NAME :

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YEAR 12 – MATHS EXT.2

REVIEW TOPIC (PAPER 1): HARDER EXT 1- INEQUALITIES

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CSSA 2000

8.

(b) (i) If $y = x^k + (c - x)^k$, where $c > 0$, $k > 0$, $k \neq 1$, show that y has a single stationary value between $x = 0$ and $x = c$, and show that this stationary value is a maximum if $k < 1$ and a minimum if $k > 1$. 7

(ii) Hence show that if $a>0$, $b>0$, $a \neq b$, then

$$\frac{a^k+b^k}{2} < \left(\frac{a+b}{2}\right)^k \quad \text{if} \quad 0<k<1 \ , \quad \text{and} \quad \frac{a^k+b^k}{2} > \left(\frac{a+b}{2}\right)^k \quad \text{if} \quad k>1 \ .$$

CSSA 2001

8.

(a) (i) Given that $y = x - \ln(\sec x + \tan x)$, $0 \le x < \frac{\pi}{2}$, show that $\frac{dy}{dx} = 1 - \sec x$. **2**

(ii) Hence show that $x < \ln(\sec x + \tan x)$ for $0 < x < \frac{\pi}{2}$. **3**

HEFFERNAN 2002

8.

(a) (i) For all real, positive numbers a and b, where $a > b$ show that **3**

$$b^2 - a^2 < 2\sqrt{ab}(b-a)$$

(ii) Hence deduce that $a > c$ given that c is a positive real number and **4**

$$\sqrt{a}(b-a)+\sqrt{c}(c-b) > \frac{c^2-a^2}{2\sqrt{b}}$$

INDEPENDENT 2001

8.

(c) It is given that if a, b, c are any three positive real numbers, then $\dfrac{a+b+c}{3} \geq \sqrt[3]{abc}$.

If $a>0$, $b>0$ and $c>0$ are real numbers such that $a+b+c=1$, use the given result to show that

(i) $\dfrac{1}{abc} \geq 27$ **1**

(ii) $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c} \geq 9$ **2**

(iii) $(1-a)(1-b)(1-c) \geq 8abc$ **2**

NEAP 2001

6.

(b) (i) α. Differentiate $y = \log_e(1+x)$, and hence draw $y = x$ and $y = \log_e(1+x)$ on one set of axis. **1**

β. Using this graph, explain why **1**

$$\log_e(1+x) < x, \text{ for all } x > 0.$$

(ii) α. Differentiate $y = \frac{x}{1+x}$, and hence draw $y = \frac{x}{1+x}$ and $y = \log_e(1+x)$ on one set of axis. 1

β. Using this graph, explain why 1

$$\frac{x}{1+x} < \log_e(1+x)\text{, for all } x > 0\text{.}$$

(iii) Use the inequalities of parts (i) and (ii) to show that **4**

$$\frac{\pi}{8} - \frac{1}{4}\log_e 2 < \int_0^1 \frac{\log_e(1+x)}{1+x^2}\,dx < \frac{1}{2}\log_e 2 .$$

NEAP 2001

7.

(c) You may assume that, for all positive real numbers a and b,

$$\sqrt{ab} \le \frac{a+b}{2}.$$

(i) Show that for all positive integers n, **1**

$$^nC_0 + {^nC_1} + \ldots + {^nC_n} = 2^n.$$

(ii) Prove that for all positive integers n, **2**

$$\left(\sqrt{^nC_1} + \sqrt{^nC_2} + \ldots + \sqrt{^nC_n}\right)^2 \le n(2^n - 1).$$

You may use the identity

$$(x_1 + x_2 + \ldots + x_n)^2 = (x_1^{\ 2} + x_2^{\ 2} + \ldots + x_n^{\ 2}) + \sum_{i<j} 2x_i x_j.$$

SOLUTIONS

CSSA 2000 Q8

(b)(i)

$y = x^k + (c-x)^k$

$\frac{dy}{dx} = k\,x^{k-1} - k\,(c-x)^{k-1}$

$\frac{dy}{dx} = 0 \Rightarrow x^{k-1} = (c-x)^{k-1} \Rightarrow \left(\frac{c-x}{x}\right)^{k-1} = 1$

$\left(\frac{c-x}{x}\right)^{k-1} = 1,\ k \neq 1 \Rightarrow \frac{c}{x} - 1 = \pm 1$

$\therefore$ since $c > 0$, $\frac{dy}{dx} = 0 \Rightarrow \frac{c}{x} = 2$

Hence y has a single stationary value at $x = \frac{1}{2}c$.

$\frac{dy}{dx} = k\,x^{k-1} - k\,(c-x)^{k-1}$

$\frac{d^2y}{dx^2} = k\,(k-1)x^{k-2} + k\,(k-1)(c-x)^{k-2}$

$x = \frac{1}{2}c \Rightarrow \frac{d^2y}{dx^2} = 2k\,(k-1)\left(\frac{1}{2}c\right)^{k-2}$

$0 < k < 1 \Rightarrow \frac{d^2y}{dx^2} < 0 \Rightarrow y$ has max. at $x = \frac{1}{2}c$

$k > 1 \Rightarrow \frac{d^2y}{dx^2} > 0 \Rightarrow y$ has min. at $x = \frac{1}{2}c$

(b)(ii)

Let $c = a + b$, $a > 0$, $b > 0$, $a \neq b$

Consider $y = x^k + (c-x)^k$, $k > 0$, $k \neq 1$.

