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# YEAR 12 – MATHS EXT.2

# REVIEW TOPIC (PAPER 1): HYPERBOLA

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## CSSA 2000 Q4

Hyperbola $\mathcal{H}$ has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and eccentricity $e$, while ellipse $\mathcal{E}$ has equation $\frac{x^2}{a^2 + b^2} + \frac{y^2}{b^2} = 1$. 15

(i) Show that $\mathcal{E}$ has eccentricity $\frac{1}{e}$.

(ii) Show that $\mathcal{E}$ passes through one focus of $\mathcal{H}$, and $\mathcal{H}$ passes through one focus of $\mathcal{E}$.

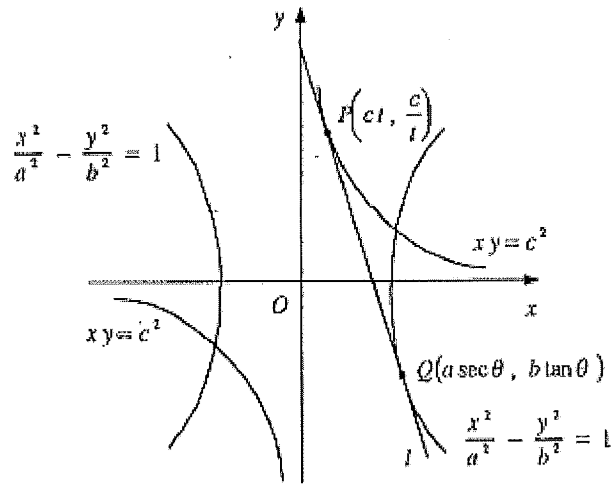
(iii) Sketch $\mathcal{H}$ and $\mathcal{E}$ on the same diagram, showing the foci S, S' of $\mathcal{H}$ and T, T' of $\mathcal{E}$, and the directrices of $\mathcal{H}$ and $\mathcal{E}$. Give the coordinates of the foci and the equations of the directrices in terms of $a$ and $e$.

(iv) If $\mathcal{H}$ and $\mathcal{E}$ intersect at $P$ in the first quadrant, show that the acute angle $\alpha$ between the tangents to the curves at $P$ satisfies $\tan\alpha = \sqrt{2}\left(e + \frac{1}{e}\right)$.

(v) What is the smallest possible acute angle between the tangents to the curves $\mathcal{H}$ and $\mathcal{E}$ at their point of intersection $P$ ?

(vi) Find the acute angle between the tangents to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at their points of intersection. Give your answer to the nearest degree.

## CSSA 2001 Q4




The line $l$ is a common tangent to the hyperbolas $xy = c^2$, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with points of contact $P$ and $Q$ respectively.

(i) Considering $l$ as a tangent to $xy = c^2$ at $P\left(ct, \frac{c}{t}\right)$, show $l$ has equation $x + t^2 y = 2ct$. 2

(ii) Considering $l$ as a tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $Q(a\sec\theta, b\tan\theta)$, show $l$ has equation $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$. 2

(iii) Deduce that $\frac{\sec\theta}{a} = \frac{-\tan\theta}{bt^2} = \frac{1}{2ct}$. 1

(iv) Write the coordinates of $Q$ in terms of $t$, $a$, $b$ and $c$, and show that $b^2t^4 + 4c^2t^2 - a^2 = 0$. **3**
Deduce that there are exactly two such common tangents to the hyperbolas.

(v) Copy the diagram and use the symmetry in the graphs to draw in the second common tangent with points of contact $R$ on $xy = c^2$ and $S$ on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. **2**
Write the coordinates of $R$ and $S$ in terms of $t$, $a$, $b$ and $c$.

(vi) Show that if $PQRS$ is a rhombus, then $b^2 = a^2$ and deduce that $t^2 < 1$. **2**

(vii) Show that if $PQRS$ is a square, then $t^4 + 2t^2 - 1 = 0$ and deduce that $2c^2 = a^2$. 3
What is the relationship between the two hyperbolas if $PQRS$ is a square ?

## NEAP 2000 Q5

(c) Consider the hyperbola $xy = c^2$ and the distinct points $P\left(ct_1, \frac{c}{t_1}\right)$ and $Q\left(ct_2, \frac{c}{t_2}\right)$ on it. **7**

(i) Show that the equation of the tangent at $\left(ct, \frac{c}{t}\right)$, where $t \neq 0$, is $x + t^2 y = 2ct$.

(ii) Show that the tangents at $P$ and $Q$ intersect at $M\left(\frac{2ct_1t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2}\right)$.

