BOARD OF STUDIES
NEW SOUTH WALES

2006

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

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Total marks – 84
Attempt Questions 1–7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Find $\displaystyle\int \frac{dx}{49+x^2}$. **2**

(b) Using the substitution $u=x^4+8$, or otherwise, find $\displaystyle\int x^3\sqrt{x^4+8}\,dx$. **3**

(c) Evaluate $\displaystyle\lim_{x\to 0}\frac{\sin 5x}{3x}$. **2**

(d) Using the sum of two cubes, simplify: **2**

$$\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}-1,$$

for $0<\theta<\dfrac{\pi}{2}$.

(e) For what values of b is the line $y=12x+b$ tangent to $y=x^3$? **3**

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Let $f(x)=\sin^{-1}(x+5)$.

(i) State the domain and range of the function $f(x)$. **2**

(ii) Find the gradient of the graph of $y=f(x)$ at the point where $x=-5$. **2**

(iii) Sketch the graph of $y=f(x)$. **2**

(b) (i) By applying the binomial theorem to $(1+x)^n$ and differentiating, show that **1**

$$n(1+x)^{n-1}=\binom{n}{1}+2\binom{n}{2}x+\cdots+r\binom{n}{r}x^{r-1}+\cdots+n\binom{n}{n}x^{n-1}.$$

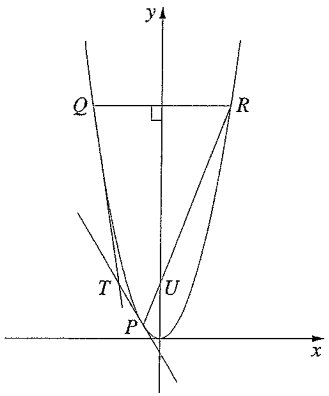
(ii) Hence deduce that **1**

$$n3^{n-1}=\binom{n}{1}+\cdots+r\binom{n}{r}2^{r-1}+\cdots+n\binom{n}{n}2^{n-1}.$$

Question 2 continues on page 5

Question 2 (continued) **Marks**

(c)



The points $P(2ap, ap^2)$, $Q(2aq, aq^2)$ and $R(2ar, ar^2)$ lie on the parabola $x^2 = 4ay$. The chord QR is perpendicular to the axis of the parabola. The chord PR meets the axis of the parabola at U.

The equation of the chord PR is $y = \frac{1}{2}(p+r)x - apr$. (Do NOT prove this.)

The equation of the tangent at P is $y = px - ap^2$. (Do NOT prove this.)

(i) Find the coordinates of U. **1**

(ii) The tangents at P and Q meet at the point T. Show that the coordinates of T are $(a(p+q),\ apq)$. **2**

(iii) Show that TU is perpendicular to the axis of the parabola. **1**

End of Question 2

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) Find $\int_0^{\frac{\pi}{4}} \sin^2 x \, dx$. **2**

(b) (i) By considering $f(x) = 3\log_e x - x$, show that the curve $y = 3\log_e x$ and the line $y = x$ meet at a point P whose x-coordinate is between 1.5 and 2. **1**

(ii) Use one application of Newton's method, starting at $x = 1.5$, to find an approximation to the x-coordinate of P. Give your answer correct to two decimal places. **2**

(c) Sophie has five coloured blocks: one red, one blue, one green, one yellow and one white. She stacks two, three, four or five blocks on top of one another to form a vertical tower.

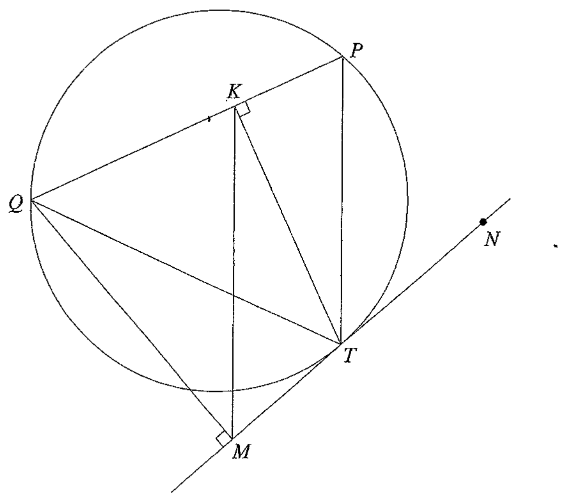
(i) How many different towers are there that she could form that are three blocks high? **1**

(ii) How many different towers can she form in total? **2**

Question 3 continues on page 7

Question 3 (continued) **Marks**

(d)



The points P, Q and T lie on a circle. The line MN is tangent to the circle at T with M chosen so that QM is perpendicular to MN. The point K on PQ is chosen so that TK is perpendicular to PQ as shown in the diagram.

(i) Show that $QKTM$ is a cyclic quadrilateral. **1**

(ii) Show that $\angle KMT = \angle KQT$. **1**

(iii) Hence, or otherwise, show that MK is parallel to TP. **2**

End of Question 3

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Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) The cubic polynomial $P(x) = x^3 + rx^2 + sx + t$, where r, s and t are real numbers, has three real zeros, 1, α and $-\alpha$.

(i) Find the value of r. **1**

(ii) Find the value of $s + t$. **2**

(b) A particle is undergoing simple harmonic motion on the x-axis about the origin. It is initially at its extreme positive position. The amplitude of the motion is 18 and the particle returns to its initial position every 5 seconds.

(i) Write down an equation for the position of the particle at time t seconds. **1**

(ii) How long does the particle take to move from a rest position to the point halfway between that rest position and the equilibrium position? **2**

(c) A particle is moving so that $\ddot{x} = 18x^3 + 27x^2 + 9x$.

Initially $x = -2$ and the velocity, v, is -6.

(i) Show that $v^2 = 9x^2(1 + x)^2$. **2**

(ii) Hence, or otherwise, show that **2**

$$\int \frac{1}{x(1+x)}\,dx = -3t.$$

(iii) It can be shown that for some constant c, **2**

$$\log_e\left(1 + \frac{1}{x}\right) = 3t + c. \qquad \text{(Do NOT prove this.)}$$

Using this equation and the initial conditions, find x as a function of t.

