



Barker College

Barker Student Number: \_\_\_\_\_

# Mathematics Extension 1

## 2006 TRIAL HIGHER SCHOOL CERTIFICATE

PM FRIDAY 11 AUGUST

Staff Involved:

- GDH\*
- WMD\*
- JM
- BTP
- GIC
- LJP
- CFR

65 copies

### General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Make sure your Barker Student Number is on ALL pages
- Board-approved calculators may be used
- A table of standard integrals is provided on page 9
- ALL necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged working

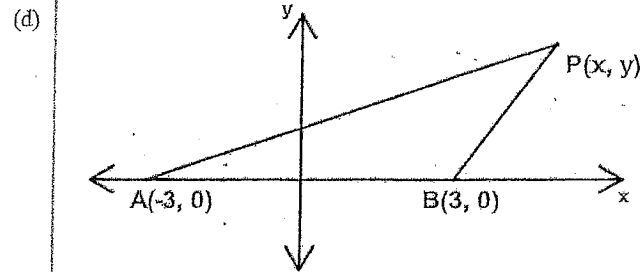
Total marks – 84

- Attempt Questions 1 – 7
- All questions are of equal value

Marks

Question 1. (12 marks) [BEGIN A NEW PAGE]

- (a) Point A has coordinates  $(-2, 4)$ . Point B has coordinates  $(10, -8)$ . Find the coordinates of the point P that divides the interval AB externally in the ratio  $3 : 2$ . 2
- (b) Find  $\int x\sqrt{2x-1} dx$  using the substitution  $u = 2x - 1$ . 3
- (c) State whether the following claim is true or false and give a reason why:  
“Because there are two types of students at Barker (day and boarder), the probability that a randomly selected Barker student is a boarder is 50%.” 1



The locus of P follows this rule:

“The gradient of PA is one unit less than the gradient of PB.”

- (i) Show that this locus has equation:  $x^2 = 6y + 9$ . 2
- (ii) Hence, or otherwise, find the coordinates of the focus of this locus. 1
- (e) Solve:  $\frac{1}{x^3} > \frac{1}{x^5}$ . 3

Marks

Question 2 (12 marks) [BEGIN A NEW PAGE]

(a) By writing  $\sin(-15^\circ)$  in the form  $\sin(A - B)$ , find the exact value of  $\sin(-15^\circ)$ .

2

(b) Consider the functions  $f(x) = \cos^{-1}(2x)$  and  $g(x) = \sin^{-1}x$ .

(i) Sketch  $f(x) = \cos^{-1}(2x)$ .

2

(ii) Prove that the  $x$ -coordinate of the point of intersection of  $f(x)$  and  $g(x)$  is  $\frac{1}{\sqrt{5}}$ .

2

(iii) Show that the gradients of the tangents to  $f(x)$  and  $g(x)$  at their point of intersection are  $-2\sqrt{5}$  and  $\frac{\sqrt{5}}{2}$  respectively.

3

(iv) Write the expansion of  $\tan(\beta - \alpha)$ .

1

(v) Hence, or otherwise, find the acute angle between  $f(x)$  and  $g(x)$  at their point of intersection (to the nearest degree).

2

Marks

Question 3 (12 marks) [BEGIN A NEW PAGE]

(a) (i) If  $y = \ln(\sin x)$ , find  $\frac{dy}{dx}$ .

1

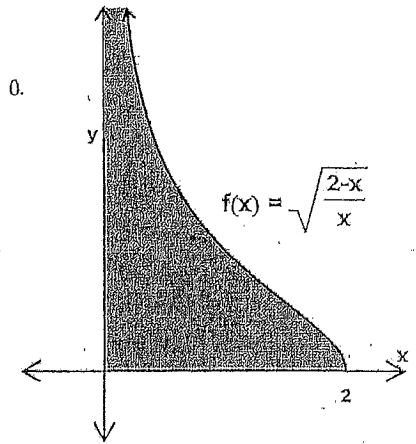
(ii) Hence, or otherwise, find  $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cot x \, dx$ .

2

(b) Find  $\int_0^{\frac{\pi}{12}} \cos^2(3x) \, dx$ .

3

(c) The curve on the right has an asymptote at  $x = 0$ . The shaded area 'goes forever' but its value has a limit. Steffi Graph was trying to find this limit but was unable to integrate  $f(x)$ . Her friend Monica suggested she use the inverse function to find the shaded area.



(i) State the domain and range of the inverse function  $f^{-1}(x)$ .

1

(ii) Roughly sketch the inverse function  $f^{-1}(x)$ .

1

(iii) Show that  $f^{-1}(x) = \frac{2}{1+x^2}$ .

2

(iv) Hence, or otherwise, find the limit that the value of the shaded area approaches.

2

Question 4 (12 marks) [BEGIN A NEW PAGE]

Marks

(a) Consider the curve:  $y = \frac{x^2 - 3}{x + 2}$

(i) Find all intercepts and the equation of the vertical asymptote.

2

(ii) Find and determine the nature of the stationary points.

3

(iii) Show that  $(x - 2) + \frac{1}{x + 2} = \frac{x^2 - 3}{x + 2}$ .

