BAULKHAM HILLS HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE

1996

MATHEMATICS

3/4 UNIT

Time allowed - two hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

* Student Number to be clearly written on the top of your front page.
* All questions may be attempted.
* Show all necessary working.
* Staple ALL questions together.

QUESTION 1:

i) Solve $\dfrac{2x+1}{x-2} \geq 1$ for x and graph the solution.

ii) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x \, dx}{\sqrt{4-\cos^2 x}}$ using $u = \cos x$

iii) Solve for x

$$\left(x+\frac{1}{x}\right)^2 - 10\left(x+\frac{1}{x}\right) + 24 = 0$$

iv) Differentiate $\tan^{-1}(\log_e x)$

v) Write down the 3rd term in

$(3a - 2b)^4$

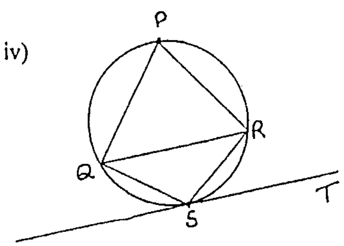
QUESTION 2: (Start a new page)

i) Sketch $y = 2\cos^{-1}\dfrac{x}{3}$

ii) Solve for θ $\quad 3\sin\theta + 4\cos\theta = 3 \quad 0 \leq \theta \leq 2\pi$ (Correct to 2dp.)

iii) The point P(6, 9) divides the interval AB in the ratio -3 : 2. Find the point B given that A is (1, 4)

iv)



ST is a tangent to the circle, and QR ∥ ST. Copy the diagram and prove that

α) SQ = SR
β) SP bisects $\widehat{QPR}$

i) If the probability of a hit in a single run is 0.1. Calculate the probability of a jet fighter getting exactly 2 hits on a target in 20 runs at the target.

ii) a) Show that the equation

$$2x^3 - 3x^2 + 0.99 = 0 \quad \text{has a root near } x = 1.$$

b) Attempt to find an improved value of this root by using Newton's Method once, starting with $x_0 = 1$.

c) Explain why this attempt fails.

iii) The polynomial $2x^3 + 3x^2 + ax - 6$ has $(x+3)$ and $(2x + b)$ as factors.

Find 'a' and 'b'

QUESTION 4: (Start a new page)

i) By using the principle of mathematical induction prove

$$\sum_{k=1}^{n} 5^k = 5\left(\frac{5^n - 1}{4}\right)$$

ii) a) What is a primitive function of $e^{f(x)} \cdot f'(x)$?

b) Using part (a) evaluate $\displaystyle\int_0^1 \frac{e^{\cos^{-1}x}\,dx}{\sqrt{1 - x^2}}$ (leave answer in irrational form)

iii) Two points P $(2ap, ap^2)$ and Q $(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$

(a) Find the equation of the tangent l to the parabola at Q.

(b) Derive the equation of the chord PQ and show that $pq = -1$ when PQ is focal chord.

(c) Find the acute angle between the tangent l and the chord PQ If $p = 3$ and $q = -0.2$.

QUESTION 5 (Start a new page)

i) The velocity of a particle moving in S.H.M. in a straight line is given by

$$v^2 = 4x - x^2 \; ms^{-1} \quad \text{where } x \text{ is displacement in } m$$

a) Find the two points between which the particle is oscillating.

b) Find the centre of the motion.

c) Find the maximum speed of the particle.

d) Find the acceleration of the particle in terms of x.

ii) Prove the identity $\dfrac{2\cos A}{\operatorname{cosec} A - 2\sin A} = \tan 2A$

iii) A person walking along a straight road observes a tower bearing 045^0 T, the angle of elevation being 5^0. After travelling a distance of 5000m, the tower bears 315^0 T and the angle of elevation is 8^0.

a) Find the height of the tower (to 0.1m).

b) Determine the angle which the road makes with a line bearing 090^0 T.

