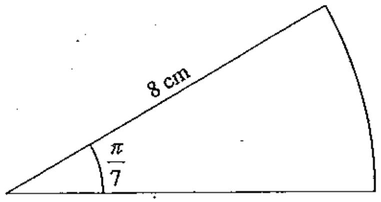
Question 11 continues on page 6

Question 11 (continued)

(f) The gradient function of a curve $y = f(x)$ is given by $f'(x) = 4x - 5$. **2**
The curve passes through the point $(2, 3)$.

Find the equation of the curve.

(g) The angle of a sector in a circle of radius 8 cm is $\frac{\pi}{7}$ radians, as shown in the diagram. **2**



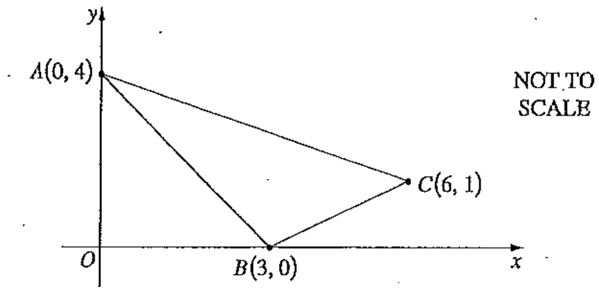
Find the exact value of the perimeter of the sector.

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet.

(a) Evaluate the arithmetic series $2 + 5 + 8 + 11 + \cdots + 1094$. **2**

(b) The points $A(0, 4)$, $B(3, 0)$ and $C(6, 1)$ form a triangle, as shown in the diagram.



(i) Show that the equation of AC is $x + 2y - 8 = 0$. **2**

(ii) Find the perpendicular distance from B to AC. **2**

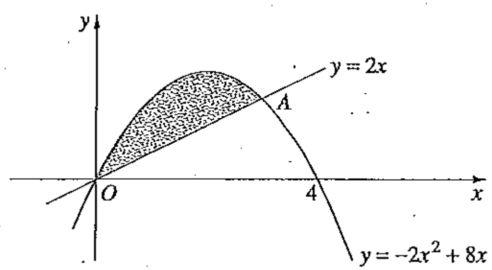
(iii) Hence, or otherwise, find the area of $\triangle ABC$. **2**

(c) A packet of lollies contains 5 red lollies and 14 green lollies. Two lollies are selected at random without replacement.

(i) Draw a tree diagram to show the possible outcomes. Include the probability on each branch. **2**

(ii) What is the probability that the two lollies are of different colours? **1**

(d) The parabola $y = -2x^2 + 8x$ and the line $y = 2x$ intersect at the origin and at the point A.



(i) Find the x-coordinate of the point A. **1**

(ii) Calculate the area enclosed by the parabola and the line. **3**

Question 13 (15 marks) Use the Question 13 Writing Booklet.

(a) (i) Differentiate $3 + \sin 2x$. **1**

(ii) Hence, or otherwise, find $\int \frac{\cos 2x}{3 + \sin 2x}\,dx$. **2**

(b) A quantity of radioactive material decays according to the equation

$$\frac{dM}{dt} = -kM,$$

where M is the mass of the material in kg, t is the time in years and k is a constant.

(i) Show that $M = Ae^{-kt}$ is a solution to the equation, where A is a constant. **1**

(ii) The time for half of the material to decay is 300 years. If the initial amount of material is 20 kg, find the amount remaining after 1000 years. **3**

(c) The displacement of a particle moving along the x-axis is given by

$$x = t - \frac{1}{1+t},$$

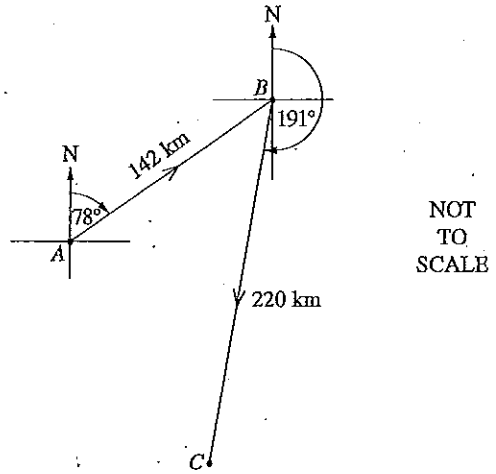
where x is the displacement from the origin in metres, t is the time in seconds, and $t \geq 0$.