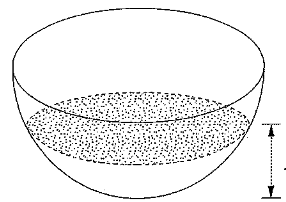
Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) Show that $y = 10e^{-0.7t} + 3$ is a solution of $\frac{dy}{dt} = -0.7(y - 3)$. **2**

(b) Let $f(x) = \log_e\left(1 + e^x\right)$ for all x. Show that $f(x)$ has an inverse. **2**

(c)



A hemispherical bowl of radius r cm is initially empty. Water is poured into it at a constant rate of k cm^3 per minute. When the depth of water in the bowl is x cm, the volume, V cm^3, of water in the bowl is given by

$$V = \frac{\pi}{3}x^2(3r - x). \qquad \text{(Do NOT prove this.)}$$

(i) Show that $\frac{dx}{dt} = \frac{k}{\pi x(2r - x)}$. **2**

(ii) Hence, or otherwise, show that it takes 3.5 times as long to fill the bowl to the point where $x = \frac{2}{3}r$ as it does to fill the bowl to the point where $x = \frac{1}{3}r$. **2**

Question 5 continues on page 11

Marks

Question 5 (continued)

(d) (i) Use the fact that $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\,\tan\beta}$ to show that **1**

$$1+\tan n\theta\,\tan(n+1)\theta=\cot\theta\left(\tan(n+1)\theta-\tan n\theta\right).$$

(ii) Use mathematical induction to prove that, for all integers $n\geq 1$, **3**

$$\tan\theta\,\tan 2\theta+\tan 2\theta\,\tan 3\theta+\cdots+\tan n\theta\,\tan(n+1)\theta=-(n+1)+\cot\theta\,\tan(n+1)\theta.$$

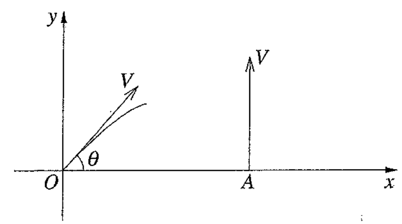
End of Question 5

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) Two particles are fired simultaneously from the ground at time $t=0$.

Particle 1 is projected from the origin at an angle θ, $0<\theta<\frac{\pi}{2}$, with an initial velocity V.

Particle 2 is projected vertically upward from the point A, at a distance a to the right of the origin, also with an initial velocity of V.



It can be shown that while both particles are in flight, Particle 1 has equations of motion:

$$x=Vt\cos\theta$$
$$y=Vt\sin\theta-\frac{1}{2}gt^2,$$

and Particle 2 has equations of motion:

$$x=a$$
$$y=Vt-\frac{1}{2}gt^2.$$

Do NOT prove these equations of motion.

Let L be the distance between the particles at time t.

Question 6 continues on page 13

Marks

Question 6 (continued)

(i) Show that, while both particles are in flight, **2**

$$L^2 = 2V^2t^2(1-\sin\theta) - 2aVt\cos\theta + a^2.$$

(ii) An observer notices that the distance between the particles in flight first decreases, then increases. **3**

Show that the distance between the particles in flight is smallest when $t = \dfrac{a\cos\theta}{2V(1-\sin\theta)}$ and that this smallest distance is $a\sqrt{\dfrac{1-\sin\theta}{2}}$.

(iii) Show that the smallest distance between the two particles in flight occurs while Particle 1 is ascending if $V > \sqrt{\dfrac{ag\cos\theta}{2\sin\theta(1-\sin\theta)}}$. **1**

(b) In an endurance event, the probability that a competitor will complete the course is p and the probability that a competitor will not complete the course is $q = 1 - p$. Teams consist of either two or four competitors. A team scores points if at least half its members complete the course.

(i) Show that the probability that a four-member team will have at least three of its members not complete the course is $4pq^3 + q^4$. **1**

(ii) Hence, or otherwise, find an expression in terms of q only for the probability that a four-member team will score points. **2**

(iii) Find an expression in terms of q only for the probability that a two-member team will score points. **1**

(iv) Hence, or otherwise, find the range of values of q for which a two-member team is more likely than a four-member team to score points. **2**

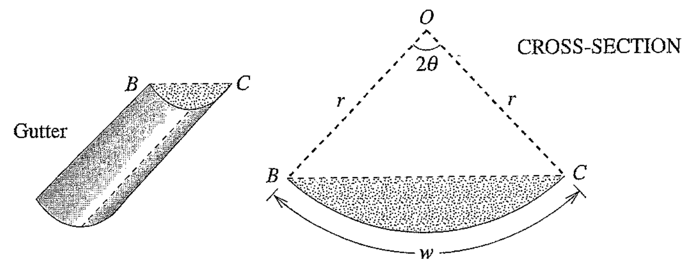
End of Question 6

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

A gutter is to be formed by bending a long rectangular metal strip of width w so that the cross-section is an arc of a circle.

Let r be the radius of the arc and 2θ the angle at the centre, O, so that the cross-sectional area, A, of the gutter is the area of the shaded region in the diagram on the right.



(a) Show that, when $0 < \theta \le \dfrac{\pi}{2}$, the cross-sectional area is **2**

$$A = r^2(\theta - \text{si } \theta\cos\ ).$$

(b) The formula in part (a) for A is true for $0 < \theta < \pi$. (Do NOT prove this.) **3**

By first expressing r in terms of w and θ, and then differentiating, show that

$$\frac{dA}{d\theta} = \frac{w^2\cos\theta(\sin\theta - \theta\cos\theta)}{2\theta^3}$$

for $0 < \theta < \pi$.

Question 7 continues on page 15

Marks

Question 7 (continued)

(c) Let $g(\theta) = \sin\theta - \theta\cos\theta$. **3**

By considering $g'(\theta)$, show that $g(\theta) > 0$ for $0 < \theta < \pi$.