QUESTION 6: (Start a new page)

i) a) Sketch the curves $y^2 = x$ and $x^2 = 8y$

b) Show that the area formed between these curves is given by

$$\int_0^4 \left(x^{\frac{1}{2}} - \frac{x^2}{8} \right) dx$$

c) The area is rotated about the x axis, find the volume of the solid of revolution so formed.

ii) Let $f(x) = \sin^{-1} x + \cos^{-1} x \qquad (0 \le x \le 1)$

Find a) $f'(x)$

b) $\int_0^1 f(x)\, dx$

iii) Newton's Law of cooling states that the rate at which a body cools is proportional to the excess of its temperature above that of its surroundings.

A sphere at a temperature of 70^0 C is placed in a container at a temperature of 20^0 C.

a) Show that $T = 20 + 50\, e^{-kt}$ is a solution of the differential equation

$$\frac{dT}{dt} = -K\,(T - 20) \quad \text{where } K \text{ is a positive constant.}$$

b) If, after 2 minutes, the temperature of the sphere is 60^0 C approx.. show that

$$T = 20 + 50\, e^{-0.11t}$$

c) Find the temperature of the sphere after 4 minutes (to the nearest degree).

QUESTION 7 (Start a new page)

i) A ball is thrown from a height 1 metre from the ground and is caught, without bouncing, 2 seconds later at a distance of 50m, also at a height of 1m. Assuming no air resistance and that gravity is approx. 10ms^{-2} you may assume $x = Vt\cos\theta$ and

$$y = -\tfrac{1}{2} gt^2 + Vt\sin\theta$$

Find a) The velocity and angle of projection of the ball.

b) The maximum height of the ball above the ground during its flight.

ii) For the expansion of $(a + bx)^n$ in ascending powers of x

a) Show that the expression for the ratio of the r^{th} and $(r+1)^{th}$ terms

is $\dfrac{T_{r+1}}{T_r} = \dfrac{(n - r + 1)}{r} \cdot \dfrac{bx}{a}$

b) Two consecutive coefficients in the expansion $(3 + x)^{15}$ are equal. Find which terms these are.

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Good work.

Question 1

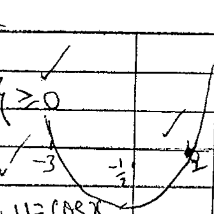
(i) $\frac{2x+1}{x-2} \geq 1$ $\quad x \neq 2$

$(2x+1)(x-2) \geq (x-2)^2$

$(x-2)\{(2x+1)-(x-2)\} \geq 0$

$\therefore (x-2)\{x+3\} \geq 0$

$x \leq -3 \quad x > 2$



(ii) $\int_0^{\frac{\pi}{2}} \frac{\sin x\,dx}{\sqrt{4-\cos^2 x}}$ let $u=\cos x$ $\quad du=-\sin x\,dx$

when $x=\frac{\pi}{2}$ $\quad u=0$

when $x=0$ $\quad u=1$

$-\int_1^0 \frac{du}{\sqrt{4-u^2}} = \int_0^1 \frac{du}{\sqrt{4-u^2}}$

$= \left[\sin^{-1}\frac{u}{2}\right]_0^1 = \frac{\pi}{6}$

(iii) $(x+\frac{1}{x})^2 - 10(x+\frac{1}{x}) + 24 = 0$

let $u = x + \frac{1}{x}$

$\therefore u^2 - 10u + 24 = 0$

$u = \frac{10 \pm \sqrt{100-4(24)}}{2} = \frac{10 \pm 2}{2} = 6$ or 4

$\therefore x + \frac{1}{x} = 6$ or $x + \frac{1}{x} = 4$

$x^2 - 6x + 1 = 0$ or $x^2 - 4x + 1 = 0$

$\therefore x = \frac{6 \pm \sqrt{36-4}}{2}$ or $x = \frac{4 \pm \sqrt{16-4}}{2}$

$x = 3 \pm 2\sqrt{2}$ or $x = 2 \pm \sqrt{3}$

(iv) $\tan^{-1}(\log_e x)$ let $u = \log_e x$

$\therefore \frac{dy}{du} = \frac{1}{1+u^2}$ $\quad \frac{du}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{x(1+(\log_e x)^2)} = \frac{1}{x+x(\ln x)^2}$