(iii) Show that if $t_1 t_2 = k$, where $k$ is a non-zero constant, then the locus of $M$ is a line passing through the origin.

## SGHS 2002 Q4

2. Find the equation of the chord of contact of the tangents to the hyperbola $x^2 - 16y^2 = 16$ from the point with coordinates $(2, -4)$ [2]

3. Find the equation of the hyperbola with foci at $(\pm 5, 0)$ and eccentricity $e = \frac{5}{4}$ [3]

## SOLUTIONS

## CSSA 2000 Q4

(i) For the hyperbola $\mathcal{H}$,

$b^2 = a^2(e^2-1)$.

$e^2 = \frac{b^2}{a^2}+1 = \frac{b^2+a^2}{a^2}$

If the ellipse $\mathcal{E}$ has eccentricity $\varepsilon$,

$b^2 = (a^2+b^2)(1-\varepsilon^2)$

$\varepsilon^2 = 1 - \frac{b^2}{a^2+b^2}$

$\therefore \varepsilon^2 = \frac{a^2}{a^2+b^2} = \frac{1}{e^2}$

Hence the ellipse $\mathcal{E}$ has eccentricity $\frac{1}{e}$.

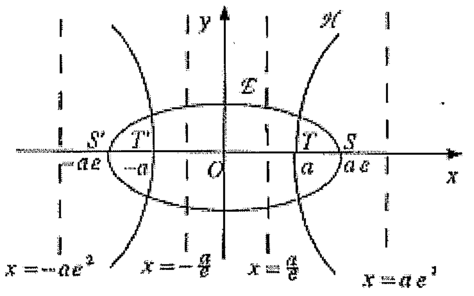
(ii) Since $a^2+b^2=a^2e^2$, the equation of the ellipse can be rewritten as $\frac{x^2}{a^2e^2}+\frac{y^2}{b^2}=1$.

One focus of $\mathcal{H}$ is $S(ae, 0)$, and this point clearly lies on the ellipse $\mathcal{E}$.

One focus of the ellipse is $T(ae\cdot\frac{1}{e},\ 0) \equiv T(a, 0)$ and this point is clearly on the hyperbola $\mathcal{H}$.

(iii) Hyperbola $\mathcal{H}$ has foci $S(ae, 0)$, $S'(-ae, 0)$ and directrices $x=\frac{a}{e}$, $x=-\frac{a}{e}$.

Ellipse $\mathcal{E}$ has foci $T(a, 0)$, $T'(-a, 0)$ and directrices $x=\frac{ae}{\left(\frac{1}{e}\right)}=ae^2$, $x=-ae^2$.




(vi) Hyperbola $\mathcal{H}$ : $\frac{x^2}{16}-\frac{y^2}{9}=1$, with eccentricity $e$ given by $9=16(e^2-1) \Rightarrow e=\frac{5}{4}$, and ellipse $\mathcal{E}$ : $\frac{x^2}{25}+\frac{y^2}{9}=1$ are two such conics.

Using the symmetry in their graphs, at all of their points of intersection, the acute angle $\alpha$ between the tangents to the curves is given by $\tan\alpha = \sqrt{2}\left(\frac{5}{4}+\frac{4}{5}\right)$

Hence $\alpha = 71°$ ( to the nearest degree)

(iv) Where the curves intersect,

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (1)$

$\frac{x^2}{a^2e^2} + \frac{y^2}{b^2} = 1 \quad (2)$

$(1)+(2) \Rightarrow \frac{x^2}{a^2e^2}(e^2+1)=2$

$e^2\times(2)-(1) \Rightarrow \frac{y^2}{b^2}(e^2+1)=e^2-1$

$b^2=a^2(e^2-1) \Rightarrow \frac{y^2}{a^2(e^2-1)}(e^2+1)=e^2-1$

$\therefore$ at $P$, $x=ae\sqrt{\frac{2}{e^2+1}}$, $y=\frac{a(e^2-1)}{\sqrt{e^2+1}}$

For the hyperbola, at $P$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{b^2}{a^2}\frac{x}{y} = (e^2-1)\frac{x}{y} = \sqrt{2}\,e$

For the ellipse, at $P$

$\frac{x^2}{a^2e^2} + \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2e^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{b^2}{a^2e^2}\frac{x}{y} = -\frac{(e^2-1)}{e^2}\frac{x}{y} = -\sqrt{2}\,\frac{1}{e}$

Hence the gradients of the tangents to $\mathcal{H}$ and $\mathcal{E}$ at $P$ are $\sqrt{2}\,e$ and $-\sqrt{2}\,\frac{1}{e}$ respectively.

