NAME :

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YEAR 12 – MATHS EXT.2

REVIEW TOPIC: COMPLEX NUMBERS

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CSSA 2000 Q3

(a) $z_1 = 1 + 2i$ and $z_2 = 3 - i$. Find the value of $z_1^2 \div \bar{z}_2$. **2**

(b) $z = \sqrt{3} + i$ **4**

(i) Write z in modulus / argument form.

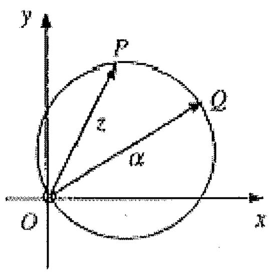
(ii) What can you say about integers n such that $z^n + (\bar{z})^n$ is rational ?

(iii) Find the smallest positive integer n such that $z^n + (\bar{z})^n$ is a negative rational number, and for this value of n, state the value of $z^n + (\bar{z})^n$.

(c) $\alpha = p + iq$ where p and q are real. **9**

(i) If z satisfies $\text{Re}(\alpha z) = 1$, show that the locus of the point P representing z in the Argand diagram is the line $px - qy = 1$.

(ii) The vector $\overrightarrow{OQ}$ represents α in the Argand diagram. If $z \neq 0$ is represented by the vector $\overrightarrow{OP}$ where P lies on the circle with diameter OQ, copy the diagram and show the vector representing $z - \alpha$.



Show that for such a complex number z, $\dfrac{z-\alpha}{z}$ is imaginary and hence $\text{Re}\left(\alpha \dfrac{1}{z}\right) = 1$.

(iii) Deduce that if z is a non-zero complex number such that the point P representing z in the Argand diagram lies on the circle with diameter OQ, where Q has coordinates (p, q), then the point representing $\frac{1}{z}$ in the same Argand diagram lies on the line $px - qy = 1$.

(iv) $z \neq 0$ satisfies the condition $|z-(1+i)| = \sqrt{2}$. Sketch the locus of the points representing z and $\frac{1}{z}$ in the same Argand diagram, and label each locus with its equation. Considering only values between $-\pi$ and π, what are the possible values of $\arg z$?

CSSA 2001 Q3

(a) In an Argand Diagram, the point P representing the complex number z moves so that $|z-(1+i)|=1$.

(i) Sketch the locus of P. 1

(ii) Shade the region where $|z-(1+i)|\le 1$ and $0<\arg(z-i)<\frac{\pi}{4}$ 1

(b) In an Argand Diagram, a regular hexagon $ABCDEF$, with the vertices taken in anticlockwise order, has its centre at the origin O and vertex A at $z=2$.

(i) Find the set of values of $\operatorname{Im}(z)$ for points z on the hexagon. 1

(ii) Find the set of values of $|z|$ for points z on the hexagon. 1

(iii) If the hexagon is rotated in a clockwise direction about the origin through an angle of $45°$, find the value in modulus / argument form of the complex number which is represented by the new position of the vertex C. 1

(c) (i) If $z = \cos\theta + i\sin\theta$, show that for positive integers n, $z^n + \frac{1}{z^n} = 2\cos n\theta$ and **3**

$z^n - \frac{1}{z^n} = 2i\sin n\theta$. Hence expand $\left(z + \frac{1}{z}\right)^4 + \left(z - \frac{1}{z}\right)^4$ to show that

$\cos^4\theta + \sin^4\theta = \frac{1}{4}(\cos 4\theta + 3)$.

(ii) By letting $x = \cos\theta$, show that the equation $8x^4 + 8\left(1 - x^2\right)^2 = 7$ has roots **2**

$\pm\cos\frac{\pi}{12}$, $\pm\cos\frac{5\pi}{12}$.

(iii) Deduce that $\cos\frac{\pi}{12}$, $\cos\frac{5\pi}{12}$ have a product of $\frac{1}{4}$ and a sum of $\sqrt{\frac{3}{2}}$. 3

(iv) Hence or otherwise find a surd expression for $\cos\frac{\pi}{12}$. 2

INDEPENDENT 2001 Q5

(b) (i) Show that the general solution of the equation $\cos 5\theta = -1$ is given by **2**

$\theta = (2n+1)\frac{\pi}{5}$, $n = 0, \pm 1, \pm 2, \ldots$.