(v) $(3a-2b)^4 = 81a^4 - 216a^3b + 216a^2b^2 - 96ab^3 + 16b^4$

3^{rd} term $216a^2b^2$ $\quad$ $\therefore {}^4C_2(3a)^2(-2b)^2$

Question 2

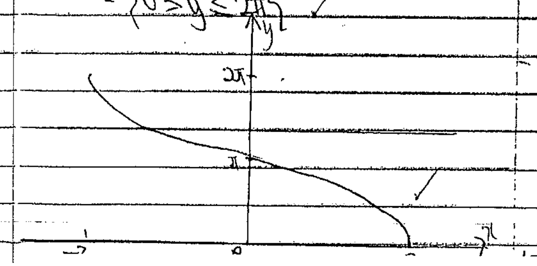
i) $y = 2\cos^{-1}\frac{x}{3}$

$Dx: \{-1 \leq \frac{x}{3} \leq 1\}$

$= \{-3 \leq x \leq 3\}$

$Ry: \{0 \leq \frac{y}{2} \leq \pi\}$

$= \{0 \leq y \leq 2\pi\}$



(ii) $3\sin\theta + 4\cos\theta = 3$ $\quad 0 \leq \theta \leq 2\pi$

$\cos\theta = \frac{1-t^2}{1+t^2}$ $\quad \sin\theta = \frac{2t}{1+t^2}$ $\quad t = \tan\frac{\theta}{2}$

$3(\frac{2t}{1+t^2}) + 4(\frac{1-t^2}{1+t^2}) = 3$

$6t + 4 - 4t^2 = 3 + 3t^2$

$7t^2 - 6t - 1 = 0$

$\frac{(7t-7)(7t+1)}{7} = (t-1)(7t+1) = 0$

$\therefore t = 1$ or $t = -\frac{1}{7}$

neglect negative value since $0 \leq \theta \leq 2\pi$ $\quad$ tan < 0 in 2nd or 4th quadrant.

$\therefore \tan\frac{\theta}{2} = 1$ $\quad \tan\frac{\theta}{2} = -\frac{1}{7}$

$\frac{\theta}{2} = \frac{\pi}{4}, \frac{5\pi}{5}\ \frac{5\pi}{4}$

$\therefore \theta = \frac{\pi}{2}, \frac{5\pi}{2}$ ie 1.57 or 3.43 (to 2 dec. pl.)

$\theta = 6.00$ rad.

(iii) $P(6,9)$ divides $-3:2$ Find B given $A(1,4)$

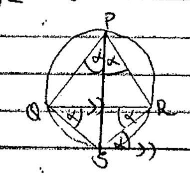
$x = \frac{mx_2+nx_1}{m+n}$ $\quad \therefore 6 = \frac{-3(x_2)+2(1)}{-1}$

$\therefore x_2 = \frac{8}{3}$

$y = \frac{my_2+ny_1}{m+n}$ $\quad \therefore 9 = \frac{-3(y_2)+2(4)}{-1}$

$\therefore y_2 = \frac{17}{3}$ $\quad \therefore B(\frac{8}{3}, \frac{17}{3})$

(iv) α)



RTP $SQ = S$

let $\angle QRS = \alpha$

$\therefore \angle QRS = \angle RS$

(alt. $\angle$s on)

Now $\angle RST = \angle RQS$ ($\angle$ in alt segment)

$\therefore SQ = SR$ (since $\triangle QSR$ is isosceles, sides of isos. $\triangle$ are eq

β) RTP SP bisects $\angle QPR$

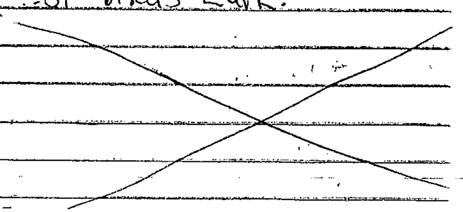
on chord QS, $\angle QRS = \angle QPS = \alpha$

($\angle$ in same segment)

on chord RS, $\angle SQR = \angle SPR = \alpha$

($\angle$ in same segment)

$\therefore$ SP bisects $\angle QPR$.