1

(iv) Hence, or otherwise, find the equation of any non-vertical asymptotes by considering what happens as  $x \rightarrow \pm\infty$ .

1

(v) Sketch the curve showing all the above features (you can assume there are no points of inflexion).

1

(b) Prove, by mathematical induction, that  $5^n + 3$  is divisible by 2 for  $n \geq 0$  where  $n$  is an integer.

4

Question 5 (12 marks) [BEGIN A NEW PAGE]

Marks

(a) A particle moves in a straight line and its position in metres at any time  $t$  seconds is given by the equation:  $x = 5\cos(2t) - 12\sin(2t)$ .

(i) Show, by differentiation, that the motion is simple harmonic.

2

(ii) State the period of the motion.

1

(iii) Express  $x$  in the form  $R\cos(2t + \alpha)$ .

2

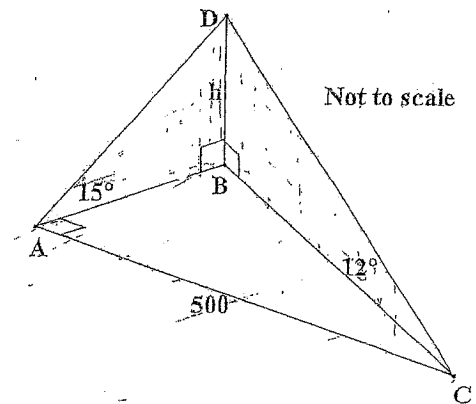
(iv) Hence, or otherwise, state the amplitude of the motion.

1

(v) Find the general solution for all the times when the particle is at the centre of the motion.

2

(b) From a point A, the top of a tower BD directly north of A has an angle of elevation of  $15^\circ$ . After walking 500 metres on a bearing of  $90^\circ$ , the top of the tower has an angle of elevation of  $12^\circ$ . Let  $h$  be the height of the tower.



(i) Give an expression for AB in terms of  $h$ .

1

(ii) Hence, find the height of the tower (to the nearest metre).

3

Question 6 (12 marks) [BEGIN A NEW PAGE]

Marks

- (a) Newton's Law of Cooling can be written with  $t$  as the subject as follows:

$$t = -\frac{1}{A} \ln\left(\frac{B-C}{D}\right).$$

Showing steps of working, make  $B$  the subject of this formula.

2

- (b) The length of each edge of a cube is  $x$  cm.

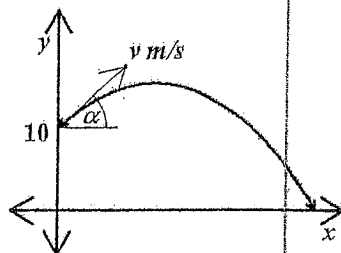
- (i) Write expressions for the surface area ( $A$ ) and volume ( $V$ ) of the cube and hence find  $\frac{dA}{dx}$  and  $\frac{dV}{dx}$ .

1

- (ii) The surface area of the cube increases at a rate of  $6 \text{ cm}^2/\text{second}$ . Find the rate of change of the volume of the cube when the length of each edge is  $5$  cm.

2

- (c) A stone is thrown from a  $10\text{m}$  high cliff with velocity  $v$  m/s at an angle of projection  $\alpha$ . The stone's horizontal displacement from the origin,  $t$  seconds after being thrown, is given by the equation  $x = vt \cos \alpha$ . Do not prove this.



- (i) Given that  $\ddot{y} = -g$ , prove that the stone's vertical displacement from the origin,  $t$  seconds after being thrown, is given by  $y = vt \sin \alpha - \frac{gt^2}{2} + 10$ .

3

- (ii) Show that  $y = x \tan \alpha - \frac{gx^2 \sec^2 \alpha}{2v^2} + 10$ .

1

- (iii) Given that  $\alpha = 45^\circ$ ,  $g = 10 \text{ m/s}^2$  and  $v = 15 \text{ m/s}$ , how far from the base of the cliff does the stone land?

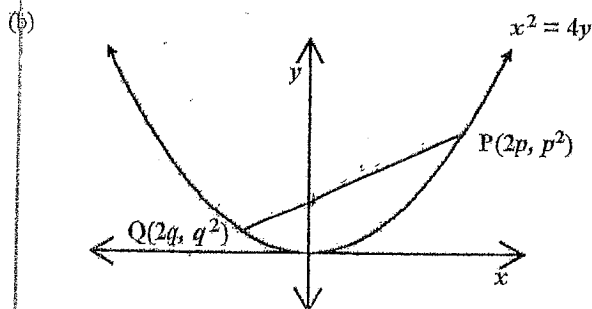
3

Marks

Question 7 (12 marks) [BEGIN A NEW PAGE]

- (a) It is known that if  $\frac{dy}{dx} = y$ , then  $y = e^x$  is a solution (since  $\frac{dy}{dx} = e^x = y$ ). If  $\frac{dy}{dx} = \frac{1}{y}$ , find a solution for  $y$ .

2



- (i) Show that the gradient of  $PQ$  is  $\frac{p+q}{2}$ .

