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YEAR 12 – EXT.2 MATHS

REVIEW TOPIC (SP2)
COMPLEX NUMBERS
(PAST HSC PAPERS 91-03)

Quest 1

(a) The complex number z and its conjugate $\bar{z}$ satisfy the equation $z\bar{z} + 2iz = 12 + 6i$. Find the possible values of z.

$\boxed{3-i, 3+3i}$

(b) $1+i$ is a root of the equation $x^2 + (a+2i)x + (5+ib) = 0$, where a and b are real. Find the values of a and b.

$\boxed{a=-3, b=-1}$

Quest 2

(a) $1-2i$ is one root of the equation $x^2+(1+i)x+k=0$. Find the other root and the value of k.

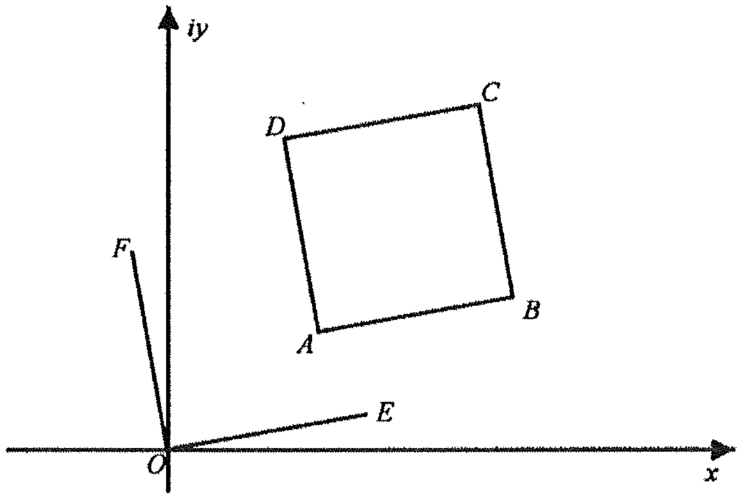
$\boxed{k=5i, x=-2+i}$

Quest 3 - HSC '91

(2) (b) Plot on an Argand diagram the points P, Q and R which correspond to the complex numbers $2i$, $\sqrt{3}-i$, and $-\sqrt{3}-i$, respectively.

Prove that P, Q, and R are the vertices of an equilateral triangle.

(d)



In the Argand diagram, $ABCD$ is a square, and OE and OF are parallel and equal to $AB$and AD respectively. The vertices A and B correspond to the complex numbers w_1 and w_2 respectively.

(a) Explain why the point E corresponds to $w_2 - w_1$.

(b) What complex number corresponds to the point F ?

$\boxed{i(w_2 - w_1)}$

(c) What complex number corresponds to the vertex D ?

$\boxed{(1-i)w_1 + iw_2}$

Quest 4 - HSC '93

(2) (e) Let P, Q and R represent the complex numbers ω_1, ω_2 and ω_3 respectively.
What geometrical properties characterize triangle PQR if $\omega_2 - \omega_1 = i(\omega_3 - \omega_1)$?
Give reasons for your answer.

Quest 5 - HSC '93

(8)*(a) Let the points $A_1, A_2, ..., A_n$ represents the nth roots of unity, $w_1, w_2, ..., w_n$,
and suppose P represents any complex number z such that $|z| = 1$.

(i) Prove that $w_1 + w_2 + ... + w_n = 0$.

(ii) Show that $PA_i^2 = (z - w_i)(\overline{z} - \overline{w}_i)$ for $i = 1, 2, \ldots, n$.

(Hint: $\overline{\alpha} - \overline{\beta} = \overline{\alpha - \beta}$ and $z\overline{z} = |z|^2$)

(iii) Prove that $\sum_{i=1}^{n} PA_i^2 = 2n$

Quest 6 - HSC '94

(2) (c) (i) On the same diagram, draw a neat sketch of the locus specified by each of the following:

(α) $|z-(3+2i)|=2$ (β) $|z+3|=|z-5|$.

(i) Hence write down all values of z which satisfy simultaneously.

$$|z-(3+2i)|=2 \quad \text{and} \quad |z+3|=|z-5|.$$

$\boxed{z=1+2i}$

(ii) Use the diagram in (i) to determine the values of k for which the simultaneous equations below have exactly one solution for IzI.

$$|z-(3+2i)|=2 \quad \text{and} \quad |z-2i|=k$$

$\boxed{k=1 \text{ or } 5}$

Quest 7 - HSC '95

(2) (c) Sketch the locus of z satisfying: **5**

(i) $\arg(z-4)=\frac{3\pi}{4}$;

(ii) $\operatorname{Im} z=|z|$.

