Diagnostic Topic Tests:
NSW Mathematics Extension 1 (Preliminary & HSC)

Test 7: The Binomial Theorem and Further Probability

Total 40 marks (Suggested time 45 minutes)

Directions to students

- Attempt ALL questions.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- The marks for each question are indicated at the start of the question.

QUESTION 1. (10 marks) — Marks

(a) Find the number of ways of choosing one or more players from a group of 4 players. **2**

(b) In how many ways can 8 girls sit at a round table if: **4**
  (i) there are no restrictions?
  (ii) two particular girls Janet and Robyn must sit next to each other?
  (iii) Janet and Robyn must be separated?

(c) There are 10 table tennis balls consisting of 2 pink, 3 orange and 5 white ones, all the same size. **4**
  (i) In how many ways can they be arranged in a row?
  (ii) In how many of these ways are the 2 pink ones separated from each other?

QUESTION 2. (8 marks)

(a) A committee of 5 is to be chosen from a group of 9 people which includes Mr Blink and Mrs Blink. **6**
  (i) If there are no restrictions, how many committees are possible?
  Find the probability that:
  (ii) both Mr and Mrs Blink are selected.
  (iii) only one of the Blinks is selected.
  (iv) neither Mr Blink nor Mrs Blink is selected.

(b) How many parallelograms are formed by a set of 4 parallel lines intersecting another set of 7 parallel lines? **2**

Diagnostic Topic Tests are issued by NEAP to individual schools and may be photocopied for the use of students of that school only. They may not be placed on the school intranet or otherwise reproduced or distributed.
Copyright © 2000 NEAP A.C.N. 006 302 401 PO Box 214 St Leonards NSW 2065 Tel: (02) 9438 1386 Fax: (02) 9438 1385 DTTN ME7QFM

QUESTION 3. (15 marks) — Marks

(a) Find the value of n if ${}^{n+1}P_2 = 4n + 10$. **3**

(b) Find the co-efficient of x^3 in the expansion of $(1-3x)^2(1+2x)^5$. **3**

(c) (i) Write down the expression for the $(r+1)^{\text{th}}$ term in the expansion of $\left(x^2 - \frac{1}{2x}\right)^{11}$. **4**
  (ii) Hence find the co-efficient of x^7 in the expansion of $\left(x^2 - \frac{1}{2x}\right)^{11}$.

(d) (i) Write down expressions for both the r^{th} term (T_r) and the $(r+1)^{\text{th}}$ term (T_{r+1}) in the expansion of $(3+2x)^8$. **5**
  (ii) Hence find an expression for $\frac{T_{r+1}}{T_r}$ in simplest form.

QUESTION 4. (7 marks)

(a) A biased coin has a probability of 0.7 of coming up tails. If this coin is tossed 8 times, find the probability of getting: **3**
  (i) 8 tails.
  (ii) exactly 5 tails.

(b) Consider the expansion of: $(1+x)^{2n} = \sum_{r=0}^{2n} \binom{2n}{r} x^r$ **4**

By substituting two suitable values of x, prove that:

$$\binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \ldots + \binom{2n}{2n} = 2^{2n-1}.$$

Copyright © 2000 NEAP

Test 7: The Binomial Theorem and Further Probability

Suggested Solutions

QUESTION 1.

(a) A player must be chosen or rejected. As each can be dealt with in two ways, the number of ways of treating the 4 players is $2 \times 2 \times 2 \times 2 = 2^4$. But this would include the case in which all prospective players are rejected.

$\therefore$ The total number of combinations is : $2^4 - 1 = 15$

Alternatively:

Consider the number of groups of 1, 2, 3, 4 players.

i.e. number of combinations $= {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4$

$= 4 + 6 + 4 + 1$

$= 15$ 2

(b) Anticlockwise and clockwise arrangements must be regarded as distinct.

(i) Number of arrangements of 8 girls $= (8-1)!$

$= 7!$

$= 5040$ 1

Note: n different objects can be arranged in a circle in $(n-1)!$ ways.