(d) Show that there is exactly one value of θ in the interval $0 < \theta < \pi$ for which $\frac{dA}{d\theta} = 0$. **2**

(e) Show that the value of θ for which $\frac{dA}{d\theta} = 0$ gives the maximum cross-sectional area. Find this area in terms of w. **2**

End of paper

STANDARD INTEGRALS

$\int x^n\,dx$	$= \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$
$\int \frac{1}{x}\,dx$	$= \ln x, \quad x > 0$
$\int e^{ax}\,dx$	$= \frac{1}{a}e^{ax}, \quad a \neq 0$
$\int \cos ax\,dx$	$= \frac{1}{a}\sin ax, \quad a \neq 0$
$\int \sin ax\,dx$	$= -\frac{1}{a}\cos ax, \quad a \neq 0$
$\int \sec^2 ax\,dx$	$= \frac{1}{a}\tan ax, \quad a \neq 0$
$\int \sec ax \tan ax\,dx$	$= \frac{1}{a}\sec ax, \quad a \neq 0$
$\int \frac{1}{a^2+x^2}\,dx$	$= \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$
$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$	$= \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$
$\int \frac{1}{\sqrt{x^2-a^2}}\,dx$	$= \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0$
$\int \frac{1}{\sqrt{x^2+a^2}}\,dx$	$= \ln\left(x + \sqrt{x^2+a^2}\right)$

NOTE : $\ln x = \log_e x, \quad x > 0$

2006 Higher School Certificate Solutions

MATHEMATICS EXTENSION 1

QUESTION 1

(a) $\displaystyle\int \frac{dx}{49+x^2} = \int \frac{1}{7^2+x^2}\,dx$

$= \frac{1}{7}\tan^{-1}\frac{x}{7}+C$ (from Standard Integrals).

(b) Given $u = x^4+8$,

$\frac{du}{dx} = 4x^3$

$\therefore\ du = 4x^3\,dx.$

Now $\displaystyle\int x^3\sqrt{x^4+8}\,dx = \frac{1}{4}\int \sqrt{x^4+8}\,.4x^3\,dx$

$= \frac{1}{4}\int u^{\frac{1}{2}}\,du$

$= \frac{1}{4}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right)+C$

$= \frac{1}{4}\cdot\frac{2}{3}u^{\frac{3}{2}}+C$

$= \frac{1}{6}.u^{\frac{3}{2}}+C$

$= \frac{1}{6}(x^4+8)^{\frac{3}{2}}+C.$

(c) $\displaystyle\lim_{x\to 0}\frac{\sin 5x}{3x} = \frac{5}{3}\lim_{x\to 0}\frac{\sin 5x}{5x}$

$= \frac{5}{3}\times 1 \quad \left(\text{since } \lim_{x\to 0}\frac{\sin x}{x} = 1\right)$

$= \frac{5}{3}.$

(d) $a^3+b^3 = (a+b)(a^2-ab+b^2).$

$\dfrac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}-1$

$= \dfrac{(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)}{(\sin\theta+\cos\theta)}-1$

$= \sin^2\theta-\sin\theta\cos\theta+\cos^2\theta-1$

$= 1-\sin\theta\cos\theta-1 \quad (\text{since } \sin^2\theta+\cos^2\theta = 1)$

$= -\sin\theta\cos\theta.$

(e) For $y = 12x+b$ to be a tangent to $y = x^3$, the gradients must be the same at the point of contact.

The gradient of the tangent to $y = x^3$ is given by $\frac{dy}{dx} = 3x^2$.

The gradient of $y = 12x+b$ is 12,

$\therefore\ 3x^2 = 12$

$x^2 = 4$

$\therefore\ x = \pm 2.$

When $x = 2$, $y = (2)^3 = 8.$

Substituting into $y = 12x+b$

$8 = 12(2)+b$

$\therefore\ b = -16.$

When $x = -2$, $y = (-2)^3 = -8.$

$\therefore\ -8 = 12(-2)+b$

$\therefore\ b = 16.$

$\therefore\ b = \pm 16.$

QUESTION 2

(a) $f(x) = \sin^{-1}(x+5).$

(i) Domain: $-1 \le x+5 \le 1$

$\therefore\ -6 \le x \le -4$

Range: $-\frac{\pi}{2} \le f(x) \le \frac{\pi}{2}$

(ii) $y = \sin^{-1}(x+5)$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x+5)^2}}\times 1$

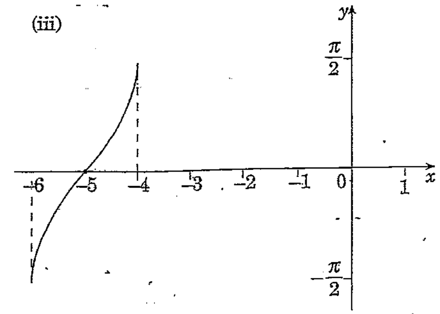
$= \frac{1}{\sqrt{1-(x+5)^2}}$

When $x = -5$, $\frac{dy}{dx} = \frac{1}{\sqrt{1-(-5+5)^2}}$

$= 1.$

$\therefore$ The gradient of $y = f(x)$ is 1 at $x = -5$.

(iii)



(b) (i) $(1+x)^n = \binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{r}x^r+\cdots+\binom{n}{n}x^n.$

Differentiate both sides with respect to x:

$n(1+x)^{n-1} = 0+\binom{n}{1}+2\binom{n}{2}x+\cdots+r\binom{n}{r}x^{r-1}+\cdots+n\binom{n}{n}x^{n-1}.$

(ii) Let $x = 2$:

$n(1+2)^{n-1} = \binom{n}{1}+2\binom{n}{2}2+\cdots+r\binom{n}{r}2^{r-1}+\cdots+n\binom{n}{n}2^{n-1}$

$n3^{n-1} = \binom{n}{1}+\cdots+r\binom{n}{r}2^{r-1}+\cdots+n\binom{n}{n}2^{n-1}.$

(c) (i) Put $x = 0$ in equation of PR (as U is the y-intercept of PR)

$y = \frac{1}{2}(p+r)0-apr$

$y = -apr.$

$\therefore\ U$ has coordinates $(0,-apr)$.

(ii) Equations of tangents at P and Q are:

$y = px-ap^2$ —①

$y = qx-aq^2$ —②

Solve simultaneously:

①−②: $0 = px-qx-ap^2+aq^2$

$0 = (p-q)x-a(p^2-q^2)$

$\therefore\ (p-q)x = a(p^2-q^2)$

$= a(p-q)(p+q)$

$\therefore\ x = a(p+q)$ as $p \ne q$.

Substituting into ①:

$y = pa(p+q)-ap^2$

$= ap^2+apq-ap^2$

$= apq.$

$\therefore\ T$ has coordinates $(a(p+q), apq)$.