Hence solve the equation $\cos 5\theta = -1$ for $0 \leq \theta \leq 2\pi$.

(ii) Use De Moivre's Theorem to show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. **3**

(iii) Find the exact trigonometric roots of the equation $16x^5 - 20x^3 + 5x + 1 = 0$. **2**

(iv) Hence find the exact values of $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$ and $\cos\frac{\pi}{5} \cdot \cos\frac{3\pi}{5}$ and factorise $16x^5 - 20x^3 + 5x + 1$ into irreducible factors over the rational numbers. **3**

NEAP 2000 Q4

(b) (i) By considering $z^9 - 1$ as a difference of two cubes, or otherwise, write 5

$$1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8$$

as a product of two polynomials with real coefficients, one of which is a quadratic.

(ii) Solve $z^9 - 1 = 0$ and hence write down the six solutions of $z^6 + z^3 + 1 = 0$.

(iii) Hence deduce that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$.

SOLUTIONS

CSSA 2000 Q3

Question 3

(a)

$z_1^2 = (1+2i)^2 = -3+4i$

$z_1^2 \div z_2 = \frac{-3+4i}{3+i}$

$= \frac{(-3+4i)(3-i)}{9+1}$

$= \frac{-5+15i}{10}$

$= \frac{-1+3i}{2}$

(b) (i)

$\sqrt{3}+i = 2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$

$z = 2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

(b) (ii)

$z^n = 2^n\left(\cos\frac{n\pi}{6}+i\sin\frac{n\pi}{6}\right)$

$\bar{z}^n = 2^n\left(\cos\frac{n\pi}{6}-i\sin\frac{n\pi}{6}\right)$

$z^n+\bar{z}^n = 2^n . 2\cos\frac{n\pi}{6}$

$\cos\frac{n\pi}{6}$ is rational when

$\frac{n\pi}{6} = k\frac{\pi}{2}$, k integral

or $\frac{n\pi}{6} = k\frac{\pi}{3}$, k integral

Hence $z^n+\bar{z}^n$ is rational when n is even or a multiple of 3.

(b)(iii) $n=4$ is the smallest positive integer for which $\cos\frac{n\pi}{6}$ is rational and negative. Hence $n=4$ and $z^4+\bar{z}^4 = 2^5\cos\frac{4\pi}{6} = -16$

(c) (i) Let $z = x+iy$

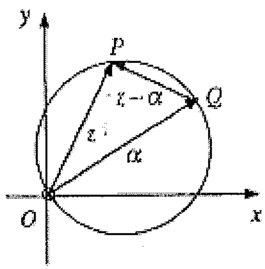
$\alpha z = (p+iq)(x+iy)$

$= (px-qy)+i(qx+py)$

$\text{Re}(\alpha z) = 1 \Rightarrow px-qy = 1$

Hence locus of P is the line $px-qy=1$.

(c)(ii)



$O\hat{P}Q = 90°$ ($\angle$ in semi – circle is rt angle.)

$\arg(z-\alpha)-\arg z = \pm\frac{\pi}{2}$

$\therefore \frac{z-\alpha}{z} = ki$, k integral

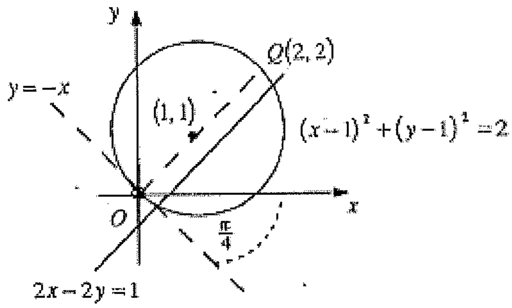
Hence $\frac{z-\alpha}{z}$ is imaginary and $\text{Re}\left(\frac{z-\alpha}{z}\right) = 0$.