1

For the remainder of the question, assume that  $PQ$  is a focal chord, passing through the focus  $F(0, 1)$ .

- (ii) Show that  $pq = -1$ .

2

- (iii) Show that the equation of  $PQ$  is  $y = \left(\frac{p^2-1}{2p}\right)x + 1$ .

1

- (iv) Let  $A$  be the area bounded by the parabola and the focal chord.

Show that  $A = \frac{1}{3} \left( p^3 + 3p + \frac{3}{p} + \frac{1}{p^3} \right)$ .

[You may assume that  $p > 0$  and  $q < 0$ ]

3

- (v) Hence, or otherwise, find the value of  $p$  that gives the minimum area found in part (iv).

3

1) (a)

A(-2, 4) B(10, -8)  
 $\frac{3}{-2}$

$P = \left( \frac{3 \times 10 - 2 \times (-2)}{3 - 2}, \frac{3 \times (-8) - 2 \times 4}{3 - 2} \right)$   
 $= (34, -32)$

1 mark for internal division with correct answer  $(5\frac{1}{2}, -3\frac{1}{5})$

(b)

$\int x \sqrt{2x-1} dx$   $u = 2x-1$   
 $\frac{du}{dx} = 2$   
 $dx = \frac{du}{2}$   
 $x = \frac{u+1}{2}$   
 $= \int \frac{(u+1)}{2} \cdot u^{\frac{1}{2}} \cdot \frac{du}{2}$   
 $= \frac{1}{4} \int (u^{\frac{3}{2}} + u^{\frac{1}{2}}) du$   
 $= \frac{1}{4} \left[ \frac{2}{5} u^{\frac{5}{2}} + \frac{2}{3} u^{\frac{3}{2}} \right] + C$   
 $= \frac{1}{10} (2x-1)^{\frac{5}{2}} + \frac{1}{6} (2x-1)^{\frac{3}{2}} + C$

No penalty for not having a constant of integration

(c) The statement is false as there are far more day students than boarders which means you are more likely to randomly select a day student.

Must justify answer to get the mark.

(d) (i)  $m_{PA} = m_{PB} - 1$

$\frac{y}{x+3} = \frac{y}{x-3} - 1$

$y(x-3) = y(x+3) - (x+3)(x-3)$

$xy - 3y = xy + 3y - (x^2 - 9)$

$x^2 - 9 = 6y$

$x^2 = 6y + 9$

(ii)  $x^2 = 4(1.5)(y+1.5)$

$\therefore$  vertex is  $(0, -1.5)$  & focal length =  $1.5u$

$\therefore$  Focus is the origin

(1)

(e)  $\frac{1}{x^3} > \frac{1}{x^5}, x \neq 0$

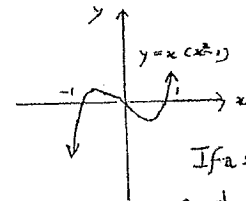
$x^6 \times \frac{1}{x^3} > x^6 \times \frac{1}{x^5}$   $\checkmark$  can also multiply by  $x^8, x^{10}$ , etc.

$x^3 > x$

$x^3 - x > 0$

$x(x^2 - 1) > 0$   
 $x(x+1)(x-1) > 0$   $\checkmark$

$\therefore -1 < x < 0$  or  $x > 1$   $\checkmark$



Ignore any irregularities with  $\leq$  or  $\geq$  instead of  $<$  and  $>$ .

If a student multiplies thru' by  $x^5$  and gets  $x < -1$  or  $x > 1$  award a total of 1 mark.

(2)

(a)  $\sin(-15^\circ)$

$= \sin(30^\circ - 45^\circ)$

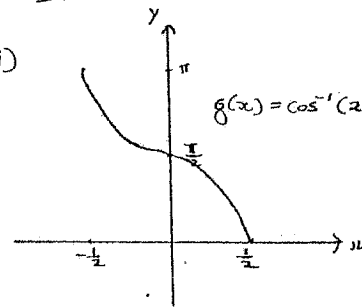
$= \sin 30^\circ \cos 45^\circ - \cos 30^\circ \sin 45^\circ$   $\checkmark$

$= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{1 - \sqrt{3}}{2\sqrt{2}}$

$\checkmark$  - ignore subsequent errors

(b) (i)



correct domain 1 mark  
 correct shape & range 1 mark

(ii) Let  $f(x) = g(x)$

i.e.  $\cos^{-1}(2x) = \sin^{-1}(x)$

Now  $\sin^2 \alpha + \cos^2 \alpha = 1$

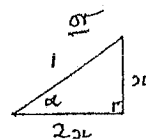
$\therefore x^2 + (2x)^2 = 1$   $\checkmark$

$5x^2 = 1$

$x^2 = \frac{1}{5}$

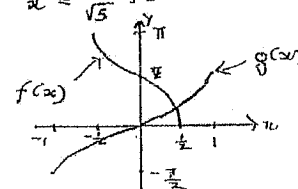
$x = \frac{1}{\sqrt{5}}$ , since  $x > 0$  - see graph  $\leftarrow$  ignore  $\pm$  in marking.

