NAME :

Centre of Excellence in Mathematics
S201 / 414 GARDENERS RD. ROSEBERY 2018
www.cemtuition.com.au

MOBILE 0412 880 475

PHONE 96663331

YEAR 12 – EXT.2 MATHS

REVIEW TOPIC (SP2)
INTEGRATION I

HSC 03

(1)

(a) Evaluate $\int_0^1 \frac{e^x}{\left(1+e^x\right)^2}\,dx$. **2**

$$\boxed{\frac{e-1}{2(1+e)}}$$

(c) By completing the square and using the table of standard integrals, find **2**

$$\int \frac{dx}{\sqrt{x^2-2x+5}}.$$

$$\boxed{\ln\left|x-1+\sqrt{x^2-2x+5}\right|+c}$$

(d) (i) Find the real numbers a and b such that **2**

$$\frac{5x^2-3x+13}{(x-1)(x^2+4)} \equiv \frac{a}{x-1}+\frac{bx-1}{x^2+4}.$$

$$\boxed{a=3, b=2}$$

(ii) Find $\int \frac{5x^2-3x+13}{(x-1)(x^2+4)}dx$. **2**

$$\boxed{3\ln|x-1|+\ln\left(x^2+4\right)-\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+c}$$

HSC 02

(1)

(b) By completing the square, find $\int \frac{dx}{x^2+2x+2}$. **2**

$\boxed{\tan^{-1}(x+1)+c}$

HSC 01

(1) (b) By completing the squares, find $\int \frac{dx}{\sqrt{x^2-4x+1}}$. **2**

$\boxed{\ln\left[(x-2)+\sqrt{x^2-4x+1}\right]+c}$

(d) Use the substitution $u = \sqrt{x-1}$ to evaluate **4**

$$\int_2^3 \frac{1+x}{\sqrt{x-1}}\,dx.$$

$$\boxed{\frac{2}{3}\left(8\sqrt{2}-7\right)}$$

(e) (i) Find the real numbers *a* and *b* such that **2**

$$\frac{5x^2-3x+1}{\left(x^2+1\right)\left(x-2\right)} \equiv \frac{ax+1}{x^2+1}+\frac{b}{x-2}.$$

$$\boxed{a=2, b=3}$$

(ii) Find $\int \frac{5x^2-3x+1}{\left(x^2+1\right)\left(x-2\right)}\,dx.$

$$\boxed{\ln\left|x^2+1\right|+\tan^{-1}x+3\ln\left|x-2\right|+c}$$

HSC 2000

(1) (b) Use the completion of squares to find **2**

$$\int \frac{4}{x^2+6x+10}\,dx$$

$$\boxed{4\tan^{-1}(x+3)+c}$$

(c) (i) Find the real numbers a, b and c such that **4**

$$\frac{9}{x^2(3-x)} \equiv \frac{ax+b}{x^2} + \frac{c}{3-x}.$$

$$\boxed{a=1, b=3, c=1}$$

(ii) Find $\int \frac{9}{x^2(3-x)}\,dx.$

$$\boxed{\ln\left|\frac{x}{3-x}\right| - \frac{3}{x} + c}$$

HSC '99 **Marks**

(1)(a) Evaluate $\int_0^1 xe^{-x^2}\,dx$. **2**

$\boxed{\frac{1}{2}\left[1-\frac{1}{e}\right]}$

(b) $\int \frac{e^x dx}{\sqrt{1-e^{2x}}}$ **2**

$\boxed{\sin^{-1}(e^x)+c}$

HSC 98

(1) (a) Evaluate $\displaystyle\int_0^3 \frac{6}{9+x^2}\,dx$

$\boxed{\frac{\pi}{2}}$

(d) Using the substitution $u^2 = 4 - x^2$, or otherwise, evaluate

$$\int_0^2 x^3\sqrt{4-x^2}\,dx$$

$\boxed{4\frac{4}{15}}$

(e) (i) Find the remainder when x^2+6 is divided by x^2+x-6.