(iii) *METHOD 1*

Gradient $TU = \dfrac{apq-(-apr)}{a(p+q)-0}$

$= \dfrac{ap(q+r)}{a(p+q)}$

$= \dfrac{p(q+r)}{p+q}.$

Since Q and R have the same y value,

then $aq^2 = ar^2$

$q = \pm r.$

But $q \ne r$ as Q and R are on opposite sides of the y-axis,

$\therefore\ q = -r.$

$\therefore$ Gradient $TU = \dfrac{p(q-q)}{p+q}$

$= 0.$

$\therefore\ TU$ is parallel to the x-axis, which makes it perpendicular to the y-axis, the axis of the parabola.

METHOD 2

Since QR is perpendicular to the y-axis,

$aq^2 = ar^2$

$q^2-r^2 = 0$

$(q-r)(q+r) = 0$

$\therefore\ q = -r, \quad (q \ne r).$

The y value of T is apq,

$\therefore$ the y value of $T = -apr.$

This is the y value of U.

Since T and U have the same y value, TU is perpendicular to the y-axis which is the axis of the parabola.

QUESTION 3

(a) $\cos 2x = 1-2\sin^2 x$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\displaystyle\int_0^{\frac{\pi}{4}}\sin^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{4}}(1-\cos 2x)\,dx$

$= \frac{1}{2}\left[x-\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\left[\frac{\pi}{4}-\frac{1}{2}\sin\frac{\pi}{2}-\left(0-\frac{1}{2}\sin 0\right)\right]$

$= \frac{\pi}{8}-\frac{1}{4}.$

(b) (i) To find P, the point of intersection, solve simultaneously: $y = 3\log_e x$ —①

$y = x$ —②

Substitute ① into ②:

$$3\log_e x = x$$
$$3\log_e x - x = 0$$

$\therefore$ The root of $f(x) = 3\log_e x - x$ gives the x-coordinate of P.

The function $f(x)$ is continuous for $x > 0$.

When $x = 1{\cdot}5$, $f(1{\cdot}5) = 3\log_e 1{\cdot}5 - 1{\cdot}5 = -0{\cdot}2836\ldots < 0.$

When $x = 2$, $f(2) = 3\log_e 2 - 2 = 0{\cdot}0794\ldots > 0.$

Since $f(1{\cdot}5) < 0$ and $f(2) > 0$, there is a root of $f(x)$ in the interval $1{\cdot}5 < x < 2$.

(ii) $f(x) = 3\log_e x - x$

$$f'(x) = 3\times\frac{1}{x} - 1 = \frac{3}{x} - 1.$$

$$x_1 = 1{\cdot}5$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 1{\cdot}5 - \left(\frac{3\log_e 1{\cdot}5 - 1{\cdot}5}{\frac{3}{1{\cdot}5} - 1}\right)$$
$$= 1{\cdot}5 - (-0{\cdot}2836\ldots)$$
$$= 1{\cdot}7836\ldots$$
$$= 1{\cdot}78 \text{ (2 decimal places)}.$$

(c) 5 blocks: 1R, 1B, 1G, 1Y, 1W

(i) 3 blocks high: $5\times4\times3 = 60$ ways

or $^5P_3 = 60$.

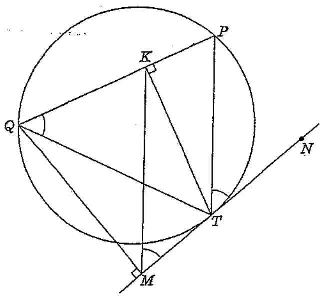
(ii) Number of different towers

2 blocks high:	$5\times4 =$	20
3 blocks high:	$5\times4\times3 =$	60
4 blocks high:	$5\times4\times3\times2 =$	120
5 blocks high:	$5\times4\times3\times2\times1 =$	120
	$=$	320

$\therefore$ Total = 320 different towers,

or total $^5P_2 + {}^5P_3 + {}^5P_4 + {}^5P_5 = 320$.

(d)



(i) $\angle QKT = 90°$ (supplementary $\angle$s, $TK \perp PQ$)

$\angle QMT = 90°$ (supplementary $\angle$s, $QM \perp TM$)

$\angle QKT + \angle QMT = 180°$.

$\therefore$ $QKTM$ is a cyclic quadrilateral, as opposite angles are supplementary.

(ii) Since $QKTM$ is a cyclic quadrilateral,

$\angle KMT = \angle KQT$ ($\angle$s in the same segment of circle $QKTM$, on KT).

(iii) $\angle PTN = \angle PQT$ ($\angle$ between tangent and chord is equal to $\angle$ in alternate segment)

But $\angle KQT = \angle PQT$

$\therefore$ $\angle PTN = \angle KMT$ (as $\angle KMT = \angle KQT$)

$MK \parallel PT$ (a pair of corresponding $\angle$s are equal).

QUESTION 4

(a) (i) $P(x) = x^3 + rx^2 + sx + t$.

The sum of the zeros $= -\dfrac{b}{a}$

$\therefore$ $1 + \alpha + (-\alpha) = -\dfrac{r}{1}$

$\therefore$ $r = -1$.

(ii) *METHOD* 1

Since 1 is a zero of $P(x)$,

then $P(1) = 0$ (Factor theorem)

$P(1) = (1)^3 + r(1)^2 + s(1) + t$

$0 = 1 + r + s + t.$

But $r = -1$,

$0 = 1 - 1 + s + t$

$\therefore$ $s + t = 0$.

METHOD 2

The product of zeros $= -\dfrac{d}{a}$.

$$1\times\alpha\times(-\alpha) = -\frac{t}{1}$$
$$\alpha^2 = t.$$

The sum of the zeros taken 2 at a time $= \dfrac{c}{a}$.

$$1\alpha + \alpha(-\alpha) + (-\alpha)1 = \frac{s}{1}$$
$$\alpha - \alpha^2 - \alpha = s$$
$$s = -\alpha^2$$
$$\therefore \quad s + t = -\alpha^2 + \alpha^2 = 0.$$

(b) (i) $x = a\cos(nt + \alpha)$.

Amplitude = 18, $\therefore$ $a = 18$.