Let  $\sin^{-1} x = \alpha$   
 $\Rightarrow x = \sin \alpha$   
 Let  $\cos^{-1}(2x) = \alpha$   
 $\Rightarrow 2x = \cos \alpha$



$x^2 + (2x)^2 = 1$   $\checkmark$

etc.



②  
(c)

(ii)  $f'(x) = \frac{-2}{\sqrt{1-(2x)^2}}$  ✓

$$f'(\frac{1}{\sqrt{5}}) = \frac{-2}{\sqrt{1-\frac{4}{5}}}$$

$$= \frac{-2}{\frac{1}{\sqrt{5}}} \quad \checkmark$$

$$= -2\sqrt{5}$$

$$g'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$g'(\frac{1}{\sqrt{5}}) = \frac{1}{\sqrt{1-\frac{1}{5}}}$$

$$= \frac{1}{\sqrt{\frac{4}{5}}}$$

$$= \frac{1}{\frac{2}{\sqrt{5}}} \quad \checkmark$$

$$= \frac{\sqrt{5}}{2}$$

(iv)  $\tan(\beta - \alpha) = \frac{\tan \beta - \tan \alpha}{1 + \tan \beta \tan \alpha}$  ✓

(v) Let the required angle be  $\phi$

$$\tan \phi = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$= \left| \frac{\frac{\sqrt{5}}{2} + 2\sqrt{5}}{1 - 5} \right| \quad \checkmark$$

$$= \frac{5\sqrt{5}}{8}$$

$$\therefore \phi = 54.41\dots^\circ$$

$$= \underline{\underline{54^\circ}} \text{ (n. deg.)} \quad \checkmark \text{ Don't worry about accuracy.}$$

③ (a) (i)  $y = \ln(\sin x)$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} \quad \checkmark$$

$$= \underline{\underline{\cot x}}$$

(ii)  $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cot x \, dx$

$$= \left[ \ln(\sin x) \right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \quad \checkmark$$

$$= \ln(\sin \frac{3\pi}{4}) - \ln(\sin \frac{\pi}{4})$$

$$= \ln(\frac{1}{\sqrt{2}}) - \ln(\frac{1}{\sqrt{2}})$$

$$= \underline{\underline{0}} \quad \checkmark$$

(b)  $\int_0^{\frac{\pi}{12}} \cos^2(3x) \, dx$

$$= \frac{1}{2} \int_0^{\frac{\pi}{12}} (\cos 6x + 1) \, dx \quad \checkmark$$

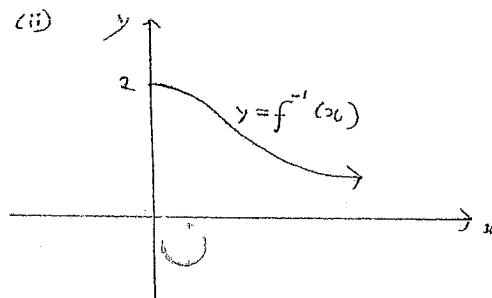
$$= \frac{1}{2} \left[ \frac{\sin 6x}{6} + x \right]_0^{\frac{\pi}{12}} \quad \checkmark$$

$$= \frac{1}{2} \left\{ \left( \frac{1}{6} + \frac{\pi}{12} \right) - (0+0) \right\} \quad \checkmark$$

$$= \underline{\underline{\frac{2+\pi}{24}}}$$

(c) (i)  $D_{f^{-1}} : x \geq 0$   
 $R_{f^{-1}} : 0 < y \leq 2$  } 1

Must have inequality signs exactly right for the mark.



y-intercept and shape required for 1 mark.

3(c)  
(iii)