Quest 8 - HSC '95

(4) (a) (i) Find the least positive integer k such that

$\cos\left(\frac{4\pi}{7}\right)+i\sin\left(\frac{4\pi}{7}\right)$ is a solution of $z^k=1$.

(ii) Show that if the complex number w is a solution of $z^n = 1$, then so is w^m, where m and n are arbitrary integers.

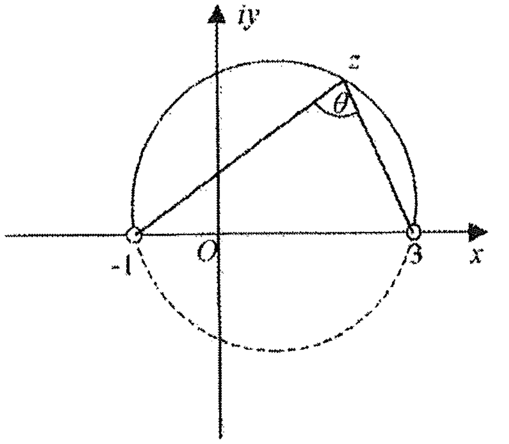
Quest 9 - HSC '96

(2) (b) On an Argand diagram, shade the region specified by both the conditions **2**

$$\text{Re}(z) \le 4 \quad \text{and} \quad |z - 4 + 5i| \le 3.$$

(d)

4



The diagram shows the locus of points z in the complex plane such that

$$\arg(z-3)-\arg(z+1)=\frac{\pi}{3}.$$

This locus is part of a circle.

The angle between the line -1 to z and from 3 to z is θ, as shown.

(iii) Explain why $\theta=\frac{\pi}{3}$.

(ii) Find the centre of the circle.

$\left(1, \frac{2}{\sqrt{3}}\right)$

Quest 10 - HSC '96

(8) (a) Let $w = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$. **7**

(i) Show that w^k is a solution of $z^9 - 1 = 0$, where k is an integer.

(ii) Prove that $w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7 + w^8 = -1$.

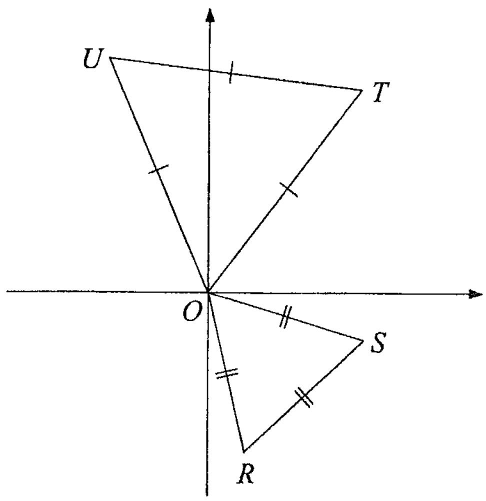
(iii) Hence show that $\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right)=\frac{1}{8}$.

Quest 11 - HSC '97

(2) (c) sketch the region where the inequalities **3**

$|z-3+i| \leq 5$ and $|z+1| \leq |z-1|$ both hold.

(7) (b) **5**



The diagram shows points O, R, S, T, and U in the complex plane. These points correspond to the complex numbers 0, r, s, t, and u respectively. The triangles ORS and OTU are equilateral. Let $\omega = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$.

(i) Explain why $u = \omega t$.

(ii) Find the complex number r in terms of s.

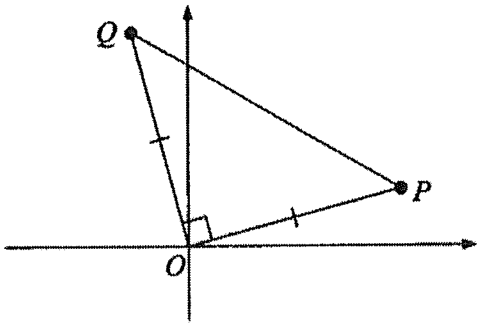
(iii) Using complex numbers, show that the lengths of RT and SU are equal.

Quest 12 - HSC '98

(2) (c) Sketch the region in the complex plane where the inequalities **2**

$|z-2+i| \leq 2$ and $\text{Im}(z) \geq 0$ both hold.

(d) **1**



The points P and Q in the complex plane correspond to the complex numbers z and w respectively. The triangle OPQ is isosceles and $\angle POQ$ is a right angle.

Show that $z^2 + w^2 = 0$.

(e) (i) By solving the equation $z^3 + 1 = 0$, find the three cube roots of -1.

$\boxed{\dfrac{1 \pm \sqrt{3}i}{2}, -1.}$

(ii) Let λ be the cube root of -1, where λ is not real.
Show that $\lambda^2 = \lambda - 1$.