(i) Show that the acceleration of the particle is always negative. **2**

(ii) What value does the velocity approach as t increases indefinitely? **1**

Question 13 continues on page 9

(d) Chris leaves island A in a boat and sails 142 km on a bearing of 078° to island B. Chris then sails on a bearing of 191° for 220 km to island C, as shown in the diagram.



(i) Show that the distance from island C to island A is approximately 210 km. **2**

(ii) Chris wants to sail from island C directly to island A. On what bearing should Chris sail? Give your answer correct to the nearest degree. **3**

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet.

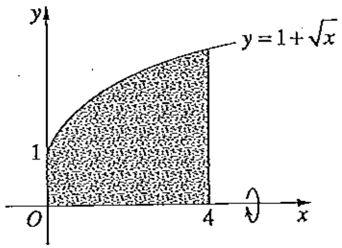
(a) Find the coordinates of the stationary point on the graph $y = e^x - ex$, and determine its nature. **3**

(b) The roots of the quadratic equation $2x^2 + 8x + k = 0$ are α and β.

(i) Find the value of $\alpha + \beta$. **1**

(ii) Given that $\alpha^2\beta + \alpha\beta^2 = 6$, find the value of k. **2**

(c) The region bounded by the curve $y = 1 + \sqrt{x}$ and the x-axis between $x = 0$ and $x = 4$ is rotated about the x-axis to form a solid. **3**



Find the volume of the solid.

(d) At the beginning of every 8-hour period, a patient is given 10 mL of a particular drug.

During each of these 8-hour periods, the patient's body partially breaks down the drug. Only $\frac{1}{3}$ of the total amount of the drug present in the patient's body at the beginning of each 8-hour period remains at the end of that period.

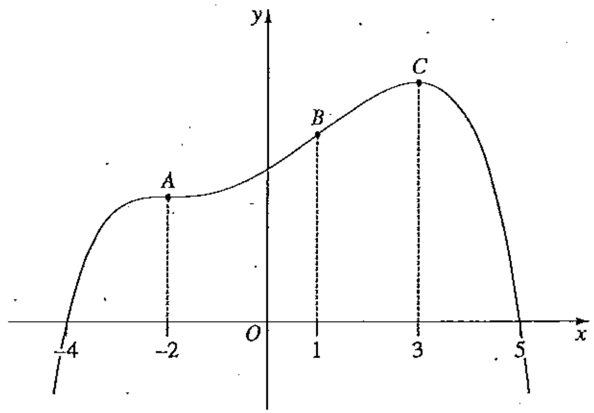
(i) How much of the drug is in the patient's body immediately after the second dose is given? **1**

(ii) Show that the total amount of the drug in the patient's body never exceeds 15 mL. **2**

Question 14 continues on page 11

(e) The diagram shows the graph of a function $f(x)$. **3**

The graph has a horizontal point of inflexion at A, a point of inflexion at B and a maximum turning point at C.



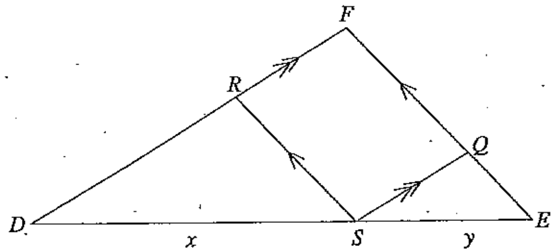
Sketch the graph of the derivative $f'(x)$.

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet.

(a) Find all solutions of $2\sin^2 x + \cos x - 2 = 0$, where $0 \le x \le 2\pi$. **3**

(b) In $\triangle DEF$, a point S is chosen on the side DE. The length of DS is x, and the length of ES is y. The line through S parallel to DF meets EF at Q. The line through S parallel to EF meets DF at R.



The area of $\triangle DEF$ is A. The areas of $\triangle DSR$ and $\triangle SEQ$ are A_1 and A_2 respectively.

(i) Show that $\triangle DEF$ is similar to $\triangle DSR$. **2**

(ii) Explain why $\dfrac{DR}{DF} = \dfrac{x}{x+y}$. **1**

(iii) Show that $\sqrt{\dfrac{A_1}{A}} = \dfrac{x}{x+y}$. **2**

(iv) Using the result from part (iii) and a similar expression for $\sqrt{\dfrac{A_2}{A}}$, deduce that $\sqrt{A} = \sqrt{A_1} + \sqrt{A_2}$. **2**

Question 15 continues on page 13

(c) The line $y = mx$ is a tangent to the curve $y = e^{2x}$ at a point P.