$$y = f(x) = \sqrt{\frac{2-x}{x}}$$

$\therefore x = \sqrt{\frac{2-y}{y}}$  is inverse function ✓

$$x^2 = \frac{2-y}{y} \quad \text{or} \quad x^2 y = 2-y$$

$$x^2 = \frac{2}{y} - 1 \quad \checkmark \quad x^2 y + y = 2$$

$$x^2 + 1 = \frac{2}{y} \quad y(x^2 + 1) = 2 \quad \checkmark$$

$$y = \frac{2}{x^2 + 1} = f^{-1}(x)$$

(iv) Consider  $\int_0^a \frac{2}{1+x^2} dx = 2 [\tan^{-1} x]_0^a$

$$= 2 \{ \tan^{-1} a - \tan^{-1} 0 \}$$

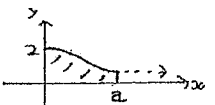
$$= 2 \tan^{-1} a$$

Req'd area =  $\lim_{a \rightarrow \infty} 2 \tan^{-1} a$

$$= 2 \lim_{a \rightarrow \infty} \tan^{-1} a$$

$$= 2 \times \frac{\pi}{2} \quad \checkmark$$

$$= \pi \text{ units}^2$$



OR  $A = \int_0^{\infty} \frac{2}{1+x^2} dx \quad \checkmark$

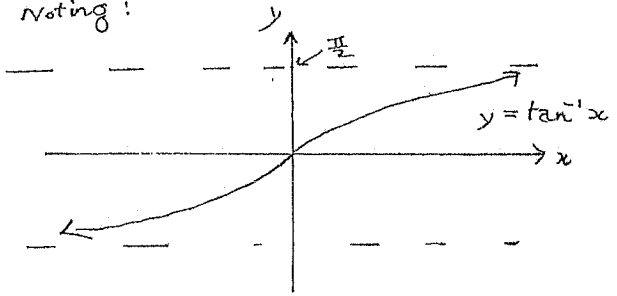
$$= 2 [\tan^{-1} x]_0^{\infty}$$

$$= 2 \{ \tan^{-1} \infty - \tan^{-1} 0 \}$$

$$= 2 \left\{ \frac{\pi}{2} - 0 \right\} \quad \checkmark$$

$$= \pi \text{ units}^2$$

Noting:



4(a)

(i)  $y = \frac{(x-\sqrt{3})(x+\sqrt{3})}{x+2}$

$\Rightarrow x = -2$  is vertical asymptote  
 when  $x = 0$ ,  $y = -\frac{3}{2}$  [y-intercept] } mark  
 when  $y = 0$ ,  $x = \pm\sqrt{3}$  [x-intercepts] ✓

(ii)  $\frac{dy}{dx} = \frac{(x+2)2x - (x^2-3)1}{(x+2)^2} \quad \checkmark$

$$= \frac{2x^2 + 4x - x^2 + 3}{(x+2)^2}$$

$$= \frac{x^2 + 4x + 3}{(x+2)^2}$$

$$= \frac{(x+1)(x+3)}{(x+2)^2}$$

$\Rightarrow$  stat. pts at  $x = -1, -3 \quad \checkmark$   
 when  $x = -1$ ,  $y = \frac{1-3}{1} = -2$   
 when  $x = -3$ ,  $y = \frac{9-3}{-1} = -6$

test  $(-1, -2)$ :

$x$	$-1\frac{1}{2}$	$-1$	$0$
$\frac{dy}{dx}$	$\frac{(-\frac{1}{2})(\frac{1}{2})}{(\frac{1}{2})^2}$	$0$	$\frac{3}{4}$
	neg.		pos.

min. t.p. } mark  
 max. t.p.

test  $(-3, -6)$ :

$x$	$-4$	$-3$	$-2\frac{1}{2}$
$\frac{dy}{dx}$	$\frac{(-3)(-1)}{(-2)^2}$	$0$	$\frac{(-1)(\frac{1}{2})}{(-\frac{1}{2})^2}$
	pos.		neg.

(iii)  $(2x-2) + \frac{1}{x+2} = \frac{(x-2)(x+2) + 1}{x+2}$

$$= \frac{x^2 - 4 + 1}{x+2} \quad \checkmark$$

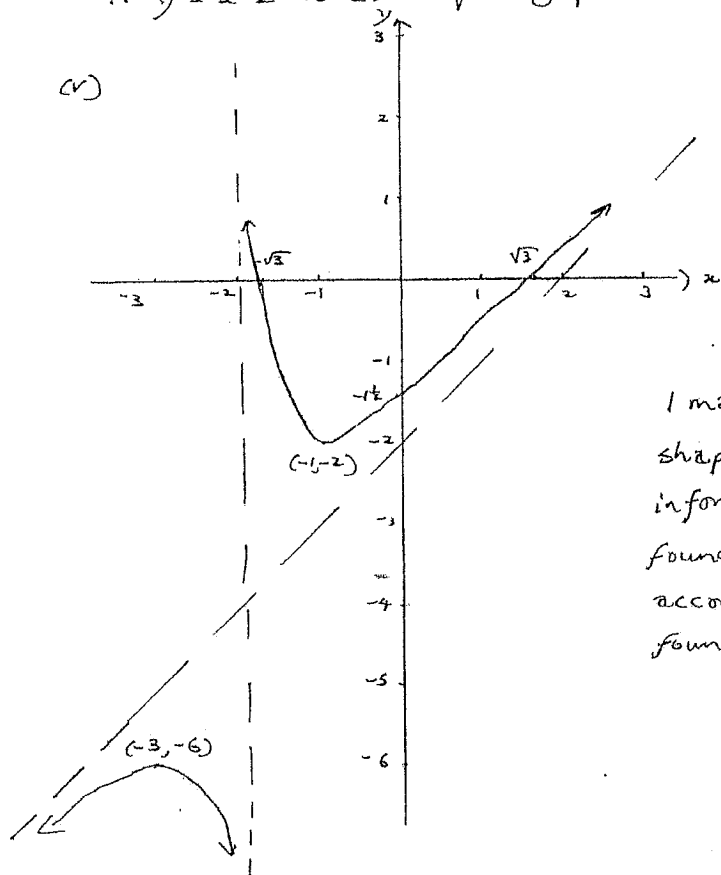
$$= \frac{x^2 - 3}{x+2}$$

④ (a) (iv)

$$\text{as } x \rightarrow \infty, (x-2) + \frac{1}{x+2} \rightarrow x-2$$

$\therefore y = x-2$  is an oblique asymptote. ✓

(v)