${}^{20}C_2$ $\quad$ Binomial Probability $(p+q)^{20}$

Question 3

(i) $0.1 \times 0.1 \times (0.9)^{18} = 1.5 \times 10^{-3} \times 190$

(ii) a) show $2x^3 - 3x^2 + 0.99 = 0$ has a root near $x=1$

$P(0) = 0.99 > 0$

$P(1) = -0.01 < 0$

$\therefore$ there's a root near 1.

b) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ $\quad$ Now $f(x_1) = -0.01$

$f'(x) = 6x^2 - 6x$ $\quad \therefore f'(x_1) = 0$

$x_2 = x_1 - \frac{-0.01}{0}$ $\quad$ but $f'(x_1) \neq 0$ $\therefore$ no sol'n

(c) Since $f'(x_1) \neq 0$ (ie denominator $\neq 0$)

$\therefore$ method doesn't work

(iii) $2x^3 + 3x^2 + ax - 6$ has $(x+3)$ and $(2x+b)$ as factors find a and b.

$P(-3) = 0$ $\quad \therefore -54 + 27 - 3a - 6 = 0$

$-3a = 33$ $\quad \therefore a = -11$

Either use trial + error method $P(-\frac{1}{2}) = 0$

or $\quad x+3 \overline{)2x^3+3x^2-11x-6}$ $\quad 2x^2-3x-2 \Rightarrow (2x+1)(x-2)$ $\quad \therefore b = 1$

Question 4.

(i) $\sum_{k=1}^{n} 5^k = 5\left(\frac{5^n-1}{4}\right)$

Step 1: Prove true for $n=1$

$5 = 5(1)$ $\quad \therefore$ True for $n=1$

Step 2: Assume true for $n=k$

Hence prove true for $n=k+1$

Let $S_k = 5\left(\frac{5^k-1}{4}\right)$ $\quad$ RTP $S_{k+1} = 5\left(\frac{5^{k+1}-1}{4}\right)$

$T_{k+1} = 5^{k+1}$

$S_{k+1} = 5\left(\frac{5^k-1}{4}\right) + 5^{k+1}$

$= 5\left(\frac{5^k-1}{4}\right) + 5\cdot5^k$

$= 5\left(\frac{5^k-1}{4} + 5^k\right) = 5\left(\frac{5^k-1+4.5^k}{4}\right)$

$= 5\left(\frac{5^{k+1}-1}{4}\right) = RHS$

$* 5\left(\frac{5.5^k-1}{4}\right) = 5\left(\frac{5^{k+1}-1}{4}\right) = RHS$

Step 3: Since true for $n=1$ and proven true for $n=k+1$ $\therefore$ true for $n=1+1=2$ and so on for all positive integers, n.

(ii)(a) $e^{f(x)} \cdot f'(x)$ $\quad \therefore \int f'(x)\,e^{f(x)} = e^{f(x)}$

(b) $\int_0^1 \frac{e^{\cos^{-1}x}\,dx}{\sqrt{1-x^2}}$

let $u = e^{\cos^{-1}x}$

$du = \frac{-e^{\cos^{-1}x}}{\sqrt{1-x^2}}\,dx$

when $x=1$ $\quad u=1$

$x=0$ $\quad u=e^{\frac{\pi}{2}}$

$-\int_{e^{\frac{\pi}{2}}}^{1} du = -[u]_{e^{\frac{\pi}{2}}}^{1} = -1 + e^{\frac{\pi}{2}}$