(i) Sketch the line and the curve on one diagram. **1**

(ii) Find the coordinates of P. **3**

(iii) Find the value of m. **1**

End of Question 15

Please turn over

Question 16 (15 marks) Use the Question 16 Writing Booklet.

(a) Use Simpson's Rule with five function values to show that **3**

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sec x \, dx \approx \frac{\pi}{9}\left(3+\frac{8}{\sqrt{3}}\right).$$

(b) At the start of a month, Jo opens a bank account and makes a deposit of \$500. At the start of each subsequent month, Jo makes a deposit which is 1% more than the previous deposit.

At the end of every month, the bank pays interest of 0.3% (per month) on the balance of the account.

(i) Explain why the balance of the account at the end of the second month is **2**

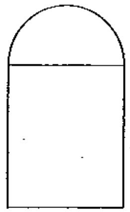
$$\$500(1.003)^2 + \$500(1.01)(1.003).$$

(ii) Find the balance of the account at the end of the 60th month, correct to the nearest dollar. **3**

Question 16 continues on page 15

(c) The diagram shows a window consisting of two sections. The top section is a semicircle of diameter x m. The bottom section is a rectangle of width x m and height y m.

The entire frame of the window, including the piece that separates the two sections, is made using 10 m of thin metal.



The semicircular section is made of coloured glass and the rectangular section is made of clear glass.

Under test conditions the amount of light coming through one square metre of the coloured glass is 1 unit and the amount of light coming through one square metre of the clear glass is 3 units.

The total amount of light coming through the window under test conditions is L units.

(i) Show that $y = 5 - x\left(1+\frac{\pi}{4}\right)$. **2**

(ii) Show that $L = 15x - x^2\left(3+\frac{5\pi}{8}\right)$. **2**

(iii) Find the values of x and y that maximise the amount of light coming through the window under test conditions. **3**

End of paper

2014 Higher School Certificate Solutions Mathematics

SECTION I

Summary

1	A	3	D	5	B	7	B	9	A
2	B	4	C	6	D	8	C	10	D

SECTION I

1 (A) $\frac{\pi^2}{6}=1.644934067\ldots$

$=1.64 \quad (3 \text{ s.f.}).$

2 (B) Compare $y=(x-1)^2$ with $y=x^2$

Replacing x with $x-1$ moves the graph of $y=x^2$ one unit to the right.

3 (D) $\log_2(x-1)=8$

$x-1=2^8$

$x=2^8+1$

$=257.$

4 (C) $\int e^{2x}dx=\frac{1}{2}\int 2e^{2x}dx$

$=\frac{1}{2}e^{2x}+c$

$=\frac{e^{2x}}{2}+c$

5 (B) $2x-3y=8$

$3y=2x-8$

$y=\frac{2}{3}x-\frac{8}{3}$

The gradient of this line is $\frac{2}{3}$.

The perpendicular gradient is $-\frac{3}{2}$.

Thus the required line is:

$y-0=-\frac{3}{2}(x-2)$

$2y=-3x+6$

$3x+2y=6.$

6 (D) $8x^3+27=(2x)^3+3^3$

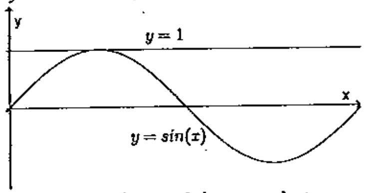
$=(2x+3)(4x^2-6x+9).$

7 (B) $(\sin x-1)(\tan x+2)=0$

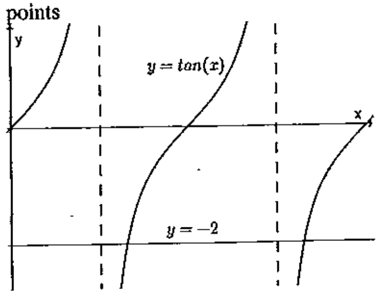
$\sin x=1$ or $\tan x=-2$

Consider the graphs:

$y=\sin x$ and $y=1$ intersect in one point



$y=\tan x$ and $y=-2$ intersect in two points



But the $\frac{\pi}{2}$ solution from the $\sin x=1$ equation makes $\tan x$ undefined and is consequently NOT a solution

There are only 2 solutions.