Period: $\dfrac{2\pi}{n} = 5$ (as the initial position is an endpoint)

$\therefore$ $n = \dfrac{2\pi}{5}$.

So $x = 18\cos\left(\dfrac{2\pi}{5}t + \alpha\right)$.

Given $x = 18$ when $t = 0$,

$$18 = 18\cos\left[\frac{2\pi}{5}(0) + \alpha\right]$$
$$\cos\alpha = 1$$
$$\alpha = 0.$$

So $x = 18\cos\left(\dfrac{2\pi}{5}t\right)$.

Note if $x = a\sin(nt + \alpha)$,

then $x = 18\sin\left(\dfrac{2\pi}{5}t + \dfrac{\pi}{2}\right)$.

(ii) Find t when $x = 9$.

$$9 = 18\cos\frac{2\pi}{5}t$$
$$\cos\frac{2\pi}{5}t = \frac{1}{2}$$
$$\frac{2\pi}{5}t = \frac{\pi}{3} \quad \text{(need the first time only)}$$
$$t = \frac{5}{6}.$$

$\therefore$ It takes $\dfrac{5}{6}$ seconds.

(c) (i) $\ddot{x} = 18x^3 + 27x^2 + 9x$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 18x^3 + 27x^2 + 9x$$
$$\frac{1}{2}v^2 = \frac{18}{4}x^4 + \frac{27}{3}x^3 + \frac{9}{2}x^2 + c$$
$$\frac{1}{2}v^2 = \frac{9}{2}x^4 + 9x^3 + \frac{9}{2}x^2 + c.$$

When $x = -2$, $v = -6$.

$$\frac{1}{2}(-6)^2 = \frac{9}{2}(-2)^4 + 9(-2)^3 + \frac{9}{2}(-2)^2 + c$$
$$18 = 18 + c$$
$$c = 0.$$
$$\therefore \quad \frac{1}{2}v^2 = \frac{9}{2}x^4 + 9x^3 + \frac{9}{2}x^2$$
$$v^2 = 9x^4 + 18x^3 + 9x^2$$
$$= 9x^2(x^2 + 2x + 1)$$
$$= 9x^2(x + 1)^2$$
$$= 9x^2(1 + x)^2, \text{ as required.}$$

(ii) $v = 3x(x+1)$ or $v = -3x(x+1)$

When $x = -2$,

$v = 3(-2)(-2+1)$ or $v = -3(-2)(-2+1)$

$= -6\times-1$ $\quad$ $= 6\times-1$

$= 6.$ $\quad$ $= -6.$

Since initial conditions are $x = -2$, $v = -6$, then $v = -3x(x+1)$.

So $\dfrac{dx}{dt} = -3x(x+1)$

$$\frac{dt}{dx} = \frac{-1}{3x(x+1)}$$

$$t = -\frac{1}{3}\int\frac{dx}{x(x+1)}$$

$\therefore$ $\displaystyle\int\frac{1}{x(1+x)}\,dx = -3t$, as required.

(iii) $\log_e\left(1 + \dfrac{1}{x}\right) = 3t + c.$

When $t = 0$, $x = -2$,

$$\log_e\left(1 + \frac{1}{-2}\right) = 3(0) + c$$
$$c = \log_e\left(\frac{1}{2}\right)$$
$$= -\log_e 2.$$

So $\log_e\left(1 + \dfrac{1}{x}\right) = 3t - \log_e 2.$

Need to find x in terms of t:

$$\log_e\left(1 + \frac{1}{x}\right) + \log_e 2 = 3t$$
$$\log_e\left[2\left(1 + \frac{1}{x}\right)\right] = 3t$$
$$\log_e\left(2 + \frac{2}{x}\right) = 3t$$
$$2 + \frac{2}{x} = e^{3t}$$
$$\frac{2}{x} = e^{3t} - 2$$
$$\frac{x}{2} = \frac{1}{e^{3t} - 2}$$
$$x = \frac{2}{e^{3t} - 2}.$$

QUESTION 5

(a) $y = 10e^{-0.7t} + 3$

$$\frac{dy}{dt} = -0{\cdot}7\times10e^{-0.7t}$$
$$-0{\cdot}7(y - 3) = -0{\cdot}7(10e^{-0.7t} + 3 - 3)$$
$$= -0{\cdot}7\times10e^{-0.7t}$$
$$= \frac{dy}{dt}.$$

$\therefore$ $y = 10e^{-0.7t} + 3$ is a solution of $\dfrac{dy}{dt} = -0{\cdot}7(y - 3)$.

(b) *METHOD* 1

$f(x) = \log_e(1 + e^x)$ for all x

$$f'(x) = \frac{1}{1 + e^x}\cdot e^x$$
$$= \frac{e^x}{1 + e^x}.$$

$e^x > 0$ for all values of x,

$\therefore \quad \dfrac{e^x}{1+e^x} > 0$ for all x.

$\therefore f(x)$ is monotonic increasing.

This means that for every value of y there is only one x. Any horizontal line will cut this graph only once.

$\therefore$ An inverse function exists.

METHOD 2

Every function has an inverse.

For $f(x) = y = \log_e(1+e^x)$, to find the inverse $f^{-1}(x)$, interchange x and y.

$$x = \log_e(1+e^y)$$

Now find y in terms of x.

Raising both sides to the power of e:

$$e^x = 1+e^y$$

$$\therefore \quad e^y = e^x - 1.$$

Taking $\log_e$ of both sides:

$$y = \log_e(e^x - 1)$$

$$\therefore \quad f^{-1}(x) = \log_e(e^x-1).$$

This inverse is a function for $x > 0$.