(ii) If Janet and Robyn must sit together, we can assign them two definite adjacent chairs and then arrange the other 6 girls in $6!$ ways. But Janet and Robyn can be arranged in $2! = 2$ ways. Therefore the total number of different ways is:

$2 \times 6! = 2 \times 720 = 1440$ 2

(iii) Number of arrangements in which they are separated $= 5040 - 1440$

$= 3600$ 1

Alternatively:

If Janet is seated first, then Robyn can be seated in 5 different places and the remaining girls in $6!$ ways.

$\therefore$ number of arrangements $= 5 \times 6! = 3600$.

(c) 2 pink, 3 orange 5 white

(i) Number of different arrangements $= \dfrac{10!}{2!\,3!\,5!}$

$= 2520$ 2

(ii) Consider first the number of arrangements with the 2 pink balls together.

These two pink balls are identical and for the purposes of arranging we treat them as one ball. e.g.

So we are arranging nine things.

Number of different arrangements $= \dfrac{9!}{3!\,5!}$

$= 504$

Number of arrangements with the pink balls separated equals the total number of arrangements minus the number of arrangements with the pink balls together. i.e. $2520 - 504$

$= 2016$ 2

Alternatively:

The two pink balls can be separated in 36 different ways and the remaining 8 balls can be arranged in $\dfrac{8!}{3!\,5!}$ ways.

No. of arrangements $= 36 \times \dfrac{8!}{3! \times 5!}$

$= 2016.$

QUESTION 2.

(a) (i) The number of possible committees $= {}^9C_5 = 126$ 1

(ii) If Mr and Mrs Blink are selected, we require 3 more committee members from the 7 remaining possibilities. i.e. ${}^7C_3$.

P (Mr & Mrs Blink) $= \dfrac{{}^7C_3}{{}^9C_5} = \dfrac{35}{126} = \dfrac{5}{18}$ 2

Note: ${}^nC_r$ and $\binom{n}{r}$ have the same meaning.

(iii) Either Mr is selected or Mrs is selected but not both.

Probability $= \dfrac{{}^7C_4}{{}^9C_5} + \dfrac{{}^7C_4}{{}^9C_5}$

$= \dfrac{2 \times 35}{126}$

$= \dfrac{5}{9}$ 2

Note: If Mr is included and Mrs is excluded we need to choose 4 from 7 etc.

(iv) Probability neither is selected $= \dfrac{{}^7C_5}{{}^9C_5} = \dfrac{1}{6}$ **1**

Alternatively:
Probability neither is selected
$= 1 -$ probability at least one is selected
$= 1 - \left(\dfrac{5}{18} + \dfrac{5}{9}\right)$
$= \dfrac{1}{6}$

(b) Each combination of 2 lines out of 4 can intersect each combination of 2 lines out of 7 to form a parallelogram.

Number of parallelograms $= {}^4C_2 \times {}^7C_2$
$= 126$ **2**

QUESTION 3.

(a)
$${}^{n+1}P_2 = 4n + 10$$
$$(n+1)(n) = 4n + 10$$
$$n^2 + n = 4n + 10$$
$$n^2 - 3n - 10 = 0$$
$$(n-5)(n+2) = 0$$
$$n = -2 \text{ or } 5$$

$n = -2$ is rejected since the symbol ${}^{n+1}P_2$ is not defined for a negative value of n. Hence $n = 5$. **3**

(b) $(1-3x)^2(1+2x)^5$
$= (1 - 6x^2 + 9x^2)[1 + {}^5C_1(2x) + {}^5C_2(2x)^2 + {}^5C_3(2x)^3 + \ldots$
$= (1 - 6x + 9x^2)(1 + 10x + 40x^2 + 80x^3 + \ldots)$

In this expansion, the only terms in x^3 are:
$1 \times 80x^3, -6x \times 40x^2, 9x^2 \times 10x$

$\therefore$ coefficient of $x^3 = 1 \times 80 - 6 \times 40 + 9 \times 10$
$= 80 - 240 + 90$
$= -70$ **3**