8 (C) Ignoring x:

$T_n=ar^{n-1}$ with $a=3, r=-2$

$3072=3(-2)^{n-1}$

$1024=(-2)^{n-1}$

$n=11$

To find the actual term:

$T_n=ar^{n-1}$ with $a=3x, r=-2x$

$=3x(-2x)^{11-1}$

$=3x\times 1024x^{10}$

$=3072x^{11}.$

9 (A) At P, the gradient is negative so the velocity is negative. Also at P, the curve is concave up so the acceleration is positive.

10 (D) $\text{P}(\text{at least 1 finishes})=1-\text{P}(\text{none finish})$

$=1-\frac{3}{4}\times\frac{5}{6}\times\frac{3}{5}$

$=\frac{5}{8}.$

SECTION II

Question 11

(a) $\frac{1}{\sqrt{5}-2}=\frac{1}{\sqrt{5}-2}\times\frac{\sqrt{5}+2}{\sqrt{5}+2}$

$=\frac{\sqrt{5}+2}{(\sqrt{5})^2-2^2}$

$=\frac{\sqrt{5}+2}{5-4}$

$=\sqrt{5}+2.$

(b) $3x^2+x-2=(3x-2)(x+1).$

(c) Using the quotient rule:

$\frac{d}{dx}\left(\frac{x^3}{x+1}\right)=\frac{(x+1)(3x^2)-(x^3)(1)}{(x+1)^2}$

$=\frac{3x^3+3x^2-x^3}{(x+1)^2}$

$=\frac{2x^3+3x^2}{(x+1)^2}.$

(d) Using the product rule:

$\int\frac{1}{(x+3)^2}dx=\int(x+3)^{-2}dx$

$=-(x+3)^{-1}+c$

$=\frac{-1}{x+3}+c.$

(e) $\int_0^{\frac{\pi}{2}}\sin\frac{x}{2}dx=\left[-2\cos\frac{x}{2}\right]_0^{\frac{\pi}{2}}$

$=-2\cos\frac{\pi}{4}-(-2\cos 0)$

$=-2\left(\frac{1}{\sqrt{2}}\right)-(-2)$

$=2-\sqrt{2}.$

(f) $f'(x)=4x-5$

$f(x)=2x^2-5x+c$

Substitute (2,3) in $f(x)$:

$3=2(2)^2-5(2)+c$

$c=5$

$\therefore\ f(x)=2x^2-5x+5.$

(g) $\ell=r\theta$ where ℓ is the arc length

$P=8+8+r\theta$

$=16+8\times\frac{\pi}{7}$

$=16+\frac{8\pi}{7}$

The perimeter is $\left(16+\frac{8\pi}{7}\right)$cm.

Question 12

(a) Arithmetic series with $a=2, d=3$

$$T_n = a+(n-1)d$$
$$1094 = 2+(n-1)3$$
$$1092 = 3n-3$$
$$n = 365$$

$$S_n = \frac{n}{2}[a+\ell]$$
$$S_{365} = \frac{365}{2}[2+1094]$$
$$= 200\,020$$

$\therefore$ the sum is 200 020.

(b) (i) Gradient of AC:

$$m_{AC} = \frac{1-4}{6-0}$$
$$= -\frac{3}{6}$$
$$= -\frac{1}{2}$$

Equation of AC:

$$y-4 = -\frac{1}{2}(x-0)$$
$$2(y-4) = -x$$
$$x+2y-8 = 0.$$

(ii) Using the perpendicular distance formula:

$$d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$
$$= \frac{|1(3)+2(0)+2|}{\sqrt{1^2+2^2}}$$
$$= \frac{5}{\sqrt{5}}$$
$$= \sqrt{5}$$

The distance is $\sqrt{5}$ units.

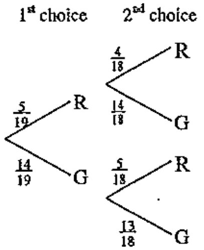
(iii) $$AC = \sqrt{(1-4)^2+(6-0)^2}$$
$$= \sqrt{45}$$
$$= 3\sqrt{5}$$

The area of $\triangle ABC$ is:

$$A = \frac{1}{2}\times 3\sqrt{5}\times\sqrt{5}$$
$$= \frac{15}{2}$$

$\therefore$ area is 7.5 unit^2.