1 mark for correct shape - (all other information has been found earlier) - according to values found above.

b) when  $n=0, 5^0+3 = 4 \mid 2$

$$\Rightarrow 5^n+3 \mid 2 \text{ for } n=0. \quad \checkmark$$

Assume  $5^k+3 = 2J, J \in \mathbb{Z}^+$

$$\text{i.e. } 5^k = 2J-3$$

Required to prove  $5^{k+1}+3 \mid 2$

$$\begin{aligned} \text{Now } 5^{k+1}+3 &= 5 \cdot 5^k+3 \quad \checkmark \\ &= 5(2J-3)+3 \text{ by induction} \\ &= 10J-12 \text{ , supposition} \end{aligned}$$

④ (b) (cont.)

$$\Rightarrow 5^{k+1}+3 \mid 2 \text{ if } 5^k+3 \mid 2 \quad (*)$$

Since  $5^n+3 \mid 2$  for  $n=0$   
it is true for  $n=1$  and  
hence true for all subsequent  
positive integral values of  $n$   
by  $(*)$ .

⑤ (a) (i)  $x = 5 \cos(2t) - 12 \sin(2t)$

$$\dot{x} = -10 \sin(2t) - 24 \cos(2t) \quad \checkmark$$

$$\begin{aligned} \ddot{x} &= -20 \cos(2t) + 48 \sin(2t) \\ &= -4(5 \cos(2t) - 12 \sin(2t)) \\ &= -4x \quad \checkmark \end{aligned}$$

which is in the form  $\ddot{x} = -n^2 x$ .  
Hence the motion is simple harmonic.

(ii) Period =  $\frac{2\pi}{2} = \pi$  seconds ✓

(iii)  $R \cos(2t + \alpha)$   
 $= R[\cos(2t)\cos\alpha - \sin(2t)\sin\alpha]$

$$\therefore \text{Let } 5 \cos(2t) - 12 \sin(2t) \equiv R \cos(2t)\cos\alpha - R \sin(2t)\sin\alpha$$

$$\Rightarrow R \cos\alpha = 5 \quad \& \quad R \sin\alpha = 12 \quad \checkmark$$

$$R^2 \cos^2\alpha + R^2 \sin^2\alpha = 5^2 + 12^2$$

$$R^2 (\cos^2\alpha + \sin^2\alpha) = 169$$

$$R = 13 \quad [R > 0]$$

$$\frac{R \sin\alpha}{R \cos\alpha} = \frac{12}{5} \Rightarrow \tan\alpha = \frac{12}{5} \quad \checkmark$$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{12}{5}\right) \quad [\alpha \text{ acute}]$$

$$\therefore x = 13 \cos\left(2t + \tan^{-1}\left(\frac{12}{5}\right)\right)$$

also  $\alpha = \sin^{-1}\left(\frac{12}{13}\right)$   
 $= \cos^{-1}\left(\frac{5}{13}\right)$

1 mark for getting both  $R$  and  $\alpha$ .



3(a)

(iv) Amplitude = 13 metres. ✓

(v) Particle is at centre of motion

when acceleration is zero

at  $x = 0$  as eqn is in

form  $\ddot{x} = -n^2x$

i.e. when  $5 \cos(2t) - 12 \sin(2t) = 0$

i.e. when  $13 \cos(2t + \tan^{-1}(\frac{12}{5})) = 0$  ✓

$\Rightarrow \cos(2t + \tan^{-1}(\frac{12}{5})) = 0$

$2t + \tan^{-1}(\frac{12}{5}) = 2n\pi \pm \cos^{-1}(0)$

$2t + \tan^{-1}(\frac{12}{5}) = 2n\pi \pm \frac{\pi}{2}$  ✓

$t = \frac{2n\pi + \frac{\pi}{2} - \tan^{-1}(\frac{12}{5})}{2}, \frac{2n\pi - \frac{\pi}{2} - \tan^{-1}(\frac{12}{5})}{2}$

choosing  $n$  to be an integer such that  $t > 0$ .

(b)(i)  $\angle ADB = 75^\circ$  ( $\angle$  sum  $\Delta$ )

$\therefore \frac{AB}{h} = \tan 75^\circ$

or  $\frac{h}{AB} = \tan 15^\circ$

or  $AB = \frac{h}{\tan 15^\circ}$

$\therefore AB = h \tan 75^\circ$  ✓

$AB = \frac{h}{\tan 15^\circ} = h \cot 15^\circ$

(ii) Similarly  $BC = h \tan 78^\circ$

By Pythagoras:  $AB^2 + AC^2 = BC^2$

$\therefore h^2 \tan^2 75^\circ + 500^2 = h^2 \tan^2 78^\circ$  ✓

$h^2 (\tan^2 78^\circ - \tan^2 75^\circ) = 250000$

$h = \frac{500}{\sqrt{\tan^2 78^\circ - \tan^2 75^\circ}}$  ✓

$= 174.55 \dots$

$= \underline{175 \text{ m}}$  (nearest metre) ✓

No penalty for accuracy/units.