$\boxed{-x+12}$

(ii) Hence, find $\displaystyle\int \frac{x^2+6}{x^2+x-6}\,dx$

$\boxed{x-3\ln|x+3|+2\ln|x-2|+c}$

HSC '97

(1) (a) Evaluate $\int_0^5 \frac{2}{\sqrt{x+4}}\,dx$ **2**

$\boxed{4}$

(c) Find $\int \frac{1}{x^2+2x+3}\,dx$ **2**

$\boxed{\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+c}$

(d) Find $\displaystyle\int \frac{4t-6}{(t+1)(2t^2+3)}\,dt$ **4**

$$\boxed{\ln\left[\frac{2t^2+3}{(t+1)^2}\right]+c}$$

HSC 96

(1) (a) Evaluate $\displaystyle\int_1^3 \frac{4}{(2+x)^2}\,dx$

$$\boxed{\frac{8}{15}}$$

(c) Find $\displaystyle\int \frac{5t^2+3}{t\left(t^2+1\right)}\,dt$

$\boxed{3\ln|t|+\ln\left(t^2+1\right)+c}$

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YEAR 12 – EXT.2 MATHS

REVIEW TOPIC (SP2)
INTEGRATION I

IMPRESSIVE

HSC 03

(1)

(a) Evaluate $\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx$. 2

$u = e^x$

$\frac{du}{dx} = e^x$ ✓

$= \int_1^e \frac{du}{(1+u)^2}$

$= \left[\frac{(1+u)^{-1}}{-1}\right]_1^e$

$= \left[\frac{-1}{1+u}\right]_1^e$ ✓

$= \frac{-1}{1+e} + \frac{1}{1+1}$

$= \frac{-1}{1+e} + \frac{1}{2}$ ✓

$= \frac{-2+1+e}{2(1+e)} = \frac{e-1}{2(1+e)}$

$\boxed{\frac{e-1}{2(1+e)}}$

(c) By completing the square and using the table of standard integrals, find 2

$$\int \frac{dx}{\sqrt{x^2-2x+5}}.$$

$= \int \frac{dx}{\sqrt{(x-1)^2+4}}$ ✓

$= \log_e\left|(x-1) + \sqrt{x^2-2x+5}\right| + c.$ ✓

$\boxed{\ln\left|x-1+\sqrt{x^2-2x+5}\right|+c}$

(d) (i) Find the real numbers a and b such that **2**

$$\frac{5x^2-3x+13}{(x-1)(x^2+4)} \equiv \frac{a}{x-1}+\frac{bx-1}{x^2+4}.$$

$$a(x^2+4)+(bx-1)(x-1) = 5x^2-3x+13$$

$x=1$:

$$5a = 5-3+13$$
$$= 15 \quad ✓$$
$$a = 3.$$

$x=-1$.

$$5a+(-b-1)(-2) = 5+3+13 \quad ✓$$
$$2b+2 = 6$$
$$2b = 4 \quad ✓$$
$$b = 2.$$
$$\therefore\ a=3,\ b=2. \quad ✓$$

$\boxed{a=3, b=2}$

(ii) Find $\displaystyle\int \frac{5x^2-3x+13}{(x-1)(x^2+4)}dx$. **2**

$$= \int \frac{3}{x-1}+\frac{2x-1}{x^2+4}\,dx$$

$$= 3\log_e|x-1| + \log_e|x^2+4| - \tfrac{1}{2}\tan^{-1}\tfrac{x}{2} \ +c. \quad ✓$$

$\boxed{3\ln|x-1|+\ln(x^2+4)-\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+c}$

HSC 02

(1)