$= e^{\frac{\pi}{2}} - 1$

(iii) (a) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$

$x^2 = 4ay$ $\quad \frac{dy}{dx} = \frac{x}{2a}$ at $Q = q$

$(y - aq^2) = q(x - 2aq)$

$y = qx - aq^2$

(b) $\frac{a(q^2-p^2)}{2a(q-p)} = \frac{q+p}{2}$

$(y - aq^2) = \frac{p+q}{2}(x - 2aq)$

$2y - 2aq^2 = px - 2apq + qx -$

if PQ a focal chord $\therefore$ goes through $S(0,a)$ sub in

$2y = px - 2apq + qx$

$\therefore 2a = -2apq$

$\therefore pq = -1$ as req'd

(c) $\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$ $\quad p=3, q=-0.2$

$m_{tangent} = q = -0.2$

$m_{chord\,PQ} = \frac{p+q}{2} = \frac{3-0.2}{2} = 1.4$

$\tan\theta = \left|\frac{-0.2+1.4}{1+(-0.2)(1.4)}\right|$

$= \left|\frac{1.2}{0.72}\right|$ $\quad \therefore \theta = 59°2'$

Question 5.

(i)(a) $v^2 = 4x - x^2$ ms^{-1}

in form $v^2 = a^2 - x^2$ $\quad$ let $v=0$ at end points, solve for x

$\therefore a = \pm 2\sqrt{x}\ x$

(b) $\dot{x} = \max$ at centre of motion

$\ddot{x} = 0$ at centre of motion

$\frac{1}{2}v^2 = 2x - \frac{x^2}{2}$

$\frac{d}{dx}(\frac{1}{2}v^2) = 2 - x = -1(x-2)$

$\therefore x = 2$

(c) $\dot{x} = \max$ at centre of motion ie $x=2$

$v^2 = 4(2) - (2)^2$

$= 4$

$\therefore v = 2ms^{-1}$ ✓

(d) $a = \frac{d}{dx}(\frac{1}{2}v^2) = 2 - x$ ✓

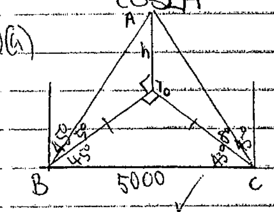
(ii) Prove $\frac{2\cos A}{\operatorname{cosec} A - 2\sin A} = \tan 2A$

LHS $= \frac{2\cos A}{\operatorname{cosec} A - 2\sin A}$ multiply by $\sin A$

$= \frac{2\sin A\cos A}{1 - 2\sin^2 A}$ ✓

$= \frac{\sin 2A}{\cos 2A} = \tan 2A = \text{RHS}$

(iii)(a)



let $AO = h$

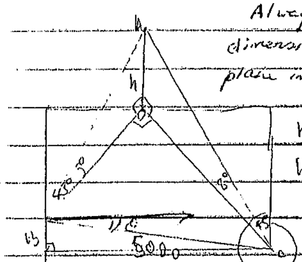
$\sin 45° = \frac{OC}{BC} = \frac{OC}{5000}$

$\therefore OC = \frac{5000}{\sqrt{2}} = OB$

Now $\tan 8° = \frac{h}{OC}$

$\tan 5° = \frac{h}{OB}$

Always express all dimensions in the horizontal plane in terms of h.



Trig again

$h = OC\tan 8°$

$h = OB\tan 5°$

$OB = h\tan 5°$ (By Trig.)

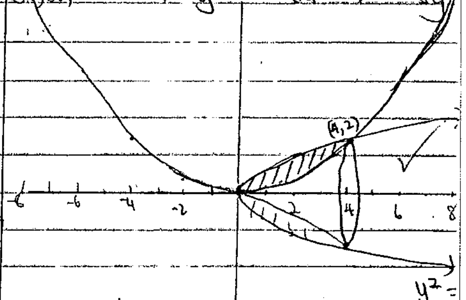
$OC = h\tan 8°$

Use Pyth. Theorem.

Find θ. $= 0$.

Question 6.