But $\text{Re}\left(\frac{z-\alpha}{z}\right) = \text{Re}\left(1-\frac{\alpha}{z}\right) = 1-\text{Re}\left(\frac{\alpha}{z}\right)$

$\therefore \text{Re}\left(\frac{z-\alpha}{z}\right) = 0 \Rightarrow \text{Re}\left(\frac{\alpha}{z}\right) = \text{Re}\left(\alpha\frac{1}{z}\right) = 1$

(c)(iii) If P lies on circle with diameter OQ, where $Q(p,q)$ represents $\alpha = p+iq$, then from (ii) $\text{Re}\left(\alpha\frac{1}{z}\right) = 1$ and hence from (i), the point representing $\frac{1}{z}$ lies on the line $px-qy=1$.

(c)(iv)



The locus of z is the circle with centre $(1,1)$ and radius $\sqrt{2}$. Its diameter is OQ as shown. Since Q has coordinates $(2,2)$, the locus of $\frac{1}{z}$ is the line $2x-2y=1$, using (iii).

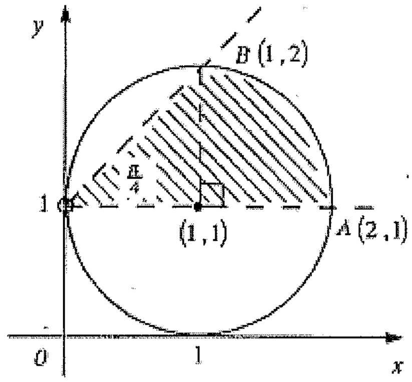
The line $y=-x$ is perpendicular to OQ and hence is tangent to the circle at O.

$\therefore -\frac{\pi}{4} < \arg z < \frac{3\pi}{4}$

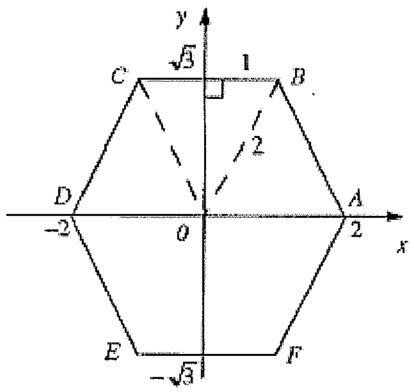
CSSA 2001 Q3

(a)

Answer (i), (ii) Locus of P is the circle centred on $(1,1)$ with radius 1 unit.



(b)



(i) $-\sqrt{3} \le \text{Im}(z) \le \sqrt{3}$

(ii) $\sqrt{3} \le |z| \le 2$

(iii) Each of the triangles ΔAOB, ΔBOC, ... is equilateral with side 2 units.

$\therefore\ A\hat{O}C = 2 \times 60° = 120°$

After rotation clockwise through 45°, OC will make an angle 75°, or $\frac{5\pi}{12}$ radians, with the positive x axis. Hence C will then represent the complex number $2\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$.

(c)

Answer

(i) Using De Moivre's Theorem,

$z = \cos\theta + i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$

$\therefore\ z^n + \frac{1}{z^n} = 2\cos n\theta, \quad z^n - \frac{1}{z^n} = 2i\sin n\theta$

$$\left(z+\frac{1}{z}\right)^4 + \left(z-\frac{1}{z}\right)^4 = 2\left(z^4 + 6z^2\cdot\frac{1}{z^2} + \frac{1}{z^4}\right)$$

$$= 2\left(z^4 + \frac{1}{z^4}\right) + 12$$

$(2\cos\theta)^4 + (2i\sin\theta)^4 = 2(2\cos 4\theta) + 12$

$16(\cos^4\theta + \sin^4\theta) = 4(\cos 4\theta + 3)$

$\therefore\quad \cos^4\theta + \sin^4\theta = \frac{1}{4}(\cos 4\theta + 3)$

(ii)