(c) (i) $\dfrac{dx}{dt} = \dfrac{dx}{dV} \times \dfrac{dV}{dt}$

Now $\quad V = \dfrac{\pi}{3}x^2(3r-x)$

$$= \pi r x^2 - \frac{\pi x^3}{3}$$

$$\frac{dV}{dx} = 2\pi r x - \pi x^2$$

$$\therefore \quad \frac{dx}{dV} = \frac{1}{2\pi r x - \pi x^2}$$

$$\frac{dV}{dt} = k \qquad \text{(given)}$$

$$\therefore \quad \frac{dx}{dt} = \frac{1}{2\pi r x - \pi x^2} \times k$$

$$\text{ie.} \quad \frac{dx}{dt} = \frac{k}{\pi x(2r-x)}.$$

(ii) Using $\dfrac{dx}{dt} = \dfrac{k}{\pi x(2r-x)}$

$$\frac{dt}{dx} = \frac{\pi x(2r-x)}{k}$$

$$= \frac{\pi}{k}(2rx - x^2)$$

$$t = \frac{\pi}{k}\int (2rx - x^2)\,dx$$

$$\therefore \quad t = \frac{\pi}{k}\left(rx^2 - \frac{x^3}{3}\right) + c.$$

When $t = 0$, $x = 0$ $\quad \therefore c = 0$

$$\therefore \quad t = \frac{\pi}{k}\left(rx^2 - \frac{x^3}{3}\right).$$

Let t_1 be the time taken to fill the bowl to a point where $x = \dfrac{2r}{3}$, and t_2 be the time taken to fill the bowl to a point where $x = \dfrac{1r}{3}$.

When $x = \dfrac{2r}{3}$,

$$t_1 = \frac{\pi}{k}\left[r\left(\frac{2r}{3}\right)^2 - \frac{\left(\frac{2r}{3}\right)^3}{3}\right]$$

$$= \frac{\pi}{k}\left(\frac{4r^3}{9} - \frac{8r^3}{81}\right)$$

$$= \frac{28\pi}{81k}r^3.$$

When $x = \dfrac{1}{3}r$,

$$t_1 = \frac{\pi}{k}\left[r\left(\frac{1}{3}r\right)^2 - \frac{\left(\frac{1}{3}r\right)^3}{3}\right]$$

$$= \frac{\pi}{k}\left(\frac{r^3}{9} - \frac{r^3}{81}\right)$$

$$= \frac{8\pi}{81k}r^3.$$

$$\therefore \quad \frac{t_1}{t_2} = \frac{\frac{28\pi}{81k}r^3}{\frac{8\pi}{81k}r^3} = \frac{28}{8} = 3{\cdot}5$$

$$\text{ie.} \quad t_1 = 3{\cdot}5\,t_2.$$

(d) (i) $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}.$

Let $\alpha = (n+1)\theta$ and $\beta = n\theta$

$$\therefore \quad \tan[(n+1)\theta - n\theta]$$

$$= \frac{\tan(n+1)\theta - \tan n\theta}{1+\tan(n+1)\theta\tan n\theta}$$

$$\tan\theta = \frac{\tan(n+1)\theta - \tan\theta}{1+\tan(n+1)\theta\tan n\theta}$$

$$\therefore \quad 1+\tan(n+1)\theta\tan n\theta$$

$$= \frac{\tan(n+1)\theta - \tan\theta}{\tan\theta}$$

$$= \cot\theta[\tan(n+1)\theta - \tan\theta].$$

(ii) *METHOD 1* $\qquad$ (for $n = 1$)

Let $S_n = \tan\theta\tan 2\theta + \tan 2\theta\tan 3\theta + \cdots + \tan n\theta\tan(n+1)\theta.$

For $n = 1$ show $S_1 = -(1+1) + \cot\theta\tan 2\theta.$

LHS $= S_1$

$$= \tan\theta\tan 2\theta$$

$$= \cot\theta(\tan 2\theta - \tan\theta) - 1, \quad \text{using result in (i),}$$

$$= \cot\theta\tan 2\theta - \cot\theta\tan\theta - 1$$

$$= \cot\theta\tan 2\theta - 1 - 1, \quad \text{as } \cot\theta = \frac{1}{\tan\theta},$$

$$= \cot\theta\tan 2\theta - 2$$

$$= \text{RHS.}$$

$\therefore$ The result is true for $n = 1$.

METHOD 2 $\qquad$ (for $n = 1$)

LHS $= S_1$

$$= \tan\theta\tan 2\theta$$

$$= \tan\theta \times \frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{2\tan^2\theta}{1-\tan^2\theta}.$$

RHS $= -2 + \cot\theta\tan 2\theta$

$$= -2 + \frac{2\tan\theta}{\tan\theta(1-\tan^2\theta)}$$

$$= \frac{-2\tan\theta(1-\tan^2\theta) + 2\tan\theta}{\tan\theta(1-\tan^2\theta)}$$

$$= \frac{2\tan^3\theta}{\tan\theta(1-\tan^2\theta)}$$

$$= \frac{2\tan^2\theta}{1-\tan^2\theta}.$$

$\therefore$ LHS = RHS.

$\therefore$ The result is true for $n = 1$.

Let k be a value for which the result is true, ie. $S_k = -(k+1) + \cot\theta\tan(k+1)\theta.$

We need to show that

$$S_{k+1} = -(k+2) + \cot\theta\tan(k+2)\theta.$$

$$S_{k+1} = S_k + T_{k+1}$$

Now $T_{k+1} = \tan(k+1)\theta\tan(k+2)\theta$

$$= \cot\theta[\tan(k+2)\theta - \tan(k+1)\theta] - 1, \quad \text{using result in (i).}$$

$$\therefore \quad S_{k+1} = -(k+1) + \cot\theta\tan(k+1)\theta + \cot\theta[\tan(k+2)\theta - \tan(k+1)\theta] - 1$$

$$= -k - 1 + \cot\theta\tan(k+1)\theta + \cot\theta\tan(k+2)\theta - \cot\theta\tan(k+1)\theta - 1$$

$$= -(k+2) + \cot\theta\tan(k+2)\theta.$$

~~$\therefore$ When the result is true for $n = k$, it is also true for $n = k+1$. Since the result is true for $n = 1$, the result is true for $n = 1+1 = 2$ and $n = 2+1 = 3$, and so on. Therefore, the result is true for all positive integer values of n.~~

$\therefore$ by the principle of mathematical induction, the result is true for all integers $n \geq 1$.

QUESTION 6

(a) (i) At any time t the coordinates of particles 1 and 2 are $\left(Vt\cos\theta, Vt\sin\theta - \frac{1}{2}gt^2\right)$ and $\left(a, Vt - \frac{1}{2}gt^2\right)$ respectively.