(b) By completing the square, find $\displaystyle\int \frac{dx}{x^2+2x+2}$. **2**

$$= \int \frac{dx}{(x+1)^2+1} \quad ✓$$

$$= \tan^{-1}(x+1)+c.$$

$\boxed{\tan^{-1}(x+1)+c}$

HSC 01

(1) (b) By completing the squares, find $\displaystyle\int \frac{dx}{\sqrt{x^2-4x+1}}$. **2**

$$= \int \frac{dx}{\sqrt{(x-2)^2-3}} \quad ✓$$

$$= \log_e\left|x-2+\sqrt{x^2-4x+1}\right| +c \quad ✓$$

$\boxed{\ln\left[(x-2)+\sqrt{x^2-4x+1}\right]+c}$

(d) Use the substitution $u=\sqrt{x-1}$ to evaluate **4**

$$\int_2^3 \frac{1+x}{\sqrt{x-1}}\,dx.$$

$u = \sqrt{x-1}$

$u^2 = x-1$

$x = u^2+1$

$\frac{dx}{du} = 2u.$ ✓

$= \int_1^{\sqrt{2}} \frac{2u}{u}(u^2+2)\,du$

$= 2\int_1^{\sqrt{2}} u^2+2\,du$ ✓

$= 2\left[\frac{u^3}{3}+2u\right]_1^{\sqrt{2}}$

$= 2\left[\frac{2\sqrt{2}}{3}+2\sqrt{2}-\frac{1}{3}-2\right]$

$= 2\left[\frac{8\sqrt{2}}{3}-\frac{7}{3}\right]$ ✓

$= \frac{2}{3}(8\sqrt{2}-7)$ ✓

$\boxed{\frac{2}{3}(8\sqrt{2}-7)}$

(e) (i) Find the real numbers a and b such that **2**

$$\frac{5x^2-3x+1}{(x^2+1)(x-2)} \equiv \frac{ax+1}{x^2+1}+\frac{b}{x-2}.$$

$b(x^2+1)+(ax+1)(x-2) = 5x^2-3x+1.$ ✓

$x=2:\quad 5b = 20-6+1$

$= 15$ ✓

$b = 3.$

$x=1\quad 2b+(-1)(a+1) = 5-3+1$

$-(a+1) = -3$

$a+1 = 3$ ✓

$a = 2.$

$\boxed{a=2, b=3}$

(ii) Find $\int \frac{5x^2-3x+1}{(x^2+1)(x-2)}\,dx.$

$= \int \frac{2x+1}{x^2+1} + \frac{3}{x-2}\,dx$ ✓

$= \log_e|x^2+1| + \tan^{-1}x + 3\log_e|x-2| + C.$ ✓

$\boxed{\ln|x^2+1|+\tan^{-1}x+3\ln|x-2|+c}$

HSC 2000

(1) (b) Use the completion of squares to find 2

$$\int \frac{4}{x^2+6x+10}\,dx$$

$$= \int \frac{4}{(x+3)^2+1} \quad ✓$$

$$= 4\tan^{-1}(x+3) + c \quad ✓$$

$\boxed{4\tan^{-1}(x+3)+c}$

(c) (i) Find the real numbers a, b and c such that 4

$$\frac{9}{x^2(3-x)} \equiv \frac{ax+b}{x^2} + \frac{c}{3-x}.$$

$$cx^2 + (3-x)(ax+b) = 9$$

$x = 0$ $\quad 3b = 9$ ✓

$\quad b = 3.$

$x = 3$ $\quad 9c = 9$

$\quad c = 1$ ✓

$x = 1$ $\quad c + 2(a+b) = 9$

$\quad a + b = 4$ ✓

$\quad a = 1$

$\therefore a = 1$ ✓

$b = 3$

$c = 1.$ ✓

$\boxed{a=1, b=3, c=1}$

(ii) Find $\int \frac{9}{x^2(3-x)}\,dx.$

$$= \int \frac{x+3}{x^2} + \frac{1}{3-x}\,dx. \quad ✓$$

$$= \int \frac{1}{x} + \frac{3}{x^2} + \frac{1}{3-x}\,dx.$$

$$= \log_e|x| - \log_e|3-x| + 3\frac{1}{x}(-1).$$