From (i), for $0 < x < c$,

$0 < k < 1 \Rightarrow$ y has a maximum value of $\left(\frac{1}{2}c\right)^k + \left(c - \frac{1}{2}c\right)^k = 2\left(\frac{1}{2}c\right)^k$ when $x = \frac{1}{2}c$.

$0 < k < 1 \Rightarrow x^k + (c-x)^k < 2\left(\frac{1}{2}c\right)^k$ for $x \neq \frac{1}{2}c$

$a^k + (c-a)^k < 2\left(\frac{1}{2}c\right)^k \quad \left(a \neq \frac{1}{2}c\right)$

$\therefore 0 < k < 1 \Rightarrow \frac{a^k + b^k}{2} < \left(\frac{a+b}{2}\right)^k$

$k > 1 \Rightarrow$ y has a minimum value of $\left(\frac{1}{2}c\right)^k + \left(c - \frac{1}{2}c\right)^k = 2\left(\frac{1}{2}c\right)^k$ when $x = \frac{1}{2}c$.

$k > 1 \Rightarrow x^k + (c-x)^k > 2\left(\frac{1}{2}c\right)^k$ for $x \neq \frac{1}{2}c$

$a^k + (c-a)^k > 2\left(\frac{1}{2}c\right)^k \quad \left(a \neq \frac{1}{2}c\right)$

$\therefore k > 1 \Rightarrow \frac{a^k + b^k}{2} > \left(\frac{a+b}{2}\right)^k$

CSSA 2001 Q8

(i) $y = x - \ln(\sec x + \tan x)$, $0 \le x < \frac{\pi}{2}$

$$\frac{dy}{dx} = 1 - \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

$$= 1 - \frac{\sec x\,(\sec x + \tan x)}{\sec x + \tan x}$$

$$= 1 - \sec x$$

(ii) $x = 0 \Rightarrow y = 0 - \ln(1+0) = 0$

$\frac{dy}{dx} = 0$ for $x = 0$, and $\frac{dy}{dx} < 0$ for $0 < x < \frac{\pi}{2}$

Hence $y = x - \ln(\sec x + \tan x)$ is a decreasing function, and hence $y < 0$, for $0 < x < \frac{\pi}{2}$.

$x < \ln(\sec x + \tan x)$ for $0 < x < \frac{\pi}{2}$.

HEFFERNAN 2002 Q8

Question 8

(a) (i)

$$a > b$$
$$a - b > 0$$
$$(a-b)^2 > 0$$
$$a^2 - 2ab + b^2 > 0$$
$$a^2 + b^2 > 2ab \qquad \textbf{(1 mark)}$$
$$a^2 + 2ab + b^2 > 4ab$$
$$(a+b)^2 > 4ab$$
$$a + b > 2\sqrt{ab} \qquad \textbf{(1 mark)}$$
$$(a+b)(b-a) < 2\sqrt{ab}(b-a) \qquad \text{since } b - a < 0$$
$$b^2 - a^2 < 2\sqrt{ab}(b-a)$$

(1 mark) for reversing inequality & stating why

(ii) We have, $\sqrt{a}(b-a) + \sqrt{c}(c-b) > \dfrac{c^2 - a^2}{2\sqrt{b}}$

So, $2\sqrt{ab}(b-a) + 2\sqrt{bc}(c-b) > c^2 - a^2$

So, $c^2 - a^2 < 2\sqrt{ab}(b-a) + 2\sqrt{bc}(c-b)$ **(1 mark)**

From part (i) we know that $a > b$ and $b^2 - a^2 < 2\sqrt{ab}(b-a)$ -(A)

Suppose that $b > c$, then, following the pattern, we have

$$c^2 - b^2 < 2\sqrt{bc}(c-b) \qquad \text{-(B)}$$

(1 mark)

Adding (A) and (B) gives $c^2 - a^2 < 2\sqrt{ab}(b-a) + 2\sqrt{bc}(c-b)$ which was given.