6 (a)  $t = -\frac{1}{A} \ln\left(\frac{B-C}{D}\right)$

$-At = \ln\left(\frac{B-C}{D}\right)$

$e^{-At} = \frac{B-C}{D}$  ✓

$B-C = D e^{-At}$

$\therefore \underline{B = C + D e^{-At}}$  ✓

(b)(i)  $A = 6x^2 \Rightarrow \frac{dA}{dx} = 12x$  } 1 mark

$V = x^3 \Rightarrow \frac{dV}{dx} = 3x^2$

← If any of this done in (i) give mark.

(ii)  $\frac{dA}{dt} = 6x^2/s$

or  $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dA} \times \frac{dA}{dt}$

$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$

$= 3x^2 \times \frac{1}{12x} \times 6$  ✓

when  $x = 5$ ,  $6 = (12 \times 5) \times \frac{dx}{dt}$

when  $x = 5$ ,  $\frac{dV}{dt} = 3 \times 5^2 \times \frac{1}{12 \times 5} \times 6$  ✓

$\frac{dx}{dt} = \frac{1}{10} \text{ cm/s}$  ✓

$= 7.5 \text{ cm}^3/\text{s}$

and  $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

or  $\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$

$= (3 \times 5^2) \times \frac{1}{10}$  ✓

$= \frac{1}{12x} \times 6$

$= \underline{7.5 \text{ cm}^3/\text{s}}$

$= \frac{1}{2x}$  ✓

(c)  $\ddot{y} = -g$

$\dot{y} = \int -g dt$

$= -gt + c_1$

when  $t = 0$ ,  $\dot{y} = v \sin \alpha$

$v \sin \alpha = c_1$  ✓

$\therefore \dot{y} = -gt + v \sin \alpha$

$y = \int (-gt + v \sin \alpha) dt$

$= -\frac{gt^2}{2} + vt \sin \alpha + c_2$  ✓

when  $t = 0$ ,  $y = 10$ :

$10 = c_2$  ✓

$\therefore y = vt \sin \alpha - \frac{gt^2}{2} + 10$

$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

$= 3x^2 \times \frac{1}{2x}$

$= \frac{3x}{2}$

when  $x = 5$ ,  $\frac{dV}{dt} = \frac{3 \times 5}{2}$  ✓

$= 7.5 \text{ cm}^3/\text{s}$

Students must demonstrate by substitution how they arrive at both the constants of integration - do not award full marks to solutions that skip any key steps in the process.

6(c)(ii)

$$x = vt \cos \alpha \Rightarrow t = \frac{x}{v \cos \alpha}$$

$$\therefore y = v \times \frac{x}{v \cos \alpha} \times \sin \alpha - \frac{g \times \left(\frac{x}{v \cos \alpha}\right)^2}{2} + 10 \checkmark$$

$$= x \tan \alpha - \frac{g x^2}{2 v^2 \cos^2 \alpha} + 10$$

$$= x \tan \alpha - \frac{g x^2 \sec^2 \alpha}{2 v^2} + 10$$

$\alpha = 45^\circ, g = 10 \text{ m/s}^2, v = 15 \text{ m/s}$   
 (ii)  $\Rightarrow y = x - \frac{10 x^2 \times x}{2 \times 225} + 10 \checkmark$  [substituting & calculating the value of  $\sec^2 45^\circ$ ]

When  $y = 0, 0 = x - \frac{10 x^2}{225} + 10$

$$\frac{10 x^2}{225} - x - 10 = 0$$

$$2x^2 - 45x - 450 = 0 \checkmark$$

$$(2x + 15)(x - 30) = 0$$

$\therefore x = 30 (x > 0) \checkmark$  ← Must exclude other

So stone lands 30 m from base of cliff. solution to quadratic.

7(a)  $\frac{dx}{dy} = y$  if  $\frac{dy}{dx} = \frac{1}{y}$

$$\therefore \int \frac{dx}{dy} dy = \int y dy \checkmark$$

$$x = \frac{y^2}{2} + \text{a constant}$$

$$y^2 = 2x + \text{a constant}$$

$\therefore y = \pm \sqrt{2x} + c \checkmark \rightarrow$  accept  $y = \sqrt{2x}$   
 or  $y = -\sqrt{2x}$   
 {with or without a constant}

(b) (i)  $m_{PQ} = \frac{p^2 - q^2}{2p - 2q}$   
 $= \frac{(p+q)(p-q)}{2(p-q)} \checkmark$   
 $= \frac{p+q}{2}$

7(b)

(ii) Since PQ passes through  $F_3$

$$m_{PF} = m_{PQ}$$

$$\therefore \frac{p^2 - 1}{2p} = \frac{p+q}{2} \checkmark$$

$$\cancel{2p^2 - 2} = \cancel{2p^2} + 2pq \checkmark$$

$$\therefore pq = -1$$

(iii) As above  $m_{PQ} = m_{PF}$   
 $= \frac{p^2 - 1}{2p}$

OR  $y - 1 = \left(\frac{p+q}{2}\right)(x - 0)$

$$y = \left(\frac{p+q}{2}\right)x + 1$$

$$= \left(\frac{p - \frac{1}{p}}{2}\right)x + 1 \checkmark$$

$$= \left(\frac{p^2 - 1}{2p}\right)x + 1$$

Also PQ passes through  $(0, 1)$  both for mark  
 so its y-intercept is 1.