(c) $\left(x^2 - \dfrac{1}{2x}\right)^{11}$

Note: In the expansion of $(a+x)^n$,
$T_{r+1} = {}^nC_r\, a^{n-r}\, x^r$

(i)
$$T_{r+1} = {}^{11}C_r(x^2)^{11-r}\left(-\frac{1}{2x}\right)^r$$
$$= (-1)^r\, {}^{11}C_r\, x^{22-2r}\left(\frac{1}{2}\right)^r\left(\frac{1}{x}\right)^r$$
$$= (-1)^r\, {}^{11}C_r\, x^{22-2r}\, 2^{-r}\, x^{-r}$$
$$= (-1)^r\, {}^{11}C_r\, 2^{-r}\, x^{22-3r}$$ **2**

(ii) Since the index of the required power is 7
$$22 - 3r = 7$$
$$3r = 15$$
$$r = 5$$

Hence the coefficient of $x^7 = (-1)^5\, {}^{11}C_5\, 2^{-5}$
$= -\dfrac{231}{16}$

2

(d) (i) $(3+2x)^8$

Note: In the expansion of $(a+x)^n$,
$T_{r+1} = {}^nC_r a^{n-r} x^r$

$$T_{r+1} = {}^8C_r\, 3^{8-r}\, (2x)^r$$
$$T_{r+1} = {}^8C_r\, 3^{8-r}\, 2^r x^r$$
$$T_r = {}^8C_{r-1}\, 3^{8-(r-1)}\, (2x)^{r-1}$$
$$= {}^8C_{r-1}\, 3^{9-r}\, 2^{r-1}\, x^{r-1}$$ **2**

(ii)

Note: ${}^nC_r = \dfrac{n!}{r!(n-r)!}$

$$\frac{T_{r+1}}{T_r} = \frac{{}^8C_r\, 3^{8-r}\, 2^r\, x^r}{{}^8C_{r-1}\, 3^{9-r}\, 2^{r-1}\, x^{r-1}}$$
$$= \frac{{}^8C_r}{{}^8C_{r-1}} \times \frac{2x}{3}$$
$$= \frac{\dfrac{8!}{r!(8-r)!}}{\dfrac{8!}{(r-1)!(9-r)!}} \times \frac{2x}{3}$$
$$= \frac{8!}{r!(8-r)!} \times \frac{(r-1)!(9-r)!}{8!} \times \frac{2x}{3}$$
$$= \frac{8!}{r(r-1)!(8-r)!} \times \frac{(r-1)!(9-r)(8-r)!}{8!} \times \frac{2x}{3}$$
$$= \frac{9-r}{r} \times \frac{2x}{3}$$
$$= \frac{2x(9-r)}{3r}.$$ **3**

QUESTION 4.

(a) This is a binomial probability experiment. The terms in the expansion of $(q+p)^8$ give the probabilities of the various outcomes. If $q=0.7$ is the probability of getting a tail and $p=0.3$ is the probability of getting a head then:

(i) Probability of 8 tails $= q^8 = (0.7)^8$

$= 0.0576$ (4 d.p.) **1**

(ii) Probability of exactly 5 tails $= {}^8C_5(0.7)^5(0.3)^3$

$= 0.2541$ (4 d.p.) **2**

(b) $(1+x)^{2n} = \sum_{r=0}^{2n} \binom{2n}{r} x^r$

$(1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + \binom{2n}{2}x^2 + \binom{2n}{3}x^3 + \dots + \binom{2n}{2n}x^{2n} \dots (A)$

When $x=1$, (A) becomes

$2^{2n} = \binom{2n}{0} + \binom{2n}{1} + \binom{2n}{2} + \binom{2n}{3} + \dots + \binom{2n}{2n} \dots (B)$

When $x=-1$, (A) becomes

$0 = \binom{2n}{0} - \binom{2n}{1} + \binom{2n}{2} - \binom{2n}{3} + \dots + \binom{2n}{2n} \dots (C)$

Add (B) and (C)

$2^{2n} = 2\binom{2n}{0} + 2\binom{2n}{2} + 2\binom{2n}{4} + \dots + 2\binom{2n}{2n}$

Divide both sides by 2

$\frac{2^{2n}}{2} = \binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \dots + \binom{2n}{2n}$

$2^{2n-1} = \binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \dots + \binom{2n}{2n}$

$\therefore \binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \dots + \binom{2n}{2n} = 2^{2n-1}$ **4**

Copyright © 2000 NEAP