$\tan\alpha = \left|\frac{\sqrt{2}\,e-\left(-\sqrt{2}\,\frac{1}{e}\right)}{1+\sqrt{2}\,e\left(-\sqrt{2}\,\frac{1}{e}\right)}\right| = \sqrt{2}\left|\frac{e+\frac{1}{e}}{1-2}\right|$

$\therefore \tan\alpha = \sqrt{2}\left(e+\frac{1}{e}\right)$

(v) For the hyperbola $\mathcal{H}$, $e>1$

$\left(e+\frac{1}{e}\right)^2 = \left(e-\frac{1}{e}\right)^2+4 \Rightarrow \left(e+\frac{1}{e}\right)^2 > 4$

and $\left(e+\frac{1}{e}\right)^2 \to 4$ as $e \to 1^+$.

$\therefore \left(e+\frac{1}{e}\right) > 2$ and $\left(e+\frac{1}{e}\right) \to 2$ as $e\to 1^+$

Hence $\tan\alpha > 2\sqrt{2} \Rightarrow \alpha > \tan^{-1}(2\sqrt{2})$,

and $\alpha \to \tan^{-1}(2\sqrt{2})$ as $e \to 1^+$.

## CSSA 2001 Q4

**Answer**

(i)

$$\left.\begin{aligned} x = ct &\Rightarrow \frac{dx}{dt} = c \\ y = \frac{c}{t} &\Rightarrow \frac{dy}{dt} = \frac{-c}{t^2} \end{aligned}\right\} \quad \therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = -\frac{1}{t^2}$$

Hence tangent $l$ has gradient $-\frac{1}{t^2}$ and equation $x + t^2 y = k$, $k$ constant, where $P\left(ct, \frac{c}{t}\right)$ lies on $l \Rightarrow ct + ct = k$. Hence $l$ has equation $x + t^2 y = 2ct$.

(ii)

$$x = a\sec\theta \Rightarrow \frac{dx}{d\theta} = a\sec\theta\tan\theta$$

$$y = b\tan\theta \Rightarrow \frac{dy}{d\theta} = b\sec^2\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{b\sec\theta}{a\tan\theta}$$

Hence tangent $l$ has gradient $\frac{b\sec\theta}{a\tan\theta}$ and equation $x\,b\sec\theta - y\,a\tan\theta = k$, $k$ constant, where $Q(a\sec\theta,\ b\tan\theta)$ lies on $l$
$\Rightarrow k = ab\sec^2\theta - ab\tan^2\theta = ab$. Hence $l$ has equation $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

(iii) Comparing the two forms of the equation of line $l$, the coefficients must be in proportion. Hence

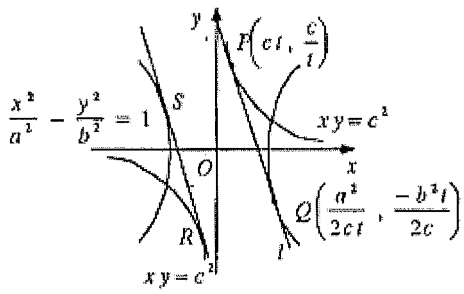
$$\frac{\left(\frac{\sec\theta}{a}\right)}{1} = \frac{\left(\frac{-\tan\theta}{b}\right)}{t^2} = \frac{1}{2ct} \qquad \therefore \frac{\sec\theta}{a} = \frac{-\tan\theta}{bt^2} = \frac{1}{2ct}.$$

(iv)

$$\left.\begin{aligned} &Q(a\sec\theta,\ b\tan\theta) \\ &\equiv Q\left(\frac{a^2}{2ct},\ \frac{-b^2t}{2c}\right) \end{aligned}\right\}, \qquad \left.\begin{aligned} &\sec^2\theta - \tan^2\theta = 1 \\ &\left(\frac{a}{2ct}\right)^2 - \left(\frac{-bt}{2c}\right)^2 = 1 \end{aligned}\right\} \Rightarrow \qquad \begin{aligned} a^2 - b^2t^4 &= 4c^2t^2 \\ b^2t^4 + 4c^2t^2 - a^2 &= 0 \end{aligned}$$

This quadratic in $t^2$ has discriminant $\Delta = 16c^4 + 4a^2b^2 > 0$, and hence has two real roots which are opposite in sign (since their product is negative). But $t^2 > 0$, hence there is exactly one solution for $t^2$, and two solutions for $t$ which are opposites of each other. Each such value of $t$ gives a common tangent $l$ to the two hyperbolas.