Hence using  $y = mx + b$ ,  
 PQ is  $y = \left(\frac{p^2 - 1}{2p}\right)x + 1$ .

(iv)  $x^2 = 4y \Rightarrow y = \frac{x^2}{4}$

&  $pq = -1 \Rightarrow q = -\frac{1}{p}$

$\therefore A = \int_{-\frac{2}{p}}^{2p} \left\{ \left(\frac{p^2 - 1}{2p}\right)x + 1 - \frac{x^2}{4} \right\} dx \checkmark$

$$= \left[ \left(\frac{p^2 - 1}{4p}\right)x^2 + x - \frac{x^3}{12} \right]_{-\frac{2}{p}}^{2p}$$

$$= \left\{ \left(\frac{p^2 - 1}{4p}\right) \times 4p^2 + 2p - \frac{8p^3}{12} \right\} - \left\{ \left(\frac{p^2 - 1}{4p}\right) \times \frac{4}{p^2} - \frac{2}{p} + \frac{8}{12p^3} \right\} \checkmark$$

$$= p^3 - p + 2p - \frac{2p^3}{3} - \frac{1}{p} + \frac{1}{p^3} + \frac{2}{p} - \frac{2}{3p^3}$$

$$= \frac{p^3}{3} + p + \frac{1}{p} + \frac{1}{3p^3} \checkmark$$

$$= \frac{1}{3} \left( p^3 + 3p + \frac{3}{p} + \frac{1}{p^3} \right)$$

(b)  
(iv) (cont.)

Alternatively:

$$\begin{aligned}
 A &= \int_{2q}^{2p} \left\{ \left( \frac{p^2-1}{2p} \right) x + 1 - \frac{x^2}{4} \right\} dx \\
 &= \left[ \left( \frac{p^2-1}{4p} \right) x^2 + x - \frac{x^3}{12} \right]_{2q}^{2p} \quad \checkmark \\
 &= \left\{ \left( \frac{p^2-1}{4p} \right) \times 4p^2 + 2p - \frac{8p^3}{12} \right\} - \left\{ \left( \frac{p^2-1}{4p} \right) \times 4q^2 + 2q - \frac{8q^3}{12} \right\} \\
 &= p^3 - p + 2p - \frac{2p^3}{3} - p^2 + \frac{q^2}{p} - 2q + \frac{2q^3}{3} \\
 &= \frac{p^3}{3} + p - p \times \left( -\frac{1}{p} \right)^2 + \frac{\left( -\frac{1}{p} \right)^2}{p} - 2 \left( -\frac{1}{p} \right) + \frac{2 \left( -\frac{1}{p} \right)^3}{3} \quad \left\{ \begin{array}{l} \text{since} \\ q = -\frac{1}{p} \end{array} \right\} \checkmark \\
 &= \frac{p^3}{3} + p - \frac{1}{p} + \frac{1}{p^3} + \frac{2}{p} - \frac{2}{3p^3} \\
 &= \frac{p^3}{3} + p + \frac{1}{p} + \frac{1}{3p^3} \quad \checkmark \\
 &= \frac{1}{3} \left( p^3 + 3p + \frac{3}{p} + \frac{1}{p^3} \right)
 \end{aligned}$$

$$(v) A = \frac{1}{3} (p^3 + 3p + 3p^{-1} + p^{-3})$$

$$\begin{aligned}
 \frac{dA}{dp} &= \frac{1}{3} (3p^2 + 3 - 3p^{-2} - 3p^{-4}) \quad \checkmark \\
 &= p^2 + 1 - \frac{1}{p^2} - \frac{1}{p^4}
 \end{aligned}$$

$$\text{when } \frac{dA}{dp} = 0, \quad p^2 + 1 - \frac{1}{p^2} - \frac{1}{p^4} = 0$$

$$\therefore p^6 + p^4 - p^2 - 1 = 0$$

$$p^4(p^2+1) - 1(p^2+1) = 0$$

$$(p^4-1)(p^2+1) = 0 \quad \checkmark$$

$$\therefore p^4-1=0 \quad \left\{ \begin{array}{l} \text{no real solns} \\ \text{for } p^2+1=0 \end{array} \right\}$$

$$\therefore p^4=1 \Rightarrow p=1 \quad (p>0)$$

$$\begin{aligned}
 \text{when } p = \frac{1}{2}, \quad \frac{dA}{dp} &= \left( \frac{1}{2} \right)^2 + 1 - \frac{1}{\left( \frac{1}{2} \right)^2} - \frac{1}{\left( \frac{1}{2} \right)^4} \\
 &= \frac{1}{4} + 1 - 4 - 16 < 0
 \end{aligned}$$

$$\text{when } p = 2, \quad \frac{dA}{dp} = 4 + 1 - \frac{1}{4} - \frac{1}{16} > 0$$

min. tp