(c) (i)



(ii) $P(\text{different colours}) = P(RG) \text{ or } P(GR)$

$$= \frac{5}{19}\times\frac{14}{18}+\frac{14}{19}\times\frac{5}{18}$$
$$= \frac{70}{171}.$$

(d) (i) Equating the two equations at P:

$$2x = -2x^2+8x$$
$$2x^2-6x = 0$$
$$2x(x-3) = 0$$
$$x = 0 \text{ or } x = 3$$

$\therefore$ the x-coordinate of A is $x=3$.

(ii) Area between two curves:

$$A = \int_0^3 (-2x^2+8x)-(2x)\,dx$$
$$= \int_0^3 -2x^2+6x\,dx$$
$$= \left[-\frac{2x^3}{3}+3x^2\right]_0^3$$
$$= -\frac{2(3)^3}{3}+3(3)^2-(0+0)$$
$$= 9$$

$\therefore$ the area is 9 unit^2.

Question 13

(a) (i) $$\frac{d}{dx}(3+\sin 2x) = 2\cos 2x$$

(ii) Using the result from part (i) and noting that integrand is of the form

$$\int \frac{f'(x)}{f(x)}dx = \ln[f(x)]+c$$

$$\int \frac{\cos 2x}{3+\sin 2x}\,dx = \frac{1}{2}\int\frac{2\cos 2x}{3+\sin 2x}\,dx$$
$$= \frac{1}{2}\ln(3+\sin 2x)+c.$$

(b) (i) $$M = Ae^{-kt}$$
$$\frac{dM}{dt} = Ae^{-kt}\times -k$$
$$= -kM.$$

(ii) When $t=0$: $\quad M = A = 20$

When $t=300$: $\quad M = \frac{A}{2} = 10$

$\therefore M = 20e^{-kt}$

Substitute $t=300, M=10$

$$10 = 20e^{-300k}$$
$$\frac{1}{2} = e^{-300k}$$
$$\ln\frac{1}{2} = -300k$$
$$k = -\frac{1}{300}\ln\frac{1}{2}$$

Substitute this and t = 1000 into

$$M = 20e^{-kt}$$
$$M = 20e^{-\left(-\frac{1}{300}\ln\frac{1}{2}\right)\times 1000}$$
$$= 20e^{\frac{10}{3}\ln\frac{1}{2}}$$
$$= 1.984251315...$$

$\therefore$ 1.98 kg remains (3 s.f.).

(c) (i) $$x = t-\frac{1}{1+t}$$
$$= t-(1+t)^{-1}$$
$$\frac{dx}{dt} = 1+(1+t)^{-2}$$
$$\frac{d^2x}{dt^2} = -2(1+t)^{-3}$$
$$= -\frac{2}{(1+t)^3}$$

Since $t>0$ then $-\frac{2}{(1+t)^3}<0$

So the acceleration is always negative.

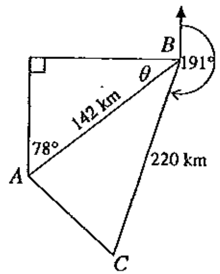
(ii) $$v = 1+(1+t)^{-2}$$
$$= 1+\frac{1}{(1+t)^2}$$

As $t\to\infty$, $\frac{1}{(1+t)^2}\to 0$

$$\lim_{t\to\infty}\left[1+\frac{1}{(1+t)^2}\right] = 1+0$$
$$= 1$$

$\therefore$ velocity approaches 1 ms^{-1}.

(d) (i)



$$\theta = 90°-78° \quad \text{(angle sum of triangle)}$$
$$= 12°$$
$$\angle ABC = 270°-12°-191°$$
$$= 67°$$

By the cosine rule in $\triangle ABC$:

$AC^2 = 142^2 + 220^2 - 2(142)(220)\cos 67°$

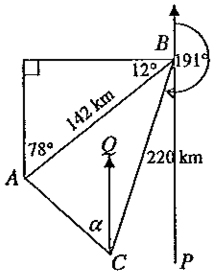
$AC^2 = 44151.119...$

$AC = \sqrt{44151.119...}$

$= 210.12...$

$\therefore$ the distance is 210 km as required.