By the distance formula:

$$L^2 = (Vt\cos\theta - a)^2 + \left(Vt\sin\theta - \frac{1}{2}gt^2 - Vt + \frac{1}{2}gt^2\right)^2$$

$$= V^2t^2\cos^2\theta - 2aVt\cos\theta + a^2 + (Vt\sin\theta - Vt)^2$$

$$= V^2t^2\cos^2\theta - 2aVt\cos\theta + a^2 + V^2t^2\sin^2\theta - 2V^2t^2\sin\theta + V^2t^2$$

$$= V^2t^2(\cos^2\theta + \sin^2\theta - 2\sin\theta + 1) - 2aVt\cos\theta + a^2$$

$$= V^2t^2(2 - 2\sin\theta) - 2aVt\cos\theta + a^2$$

$$= 2V^2t^2(1-\sin\theta) - 2aVt\cos\theta + a^2,$$

as required.

(ii) *METHOD 1*

$$L^2 = 2V^2(1-\sin\theta)t^2 - 2aV\cos\theta t + a^2$$

The right-hand side of L^2 is a quadratic in t, which has a minimum value since $2V^2(1-\sin\theta) > 0$ for $0 < \theta < \dfrac{\pi}{2}$.

This minimum occurs when

$$t = -\frac{1}{2} \times \frac{-2aV\cos\theta}{2V^2(1-\sin\theta)}$$

$$= \frac{a\cos\theta}{2V(1-\sin\theta)}.$$

NB: For $y = ax^2 + bx + c$, the minimum value if $a > 0$ occurs when $x = -\dfrac{b}{2a} = -\dfrac{1}{2} \times \dfrac{b}{a}$.

Substituting this into L^2, we get

$$L^2 = 2V^2(1-\sin\theta)\left(\frac{a^2\cos^2\theta}{4V^2(1-\sin\theta)^2}\right) - 2aV\cos\theta\left(\frac{a\cos\theta}{2V(1-\sin\theta)}\right) + a^2$$

$$= \frac{a^2\cos^2\theta}{2(1-\sin\theta)} - \frac{a^2\cos^2\theta}{1-\sin\theta} + a^2$$

$$= -\frac{a^2\cos^2\theta}{2(1-\sin\theta)} + a^2$$

$$= \frac{-a^2(1-\sin^2\theta)}{2(1-\sin\theta)} + a^2$$

$$= \frac{-a^2(1-\sin\theta)(1+\sin\theta)}{2(1-\sin\theta)} + a^2$$

$$= \frac{-a^2(1+\sin\theta) + 2a^2}{2}$$

$$= \frac{a^2(1-\sin\theta)}{2}$$

$\therefore L = a\sqrt{\dfrac{1-\sin\theta}{2}} \qquad (L > 0).$

METHOD 2

$\dfrac{d}{dt}(L^2) = 4V^2 t(1-\sin\theta) - 2aV\cos\theta$

Stationary points occur when $\dfrac{d}{dt}(L^2) = 0.$

$4V^2 t(1-\sin\theta) - 2aV\cos\theta = 0$

$t = \dfrac{2aV\cos\theta}{4V^2(1-\sin\theta)} = \dfrac{a\cos\theta}{2V(1-\sin\theta)}.$

And $\dfrac{d^2}{dt^2}(L^2) = 4V^2(1-\sin\theta) > 0$, since $V^2 > 0$

and $1-\sin\theta > 0$ for $0 < \theta < \dfrac{\pi}{2}$.

Substituting $t = \dfrac{a\cos\theta}{2V(1-\sin\theta)}$ into L^2,

shown in Method 1, gives $L = a\sqrt{\dfrac{1-\sin\theta}{2}}.$

(iii) Particle 1 is ascending when $\dot{y} > 0$.

$\dot{y} = V\sin\theta - gt$

$\therefore \quad V\sin\theta - gt > 0$

$t < \dfrac{V\sin\theta}{g}.$

$\therefore$ Smallest distance occurs while particle 1 is ascending if

$\dfrac{a\cos\theta}{2V(1-\sin\theta)} < \dfrac{V\sin\theta}{g}$

$\dfrac{ag\cos\theta}{2(1-\sin\theta)\sin\theta} < V^2$

$\therefore V > \sqrt{\dfrac{ag\cos\theta}{2\sin\theta(1-\sin\theta)}} \quad (V > 0).$

(b) Let X be the number of competitors from a 4 member team who do not complete the course.

$P(X = k) = \dbinom{4}{k} p^{4-k} q^k$

(i) P(at least 3 competitors do not complete the course)

$= P$(exactly 3 do not complete the course) $+ P$(all 4 do not complete the course)

$= P(X=3) + P(X=4)$

$= \dbinom{4}{3} p^{4-3} q^3 + \dbinom{4}{4} p^{4-4} q^4$

$= 4pq^3 + q^4$ as required.

(ii) P(4 member team scores points)

$= 1 - P$(4 member team doesn't score points)

$= 1 - P$(at least 3 competitors do not complete the course)

$= 1 - (4pq^3 + q^4)$

$= 1 - 4(1-q)q^3 - q^4$

$= 1 - 4q^3 + 4q^4 - q^4$

$= 1 - 4q^3 + 3q^4.$

(iii) P(2 member team scores points)

$= 1 - P$(2 member team doesn't score points)

$= 1 - P$(both do not complete the course)

$= 1 - q^2.$

(iv) Find q such that:

P(2 member team scores points) > P(4 member team scores points)

$1 - q^2 > 1 - 4q^3 + 3q^4$

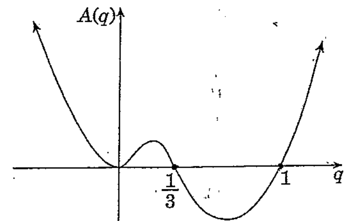
$0 > 3q^4 - 4q^3 + q^2$

$q^2(3q^2 - 4q + 1) < 0$

$q^2(3q-1)(q-1) < 0.$

Let $A(q) = q^2(3q-1)(q-1)$.

Sketch $A(q)$.

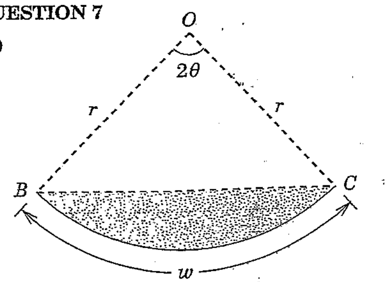


From the graph, $A(q) < 0$ for $\dfrac{1}{3} < q < 1$.