$x = \cos\theta, \qquad 8x^4 + 8(1-x^2)^2 = 7$

$1 - x^2 = \sin^2\theta \Rightarrow 8(\cos^4\theta + \sin^4\theta) = 7$

$2(\cos 4\theta + 3) = 7$

Hence equation becomes

$x = \cos\theta, \qquad \cos 4\theta = \frac{1}{2}$

$4\theta = 2n\pi \pm \frac{\pi}{3} \Rightarrow \theta = \frac{(6n\pm 1)\pi}{12}$

$n = 0, \pm 1, \pm 2, \ldots$

$x = \cos\frac{\pi}{12}, \cos\frac{5\pi}{12}, \cos\frac{7\pi}{12}, \cos\frac{11\pi}{12}$

$x = \cos\frac{\pi}{12}, \cos\frac{5\pi}{12}, \cos\left(\pi - \frac{5\pi}{12}\right), \cos\left(\pi - \frac{\pi}{12}\right)$

$\therefore\ x = \pm\cos\frac{\pi}{12}, \pm\cos\frac{5\pi}{12}$

(iii) $8x^4 + 8(1-x^2)^2 = 7$ simplifies to give

$16x^4 - 16x^2 + 1 = 0$,

with roots $\cos\frac{\pi}{12}, -\cos\frac{\pi}{12}, \cos\frac{5\pi}{12}, -\cos\frac{5\pi}{12}$.

Then $\alpha\beta\gamma\delta = \cos^2\frac{\pi}{12}\cos^2\frac{5\pi}{12} = \frac{1}{16}$

$\sum\alpha\beta = -\cos^2\frac{\pi}{12} - \cos^2\frac{5\pi}{12} = -1$

where $0 < \frac{\pi}{12} < \frac{5\pi}{12} < \frac{\pi}{2}$.

Then $\cos\frac{\pi}{12}\cos\frac{5\pi}{12} = +\sqrt{\frac{1}{16}} = \frac{1}{4}$, and

$\cos^2\frac{\pi}{12} + \cos^2\frac{5\pi}{12} + 2\cos\frac{\pi}{12}\cos\frac{5\pi}{12} = 1 + \frac{1}{2}$

$\therefore\quad \left(\cos\frac{\pi}{12} + \cos\frac{5\pi}{12}\right)^2 = \frac{3}{2}$

$\cos\frac{\pi}{12} + \cos\frac{5\pi}{12} = \sqrt{\frac{3}{2}}$

(iv) $\cos\frac{\pi}{12}, \cos\frac{5\pi}{12}$ are roots of the quadratic equation $x^2 - \sqrt{\frac{3}{2}}\,x + \frac{1}{4} = 0$.

$$x = \frac{\sqrt{\frac{3}{2}} \pm \sqrt{\frac{3}{2} - 1}}{2} = \frac{\sqrt{3} \pm 1}{2\sqrt{2}}$$

$\cos\frac{\pi}{12} > \cos\frac{5\pi}{12} \Rightarrow \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

INDEPENDENT 2001 Q5

Answer

(i)

$\cos 5\theta = -1 \Rightarrow 5\theta = (2n+1)\pi$

$\theta = (2n+1)\frac{\pi}{5}$, $n = 0, \pm 1, \pm 2, \ldots$

$0 \le \theta \le 2\pi \Rightarrow \theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$

(ii) Using the binomial expansion,

$\text{Re}\left\{(\cos\theta + i\sin\theta)^5\right\}$

$= \cos^5\theta + 10\cos^3\theta\,(i\sin\theta)^2 + 5\cos\theta\,(i\sin\theta)^4$

$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$= \cos^5\theta - 10\cos^3\theta\,(1-\cos^2\theta) + 5\cos\theta\,(1-\cos^2\theta)^2$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

Using De Moivre's Theorem,

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

Hence $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(iii) $16x^5 - 20x^3 + 5x + 1 = 0$

has solutions $x = \cos\theta$ where $\cos 5\theta = -1$.