(ii)



$\angle PBC = 191° - 180°$

$= 11°$

$\angle QCB = 11°$ (alternate $\angle$s, QC $\parallel$ BP)

By the cosine rule in $\triangle ABC$:

$\cos\angle ACB = \dfrac{AC^2 + 220^2 - 142^2}{2(AC)(220)}$

$= 0.782956723...$

using the full value for AC

$\angle ACB = 38.46790685...°$

$= 38°$ (nearest degree)

$\therefore\ \alpha = 38° - 11°$

$= 27°$

Bearing $= 360° - 27°$

$= 333°$.

Question 14

(a) (i) $y = e^x - ex$

$\dfrac{dy}{dx} = e^x - e$

$\dfrac{d^2y}{dx^2} = e^x$

Stationary points occur when $\dfrac{dy}{dx} = 0$:

$\therefore\ e^x - e = 0$

$e^x = e$

$x = 1$

When $x = 1$: $\quad y = e^1 - e = 0$

$\dfrac{d^2y}{dx^2} = e^1 > 0$

$\therefore$ the stationary point is $(1,0)$ and it is a local minimum.

(b) (i) $\alpha + \beta = -\dfrac{b}{a}$ where $a = 2$ and $b = 8$

$= -\dfrac{8}{2}$

$= -4$.

(ii) $\alpha^2\beta + \alpha\beta^2 = 6$

$\alpha\beta(\alpha + \beta) = 6$

$\dfrac{k}{2}(-4) = 6$ from (i)

$-2k = 6$

$k = -3$.

(c) $V = \pi\displaystyle\int_0^4 \left(1 + \sqrt{x}\right)^2 dx$

$= \pi\displaystyle\int_0^4 \left(1 + 2x^{\frac{1}{2}} + x\right) dx$

$= \pi\left[x + 2.\tfrac{2}{3}x^{\frac{3}{2}} + \tfrac{1}{2}x^2\right]_0^4$

$= \pi\left[4 + \tfrac{4}{3}(4)^{\frac{3}{2}} + \tfrac{1}{2}(4)^2 - (0 + 0 + 0)\right]$

$= \dfrac{68\pi}{3}$

$\therefore$ the volume is $\dfrac{68\pi}{3}$ units^3.

(d) (i) First dose: 10 mL

Second dose: $\left(\dfrac{1}{3}\times 10 + 10\right) = 13\dfrac{1}{3}$

$\therefore\ 13\dfrac{1}{3}$ mL remains.

(ii) Third dose:

$\left[10 + \dfrac{1}{3}\left(10 + \dfrac{1}{3}\times 10\right)\right] = 10 + \dfrac{10}{3} + \dfrac{10}{3^2}$

This forms a geometric series with

$a = 10,\ r = \dfrac{1}{3}$.

$S_\infty = \dfrac{a}{1 - r}$

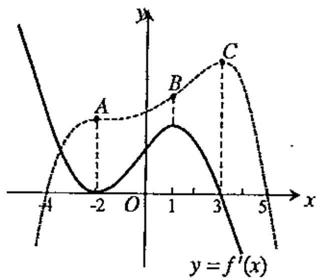
$= \dfrac{10}{1 - \frac{1}{3}}$

$= 10 \times \dfrac{3}{2}$

$= 15$

$\therefore$ there will never be more than 15 mL of the drug in the patient's body.

(e)



Question 15

(a) $2\sin^2 x + \cos x - 2 = 0$

$2(1 - \cos^2 x) + \cos x - 2 = 0$

$2 - 2\cos^2 x + \cos x - 2 = 0$

$2\cos^2 x - \cos x = 0$

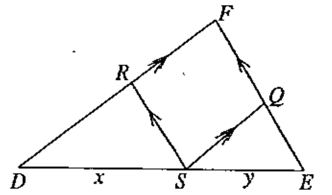
$\cos x(2\cos x - 1) = 0$

$\therefore\ \cos x = 0$ or $\cos x = \dfrac{1}{2}$

$x = \dfrac{\pi}{2}, \dfrac{3\pi}{2} \qquad x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

The solutions are $x = \dfrac{\pi}{3}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{3}$.