$\therefore$ The two-member team is more likely to score points than the four-member team when $\dfrac{1}{3} < q < 1$.

QUESTION 7

(a)



Cross-sectional area = area of segment

$\therefore \quad A = \dfrac{1}{2}r^2 . 2\theta - \dfrac{1}{2}r^2\sin 2\theta$

$= r^2\theta - \dfrac{1}{2}r^2 . 2\sin\theta\cos\theta$

$= r^2\theta - r^2\sin\theta\cos\theta$

$\therefore \quad A = r^2(\theta - \sin\theta\cos\theta).$

(b) Using $\ell = r\theta$, $w = r \times 2\theta$,

$\therefore \quad r = \dfrac{w}{2\theta}.$

Substituting $r = \dfrac{w}{2\theta}$ into A,

$A = \left(\dfrac{w}{2\theta}\right)^2(\theta - \sin\theta\cos\theta)$

$= \dfrac{w^2}{4\theta^2}(\theta - \sin\theta\cos\theta).$

METHOD 1 Using quotient rule:

$A = \dfrac{w^2}{4}\left(\dfrac{\theta - \frac{1}{2}\sin 2\theta}{\theta^2}\right)$ as $\sin\theta\cos\theta = \dfrac{1}{2}\sin 2\theta.$

$\dfrac{dA}{d\theta} = \dfrac{w^2}{4}\left[\dfrac{\theta^2(1-\cos 2\theta) - \left(\theta - \frac{1}{2}\sin 2\theta\right).2\theta}{\theta^4}\right]$

$= \dfrac{w^2}{4}\left[\dfrac{\theta^2 - \theta^2\cos 2\theta - 2\theta^2 + \theta\sin 2\theta}{\theta^4}\right]$

$= \dfrac{w^2}{4}\left[\dfrac{\theta^2 - \theta^2(2\cos^2\theta - 1) - 2\theta^2 + \theta.2\sin\theta\cos\theta}{\theta^4}\right]$

$= \dfrac{w^2}{4}\left[\dfrac{-2\theta^2\cos^2\theta + 2\theta\sin\theta\cos\theta}{\theta^4}\right]$

$= \dfrac{w^2\cos\theta(\sin\theta - \theta\cos\theta)}{2\theta^3}.$

METHOD 2 Using product rule:

$A = \dfrac{w^2\theta^{-2}}{4}(\theta - \sin\theta\cos\theta)$

$\dfrac{dA}{d\theta} = \dfrac{w^2}{4\theta^2}\left[1 - (\cos^2\theta - \sin^2\theta)\right] + (\theta - \sin\theta\cos\theta) \times \dfrac{-2w^2\theta^{-3}}{4}$

$= \dfrac{w^2}{4\theta^2}(1 - \cos^2\theta + \sin^2\theta) - \dfrac{w^2}{2\theta^3}(\theta - \sin\theta\cos\theta),$

$= \dfrac{w^2}{4\theta^2}(2 - 2\cos^2\theta) - \dfrac{w^2}{2\theta^3}(\theta - \sin\theta\cos\theta)$

using $\sin^2\theta = 1 - \cos^2\theta$,

$= \dfrac{w^2}{2\theta^3}(\theta - \theta\cos^2\theta) - \dfrac{w^2}{2\theta^3}(\theta - \sin\theta\cos\theta)$

$= \dfrac{w^2(\theta - \theta\cos^2\theta - \theta + \sin\theta\cos\theta)}{2\theta^3}$

$= \dfrac{w^2\cos\theta(\sin\theta - \theta\cos\theta)}{2\theta^3}.$

(c) $g(\theta) = \sin\theta - \theta\cos\theta.$

$g'(\theta) = \cos\theta - (\theta \times -\sin\theta + \cos\theta \times 1)$

$= \cos\theta + \theta\sin\theta - \cos\theta$

$= \theta\sin\theta.$

For $0 < \theta < \pi$, $\theta > 0$ and $\sin\theta > 0$

$\therefore \quad g'(\theta) > 0$ for $0 < \theta < \pi$.

Since gradient of $g(\theta)$ is positive for $0 < \theta < \pi$ then $g(\theta)$ must be increasing for $0 < \theta < \pi$.

$g(0) = 0$

$\therefore \quad g(\theta) > 0$ for $0 < \theta < \pi$.

(d) $\dfrac{dA}{d\theta} = 0, \quad 0 < \theta < \pi.$

$\therefore \quad \dfrac{w^2\cos\theta(\sin\theta - \theta\cos\theta)}{2\theta^3} = 0$

Since $\sin\theta - \theta\cos\theta > 0$ for $0 < \theta < \pi$, from part (c), then only $\cos\theta = 0$.

$\theta = \dfrac{\pi}{2}$ for $0 < \theta < \pi$

$\therefore$ The only value of θ for the interval $0 < \theta < \pi$ is $\dfrac{\pi}{2}$.

(e) $\dfrac{dA}{d\theta} = \dfrac{w^2\cos\theta(\sin\theta - \theta\cos\theta)}{2\theta^3}$

Now $\dfrac{w^2(\sin\theta - \theta\cos\theta)}{2\theta^3} > 0$ for $0 < \theta < \pi$.

$\therefore$ Since $\cos\theta > 0$ for $0 < \theta < \dfrac{\pi}{2}$

$\therefore \quad \dfrac{dA}{d\theta} > 0$ for $0 < \theta < \dfrac{\pi}{2}$

and since $\cos\theta < 0$ for $\dfrac{\pi}{2} < \theta < \pi$

$\therefore \quad \dfrac{dA}{d\theta} < 0$ for $\dfrac{\pi}{2} < \theta < \pi$

$\therefore$ The maximum cross-sectional area occurs when $\theta = \dfrac{\pi}{2}$.

$\therefore$ Maximum area $= \dfrac{w^2}{4\left(\frac{\pi}{2}\right)^2}\left(\dfrac{\pi}{2} - \sin\dfrac{\pi}{2}\cos\dfrac{\pi}{2}\right)$

$= \dfrac{w^2}{2\pi}.$

END OF EXTENSION 1 SOLUTIONS