(i)(a) Sketch $y^2 = x$ and $x^2 = 8y$



(b) $A = \int_a^b y\,dx$

$= \int_0^4 \left(x^{1/2} - \frac{x^2}{8}\right)dx$ ✓

(c) $V = \pi\int_a^b y^2\,dx = \pi\int_0^4 (x^{1/2})^2 - \left(\frac{x^2}{8}\right)^2$

$= \pi\int_0^4 \left(x - \cancel{\frac{x^{5/2}}{4}} + \frac{x^4}{64}\right)dx$

$= \pi\left[\frac{x^2}{2} - \frac{x^{3/2}}{14} + \frac{x^5}{320}\right]_0^4$

$= \pi\left[8 - 9\frac{1}{7} + 3\cdot 2\right]$

$= \frac{72\pi}{35}$ units^3

(ii) $f(x) = \sin^{-1}x + \cos^{-1}x \ (0 \le x \le 1)$

a) $f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$

(b) $\int_0^1 f(x)\,dx = [c]_0^1$

$= c$

Let $\sin^{-1}x + \cos^{-1}x = c$

Choose $x = 1$

$\therefore \sin^{-1}1 + \cos^{-1}1 = c$

$\frac{\pi}{2} + 0 = c$

$c = \frac{\pi}{2}$

(iii)(a) $T = 20 + 50e^{-kt}$

$T - 20 = 50e^{-kt}$ ✓

$\therefore \frac{dT}{dt} = -k50e^{-kt} = -k(T-20)$ ✓

(b) show $T = 20 + 50e^{-0.11t}$

at $t = 2$ $T = 60$

$60 = 20 + 50e^{-k(2)}$ ✓

$\therefore k = \frac{\ln 0.8}{-2} = 0.11$

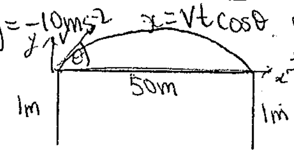
$\therefore T = 20 + 50e^{-0.11t}$ ✓

(c) at $t = 4$ $T = 20 + 50e^{-0.11(4)}$

$= 21°34'$ ✓

Question 7.

(i)(a) $g = -10ms^{-2}$ $x = Vt\cos\theta$ and $y = -\frac{1}{2}gt^2 + Vt\sin\theta$



at $t = 2$, $x = 50$, $y = 1$.

$\therefore V\cos\theta = 25$ ①

$1 = -5(4) + 2V\sin\theta + 1$ Careful here!

$V\sin\theta = \frac{20}{2}$ ② ②÷①

$\therefore \tan\theta = \frac{20}{50}$ $\therefore \theta = \cancel{22°47'}$ 21.8°

$\therefore V = \cancel{27.11549373ms^{-1}}$ $26.93ms^{-1}$

(b) max height at $t = 1$

$y = -5 + V\sin\theta + 1$ ✓

$= \cancel{6.5m}$ $6.0m$ above the ground.

(ii)(a) $(a + bx)^n$

$T_{r+1} = {}^nC_r\, a^{n-r}(bx)^r$

$T_r = {}^nC_{r-1}\, a^{n-r+1}(bx)^{r-1}$

$\therefore \frac{T_{r+1}}{T_r} = \frac{\frac{n!}{(n-r)!\,r!}}{\frac{n!}{(n-r+1)!\,(r-1)!}} \cdot a^{n-r-n+r-1}(bx)^{r-r+1}$

$= \frac{(n-r+1)!\,\cancel{(r-1)!}}{(n-r)!\ r\,\cancel{(r-1)!}}\, a^{-1}(bx)^1$

$= \frac{n-r+1}{r} \cdot \frac{bx}{a}$ as req^d.

(b) For $(3+x)^{15}$; $n = 15$, $a = 3$, $b = 1$

$\frac{T_{r+1}}{T_r} = 1 = \frac{(n-r+1)}{r} \cdot \frac{bx}{a}$

$1 = \frac{(15-r+1)}{r \cdot 3}$ comparing coefficients.

$\therefore 3r = 16 - r$

$\therefore 4r = 16$

$r = 4$

$\therefore T_5 = {}^{15}C_4\,(3)^{15-4}$

$= {}^{15}C_4 \cdot 3^{11}$