(b) (i)



In ΔDEF and ΔDSR:

$\angle FDE = \angle RDS$ (common)

$\angle DEF = \angle DSR$ (corres. $\angle$s, $RS \parallel FE$)

$\therefore\ \Delta DEF\ |||\ \Delta DSR$ (equiangular).

(ii) Using corresponding sides of similar triangles:

$\dfrac{DR}{DS} = \dfrac{DF}{DE}$

(corresponding sides of similar triangles)

$\dfrac{DR}{DF} = \dfrac{DS}{DE}$

$= \dfrac{x}{x + y}$.

(iii) $\sqrt{\dfrac{A_1}{A}} = \sqrt{\dfrac{\frac{1}{2}DS.DR\sin D}{\frac{1}{2}DF.DE\sin D}}$

$= \sqrt{\dfrac{DS.DR}{DF.DE}}$

$= \sqrt{\dfrac{DR}{DF}\times\dfrac{DS}{DE}}$

$= \sqrt{\dfrac{x}{x+y}\times\dfrac{x}{x+y}}$ using (ii)

$= \dfrac{x}{x+y}$.

(iv) Using the methods of (ii) and (iii)

$\Delta SQE\ |||\ \Delta DFE$ (equiangular)

$\therefore\ \dfrac{SE}{DE} = \dfrac{EQ}{EF}$

(corresponding sides of similar triangles)

$\dfrac{EQ}{EF} = \dfrac{y}{x + y}$

$$\frac{\text{Area } \Delta SEQ}{\text{Area } \Delta DFE} = \frac{y^2}{(x+y)^2}$$

$$\sqrt{\frac{A_2}{A}} = \frac{y}{x+y}$$

Using (iii) and the result above:

$$\sqrt{\frac{A_1}{A}} + \sqrt{\frac{A_2}{A}} = \frac{x}{x+y} + \frac{y}{x+y}$$

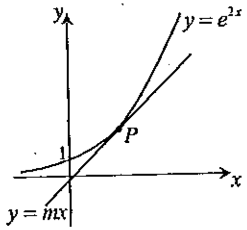
$$\frac{\sqrt{A_1}}{\sqrt{A}} + \frac{\sqrt{A_2}}{\sqrt{A}} = \frac{x+y}{x+y}$$

$$\frac{\sqrt{A_1}+\sqrt{A_2}}{\sqrt{A}} = 1$$

$$\sqrt{A_1}+\sqrt{A_2} = \sqrt{A}$$

$$\sqrt{A} = \sqrt{A_1}+\sqrt{A_2}.$$

(c) (i)



(ii) At P, the gradients are equal:

$$\frac{d}{dx}\left(e^{2x}\right) = \frac{d}{dx}(mx)$$

$$\therefore\ 2e^{2x} = m$$

$$e^{2x} = \frac{m}{2}$$

$$2x = \ln\left(\frac{m}{2}\right)$$

$$x = \frac{1}{2}\ln\left(\frac{m}{2}\right)$$

The y-value at P is:

$$y = mx$$

$$y = m\left(\frac{1}{2}\ln\left(\frac{m}{2}\right)\right)$$

$$= \frac{m}{2}\ln\left(\frac{m}{2}\right)$$

$$\therefore\ P = \left(\frac{1}{2}\ln\left(\frac{m}{2}\right), \frac{m}{2}\ln\left(\frac{m}{2}\right)\right).$$

(iii) At the point P:

$$mx = e^{2x}$$

$$m\left(\frac{1}{2}\ln\frac{m}{2}\right) = e^{2\left(\frac{1}{2}\ln\frac{m}{2}\right)}$$

$$m\left(\frac{1}{2}\ln\frac{m}{2}\right) = e^{\ln\frac{m}{2}}$$

$$m\left(\frac{1}{2}\ln\frac{m}{2}\right) = \frac{m}{2}$$

$$\frac{1}{2}\ln\frac{m}{2} = \frac{1}{2}$$

$$\ln\frac{m}{2} = 1$$

$$\frac{m}{2} = e^1$$

$$m = 2e.$$

Question 16

(a) $f(x) = \sec x$

x	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$
$f(x)$	2	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	2