So, we know now that $b > c$. **(1 mark)**

So if $b > c$ and $a > b$ (from part (i)) then $a > c$ as required. **(1 mark)**

INDEPENDENT 2001 Q8

Answer

(i)

$\sqrt[3]{abc} \le \frac{a+b+c}{3} = \frac{1}{3}$

$abc \le \frac{1}{27}$

$\frac{1}{abc} \ge 27$

(ii)

$\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge \sqrt[3]{\left(\frac{1}{a}\right)\left(\frac{1}{b}\right)\left(\frac{1}{c}\right)}$

$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge 3\sqrt[3]{\frac{1}{abc}}$

$\ge 3\sqrt[3]{27}$

$\therefore \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge 9$

(iii)

$(1-a)(1-b)(1-c)$

$= 1-(a+b+c)+(bc+ca+ab)-abc$

$= (bc+ca+ab)-abc$

$= abc\left\{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) - 1\right\}$

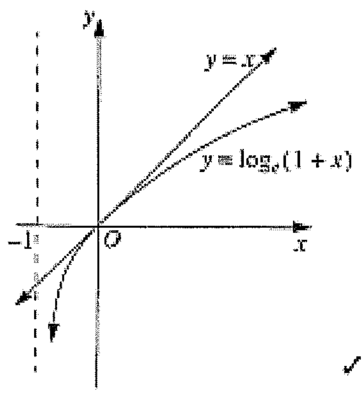
$\ge abc\,(9-1)$

$\therefore (1-a)(1-b)(1-c) \ge 8\,abc$

NEAP 2001 Q6

(b) (i) α. $y = \log_e(1+x)$

$\frac{dy}{dx} = \frac{1}{1+x}$



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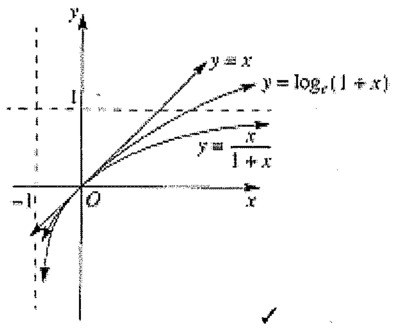
β. When $x = 0$, $\frac{dy}{dx} = 1$, so $y = x$ is a tangent at $(0, 0)$.

Since $y = \log_e(1+x)$ is concave down, it follows that its graph is below the line $y = x$ for $x > 0$. ✓

(ii) α. $y = \frac{x}{1+x}$

Using the quotient rule, $\frac{dy}{dx} = \frac{1}{(1+x)^2}$.



✓

β. When $x = 0$, $\frac{dy}{dx} = 1$, so $y = x$ is a tangent to both curves at $(0, 0)$.

But for $x > 0$, the gradient function of $y = \frac{x}{1+x}$ is less than the gradient function of $y = \log_e(1+x)$, because $\frac{1}{(1+x)^2} < \frac{1}{1+x}$ for $x > 0$.

Hence the graph of $y = \frac{x}{1+x}$ is always below the graph of $y = \log_e(1+x)$ for $x > 0$. ✓

(iii) From (i) and (ii), $\frac{x}{1+x} < \log_e(1+x) < x$ for all $x > 0$.

Hence $\frac{x}{(1+x)(1+x^2)} < \frac{\log_e(1+x)}{1+x^2} < \frac{x}{1+x^2}$ for all $x > 0$

and so $\int_0^1 \frac{x}{(1+x)(1+x^2)}\,dx < \int_0^1 \frac{\log_e(1+x)}{1+x^2}\,dx < \int_0^1 \frac{x}{1+x^2}\,dx$ for all $x > 0$. ✓

Now $\int_0^1 \frac{x}{1+x^2}\,dx = \left[\frac{1}{2}\log_e(x^2+1)\right]_0^1$

$= \frac{1}{2}\log_e 2$ ✓

Also, $\int_0^1 \frac{x}{(1+x)(1+x^2)}\,dx = \int_0^1 \left(-\frac{1}{2(x+1)} + \frac{1+x}{2(x^2+1)}\right)dx$ (partial fractions)

$= \left[-\frac{1}{2}\log_e(1+x)\right]_0^1 + \left[\frac{1}{4}\log_e(x^2+1)\right]_0^1 + \left[\frac{1}{2}\tan^{-1}x\right]_0^1$ ✓

$= -\frac{1}{2}\log_e 2 + \frac{1}{4}\log_e 2 + \frac{1}{2}\tan^{-1}1$

$= \frac{\pi}{8} - \frac{1}{4}\log_e 2$

Hence $\frac{\pi}{8} - \frac{1}{4}\log_e 2 < \int_0^1 \frac{\log_e(1+x)}{1+x^2}\,dx < \frac{1}{2}\log_e 2$ for all $x > 0$. ✓

NEAP 2001 Q7

(c) (i) The numbers ${}^nC_0, {}^nC_1, \ldots, {}^nC_n$ are defined by $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_n$.

Substituting $x = 1$, $2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$ ✓

(ii) Using the given identity,

$$\text{LHS} = {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n + \sum_{i<j} 2\sqrt{{}^nC_i}\sqrt{{}^nC_j}.$$

Using (i) and the AM/GM inequality,

$$\text{LHS} < 2^n - 1 + \sum_{i<j} 2\sqrt{{}^nC_i}\sqrt{{}^nC_j} \quad ✓$$

$$= 2^n - 1 + (n-1)({}^nC_1 + {}^nC_2 + \ldots + {}^nC_n)$$

$$= 2^n - 1 + (n-1)(2^n - 1)$$

$$= n(2^n - 1), \text{ as required.} \quad ✓$$