(v)




$$R\left(-ct, \frac{-c}{t}\right), \quad S\left(\frac{-a^2}{2ct}, \frac{b^2t}{2c}\right)$$

(vi)

$O$ is the common midpoint of diagonals $PR$ and $QS$. Hence $PQRS$ is a parallelogram.

$$\textit{gradient } PR = \frac{2c}{t} \div 2ct = \frac{1}{t^2}$$

$$\textit{gradient } QS = \frac{b^2t}{c} \div \frac{-a^2}{ct} = \frac{b^2}{a^2}(-t^2)$$

$$\therefore \textit{gradient } PR \cdot \textit{gradient } QS = -\frac{b^2}{a^2}$$

Hence if $PQRS$ is a rhombus, $PR \perp QS$ and $\textit{gradient } PR \cdot \textit{gradient } QS = -1 \Rightarrow b^2 = a^2$.

Then $t$ satisfies $a^2t^4 + 4c^2t^2 - a^2 = 0$

$$t^4 + \frac{4c^2}{a^2}t^2 = 1$$

$$\left(t^2 + \frac{2c^2}{a^2}\right)^2 = 1 + \frac{4c^4}{a^4} < \left(1 + \frac{2c^2}{a^2}\right)^2$$

Hence $t^2 < 1$

(vii) If $PQRS$ is a square, then $PQRS$ is a rhombus with $R\hat{P}Q = 45°$. Then

$$\left.\begin{array}{l} \text{gradient } PR = \frac{1}{t^2} \\ \text{gradient } PQ = \frac{-1}{t^2} \end{array}\right\} \Rightarrow 1 = \left|\frac{\left(\frac{2}{t^2}\right)}{1+\left(\frac{1}{t^2}\right)\left(\frac{-1}{t^2}\right)}\right| = \frac{-2t^2}{t^4-1} \quad (\text{since } t^2 < 1 \text{ for } PQRS \text{ a rhombus})$$

Hence $t^4 + 2t^2 - 1 = 0$. But for $PQRS$ a rhombus, $t$ satisfies $t^4 + \frac{4c^2}{a^2}t^2 - 1 = 0$.

By subtraction, $\left(\frac{4c^2}{a^2} - 2\right)t^2 = 0$. But $t^2 \neq 0$. Hence $2c^2 = a^2$.

Hence if $PQRS$ is a square (and hence a rhombus), then $b^2 = a^2$, and the two hyperbolas have equations $x^2 - y^2 = a^2$ and $xy = c^2$, where $2c^2 = a^2$.

This relationship between $c^2$ and $a^2$ means that the rectangular hyperbola $x^2 - y^2 = a^2$ rotated anticlockwise through $45°$ becomes the rectangular hyperbola $xy = c^2$.

## NEAP 2000 Q4

(c) (i) The focus is $S(ae, 0)$. The directrix is $x = \frac{a}{e}$. ✓

(ii)
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-b^2x}{a^2y}$$

So the slope of the normal at $P(x_1, y_1)$ is $\frac{a^2y_1}{b^2x_1}$ ✓

The equation of the normal is $y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1)$

$$b^2x_1(y - y_1) = a^2y_1(x - x_1) \quad ✓$$
$$a^2xy_1 - b^2x_1y = a^2x_1y_1 - b^2x_1y_1$$
$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2 \quad ✓ \text{(divide both sides by } x_1y_1)$$

(iii) $Q$ is the point $\left(\frac{a^2-b^2}{a^2}x_1, 0\right)$ or $(e^2x_1, 0)$ (from $a^2(1-e^2) = b^2$).

$$\text{so, } QS = |e^2x_1 - ae|$$
$$= e|ex_1 - a| \quad ✓$$

Also, $PM = \frac{a}{e} - x_1$

$$\text{so } e^2PM = e^2\left(\frac{a}{e} - x_1\right)$$
$$= e(a - ex_1) \quad ✓$$

Hence $QS = e^2PM$. ✓

**SGHS 2002 Q4**

2. $x^2 - 16y^2 = 16$

$\frac{x^2}{16} - y^2 = 1$  $(2, -4)$  $a^2 = 16,\ b^2 = 1$

$\frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1$

$\frac{2x}{16} + \frac{4y}{1} = 1$

$2x + 64y = 16$

$x + 32y = 8$ (2)

3. Foci at $(\pm 5, 0)$, $e = \frac{5}{4}$

$\therefore ae = 5$

$a \times \frac{5}{4} = 5 \quad \therefore a = 4$

and $b^2 = a^2(e^2 - 1)$

$= 16\left(\frac{25}{16} - 1\right)$

$b^2 = 9$ (3)

i.e. $\frac{x^2}{16} - \frac{y^2}{9} = 1$