$x = \cos\frac{\pi}{5}, \cos\frac{3\pi}{5}, \cos\pi, \cos\frac{7\pi}{5}, \cos\frac{9\pi}{5}$

$x = \cos\frac{\pi}{5}, \cos\frac{\pi}{5}, \cos\frac{3\pi}{5}, \cos\frac{3\pi}{5}, -1$

(iv)

$\sum \alpha = 0 \Rightarrow 2\left(\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}\right) - 1 = 0$

$\therefore \cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$

Product of roots is $-\frac{1}{16}$

$\therefore -\left(\cos\frac{\pi}{5}\cdot\cos\frac{3\pi}{5}\right)^2 = -\frac{1}{16}$

$\therefore \cos\frac{\pi}{5}\cdot\cos\frac{3\pi}{5} = -\frac{1}{4}$

$\left(\text{since } \cos\frac{\pi}{5} > 0,\ \cos\frac{3\pi}{5} < 0\right)$

Then $\cos\frac{\pi}{5}$, $\cos\frac{3\pi}{5}$ are roots of

the equation $4x^2 - 2x - 1 = 0$. Hence

$16x^5 - 20x^3 + 5x + 1 = (x+1)(4x^2 - 2x - 1)^2$

NEAP 2000 Q4

(b) (i) $(z^3)^3 - 1^3 = (z^3 - 1)(z^6 + z^3 + 1)$

$= (z - 1)(z^2 + z + 1)(z^6 + z^3 + 1)$

Since $z^9 - 1 = (z - 1)(z^8 + z^7 + \ldots + z + 1)$,

$z^8 + z^7 + \ldots + z + 1 = (z^2 + z + 1)(z^6 + z^3 + 1)$ ✓

(ii) $z^9 - 1 = (z^3 - 1)(z^6 + z^3 + 1)$

Let $z^9 = 1$. Let $z = \text{cis}\theta$ (where $|z| = 1$)

$\text{cis}9\theta = 1$

$9\theta = 2k\pi$, $k \in J$.

$\theta = \frac{2k\pi}{9}$, $k \in J$.

$\therefore \theta = 0, \pm\frac{2\pi}{9}, \pm\frac{4\pi}{9}, \pm\frac{6\pi}{9}, \pm\frac{8\pi}{9}$ ✓

$\theta = 0, \pm\frac{6\pi}{9}$ are roots of $z^3 - 1 = 0$. ---(1)

$\therefore$ roots of $z^6 + z^3 + 1 = 0$ are $\theta = \pm\frac{2\pi}{9}, \pm\frac{4\pi}{9}, \pm\frac{8\pi}{9}$ ✓

(iii) $\text{cis}0+\text{cis}\frac{2\pi}{9}+\text{cis}\frac{4\pi}{9}+\text{cis}\frac{6\pi}{9}+\text{cis}\frac{8\pi}{9}+\text{cis}\left(-\frac{2\pi}{9}\right)+\text{cis}\left(-\frac{4\pi}{9}\right)+\text{cis}\left(-\frac{6\pi}{9}\right)+\text{cis}\left(-\frac{8\pi}{9}\right)$ is the sum of the roots of $z^9-1=0$ which equals zero, since the coefficient of z^8 is zero.

$\text{cis}0+\text{cis}\left(\frac{6\pi}{9}\right)+\text{cis}\left(-\frac{6\pi}{9}\right)=0$ (from (1)) ✓ (sum of roots of $z^3-1=0$)

Hence $\text{cis}\frac{2\pi}{9}+\text{cis}\frac{4\pi}{9}+\text{cis}\frac{8\pi}{9}+\text{cis}\left(-\frac{2\pi}{9}\right)+\text{cis}\left(-\frac{4\pi}{9}\right)+\text{cis}\left(-\frac{8\pi}{9}\right)=2\cos\frac{2\pi}{9}+2\cos\frac{4\pi}{9}+2\cos\frac{8\pi}{9}$

$$=2\left(\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{8\pi}{9}\right)$$

$$=0$$

so $\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{8\pi}{9}=0$, i.e. $\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}=\cos\frac{\pi}{9}$ (since $\cos(\pi-x)=-\cos x$) ✓