(iii) Hence simplify $(1-\lambda)^6$.

$\boxed{\lambda^{12}}$

Quest 13 - HSC '98

(7) (a) Let $P(z) = z^8 - \frac{5}{2}z^4 + 1$. The complex number w is a root of $P(z) = 0$. **6**

(i) Show that iw and $\frac{1}{w}$ are the roots of $P(z) = 0$.

(ii) Find one of the roots of $P(z)=0$ in exact form.

$w=\pm\sqrt[4]{2}$ or $\pm\frac{1}{\sqrt[4]{2}}$

(iii) Hence find all the roots of $P(z)=0$.

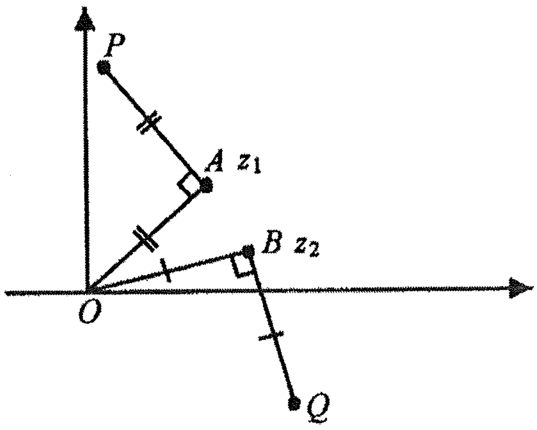
$w=\pm\sqrt[4]{2}$ or $\pm\frac{1}{\sqrt[4]{2}}$ or $\pm i\sqrt[4]{2}$ or $\pm\frac{i}{\sqrt[4]{2}}$

Quest 14 - HSC '99

(2) (c) Sketch the region in the Argand diagram where the two inequalities **2**

$|z-i|\le 2$ and $0\le\arg(z+1)\le\frac{\pi}{4}$ both hold.

(2) (e)



The points A and B in the complex plane correspont to complex numbers z_1 and z_2 respectively. Both triangles OAP and OBQ are right-angled isosceles triangles.

(i) Explain why P corresponds to the complex number $(1+i)\,z_1$.

(ii) Let M be the midpoint of PQ. What complex number corresponds to M?

$$\boxed{\frac{z_1(1+i)+z_2(1-i)}{2}}$$

Quest 15 - HSC 2002

(8)

(a) Let m be a positive integer.

(i) By using De Moivre's theorem, show that **2**

$$\sin(2m+1)\theta = \binom{2m+1}{1}\cos^{2m}\theta\sin\theta - \binom{2m+1}{3}\cos^{2m-2}\theta\sin^3\theta + \ldots + (-1)^m\sin^{2m+1}\theta.$$

(ii) Deduce that the polynomial **3**

$$p(x) \equiv \binom{2m+1}{1}x^m - \binom{2m+1}{3}x^{m-1} + \ldots + (-1)^m$$

has m distinct roots

$$\alpha_k = \cot^2\left(\frac{k\pi}{2m+1}\right) \quad \text{where } k = 1, 2, \ldots, m.$$

(iii) Prove that **2**

$$\cot^2\left(\frac{\pi}{2m+1}\right)+\cot^2\left(\frac{2\pi}{2m+1}\right)+\ldots+\cot^2\left(\frac{m\pi}{2m+1}\right)=\frac{m(2m-1)}{3}.$$

(iv) You are given that $\cot\theta<\frac{1}{\theta}$ for $0<\theta<\frac{\pi}{2}$. **2**

Deduce that: $\frac{\pi^2}{6}<\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+....+\frac{1}{m^2}\right).\frac{(2m+1)^2}{2m(2m-1)}$

Quest 16 - HSC 2003

(8)

(a) Suppose that $\omega^3 = 1$, $\omega \neq 1$, and k is a positive integer.

(i) Find the two possible values of $1 + \omega^k + \omega^{2k}$. **2**

0 or 3

(ii) Use the binomial theorem to expand $(1 + \omega)^n$ and $(1 + \omega^2)^n$, where n is a positive integer. **1**

(iii) Let ℓ be the largest integer such that $3\ell \leq n$. **2**

Deduce that

$$\binom{n}{0}+\binom{n}{3}+\binom{n}{6}+\ldots+\binom{n}{3\ell}=\frac{1}{3}\left(2^n+(1+\omega)^n+\left(1+\omega^2\right)^n\right).$$

(iv) If n is a multiple of 6, prove that **2**

$$\binom{n}{0}+\binom{n}{3}+\binom{n}{6}+\ldots+\binom{n}{n}=\frac{1}{3}\left(2^n+2\right).$$