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sec x\, dx \approx \frac{\frac{\pi}{3}-0}{6}\left[2+4\left(\tfrac{2}{\sqrt{3}}\right)+2(1)+4\left(\tfrac{2}{\sqrt{3}}\right)+\right.$$

$$= \frac{\pi}{18}\left[2+\frac{8}{\sqrt{3}}+2+\frac{8}{\sqrt{3}}+2\right]$$

$$= \frac{\pi}{18}\left[6+\frac{16}{\sqrt{3}}\right]$$

$$= \frac{\pi}{9}\left[3+\frac{8}{\sqrt{3}}\right].$$

(b) (i) Let A_n be the amount in the account at the end of month n.

$$A_1 = 500(1.003)$$

$$A_2 = [A_1 + 500(1.01)]\,(1.003)$$

$$= [500(1.003) + 500(1.01)]\,(1.003)$$

$$= 500(1.003)^2 + 500(1.01)(1.003)$$

as required.

(ii) Following the same pattern:

$$A_3 = [A_2 + 500(1.01)^2]\,(1.003)$$

$$= [500(1.003)^2 + 500(1.01)(1.003) + 500(1.01)^2]\,(1.003)$$

$$= 500(1.003)^3 + 500(1.01)(1.003)^2 + 500(1.01)^2(1.003)$$

$$A_{60} = 500(1.003)^{60} + 500(1.01)(1.003)^{59} +$$

$$\ldots + 500(1.01)^{59}(1.003)$$

This is a geometric series with

$$a = 500(1.003)^{60}$$

$$r = \frac{1.01}{1.003}$$

$$n = 60$$

$$S_n = \frac{a(r^n - 1)}{r-1}$$

$$= \frac{500(1.003)^{60}\left[\left(\frac{1.01}{1.003}\right)^{60} - 1\right]}{\left(\frac{1.01}{1.003}\right) - 1}$$

$$= 44\,404.37866\ldots$$

$\therefore$ she has \$44 404 in the account.

(c) (i) Length of metal framing is 10 m.

$$2x + 2y + \frac{1}{2}\pi x = 10$$

$$x + y + \frac{\pi}{4}x = 5$$

$$y = 5 - x + \frac{\pi}{4}x$$

$$= 5 - x\left(1 + \frac{\pi}{4}\right)$$

as required.

(ii) Area of semicircle $= \frac{1}{2}\left[\pi\left(\frac{x}{2}\right)^2\right]$

$$= \frac{\pi x^2}{8}$$

Area of rectangle $= xy$

$$L = \frac{\pi x^2}{8} + 3xy$$

$$= \frac{\pi x^2}{8} + 3x\left[5 - x\left(1 + \frac{\pi}{4}\right)\right]$$

$$L = \frac{\pi x^2}{8} + 15x - 3x^2 - \frac{3\pi x^2}{4}$$

$$= 15x - 3x^2 - \frac{5\pi x^2}{8}$$

$$= 15x - x^2\left(3 + \frac{5\pi}{8}\right) \quad \text{as required.}$$

(iii) $L = 15x - \left(3 + \frac{5\pi}{8}\right)x^2$

$$\frac{dL}{dx} = 15 - 2\left(3 + \frac{5\pi}{8}\right)x$$

Maximum light when $\frac{dL}{dx} = 0$

$$15 - 2\left(3 + \frac{5\pi}{8}\right)x = 0$$

$$15 = \left(6 + \frac{5\pi}{4}\right)x$$

$$60 = (24 + 5\pi)x$$

$$x = \frac{60}{24 + 5\pi}$$

Testing nature using the 2nd derivative:

$$\frac{dL}{dx} = 15 - 2\left(3 + \frac{5\pi}{8}\right)x$$

$$\frac{d^2L}{dx^2} = -2\left(3 + \frac{5\pi}{8}\right) < 0 \quad \therefore \text{ max}$$

The y-value is given by:

$$y = 5 - x\left(1 + \frac{\pi}{4}\right)$$

$$= 5 - \left(\frac{60}{24 + 5\pi}\right)\left(\frac{4 + \pi}{4}\right)$$

$$= 5 - 15\left(\frac{4 + \pi}{24 + 5\pi}\right)$$

$$= \frac{5(24 + 5\pi)}{24 + 5\pi} - \left(\frac{60 + 15\pi}{24 + 5\pi}\right)$$

$$= \frac{60 + 10\pi}{24 + 5\pi}$$

$\therefore$ maximum light enters when $x = \frac{60}{24 + 5\pi}$ and $y = \frac{60 + 10\pi}{24 + 5\pi}$.

End of Mathematics solutions