$$= \log_e\left|\frac{x}{3-x}\right| - \frac{3}{x} + c \quad ✓$$

$\boxed{\ln\left|\frac{x}{3-x}\right| - \frac{3}{x} + c}$

HSC '99 **Marks**

(1)(a) Evaluate $\int_0^1 xe^{-x^2}\,dx.$ 2

$u = x^2$

$\frac{du}{dx} = 2x$ ✓

$$= \frac{1}{2}\int_0^1 e^{-u}\,du$$

$$= \frac{1}{2}\left[-e^{-u}\right]_0^1$$

$$= \frac{1}{2}\left[-e^{-1} + e^0\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{e}\right] \quad ✓$$

$\boxed{\frac{1}{2}\left[1-\frac{1}{e}\right]}$

(b) $\int \frac{e^x dx}{\sqrt{1-e^{2x}}}$ 2

Let $u = e^x$

$\frac{du}{dx} = e^x.$ ✓

$$= \int \frac{du}{\sqrt{1-u^2}}$$

$$= \sin^{-1} u + c \quad ✓$$

$$= \sin^{-1}(e^x) + c.$$

$\boxed{\sin^{-1}(e^x)+c}$

HSC 98

(1) (a) Evaluate $\int_0^3 \frac{6}{9+x^2}\,dx$

$$= \frac{6}{3}\left[\tan^{-1}\frac{x}{3}\right]_0^3$$

$$= 2\left[\tan^{-1} 1\right] \checkmark$$

$$= \frac{2\pi}{4}$$

$$= \frac{\pi}{2} \checkmark$$

$\boxed{\frac{\pi}{2}}$

(d) Using the substitution $u^2 = 4-x^2$, or otherwise, evaluate

$$\int_0^2 x^3\sqrt{4-x^2}\,dx$$

$$= -\int_2^0 (4-u^2)u^2\,du$$

$$= \int_0^2 4u^2 - u^4\,du$$

$$= \left[\frac{4u^3}{3} - \frac{u^5}{5}\right]_0^2 \checkmark$$

$$= \frac{32}{3} - \frac{32}{5}$$

$$= \frac{64}{15} \checkmark$$

$$= 4\frac{4}{15}$$

$u = \sqrt{4-x^2}$ ⇒ beat to use implicit diff^n. of $u^2 = 4-x^2$

$$\frac{du}{dx} = \frac{1}{2}(4-x^2)^{-\frac{1}{2}} \times -2x \checkmark$$

$$= -\frac{x}{\sqrt{4-x^2}}$$

$$\frac{dx}{du} = -\frac{\sqrt{4-x^2}}{x} \checkmark$$

$$x^2 = 4-u^2$$

$\boxed{4\frac{4}{15}}$

(e) (i) Find the remainder when x^2+6 is divided by x^2+x-6.

$$\begin{array}{r} 1 \\ x^2+x-6\,\overline{)\,x^2 + 0 + 6} \\ x^2 + x - 6 \\ \hline -x + 12 \end{array} \checkmark$$

$$R = -x+12 \checkmark$$

$\boxed{-x+12}$

(ii) Hence, find $\int \frac{x^2+6}{x^2+x-6}\,dx$

$$= \int 1 - \frac{x-12}{x^2+x-6}\,dx \checkmark$$

$$= x - \int \frac{x-12}{(x+3)(x-2)}\,dx$$

$$\frac{A}{x+3} + \frac{B}{x-2} = \frac{x-12}{(x+3)(x-2)}$$

$$A(x-2) + B(x+3) = x-12 \checkmark$$

$x = 2$

$$5B = -10$$

$$B = -2$$

$x = -3$

$$-5A = -15$$

$$A = 3 \checkmark$$

$$\therefore \int \frac{x^2+6}{x^2+x-6} = x - \int \frac{3}{x+3} - \frac{2}{x-2}\,dx$$

$$= x - 3\log_e(x+3) + 2 \checkmark \log_e(x-2) + C$$

$\boxed{x-3\ln|x+3|+2\ln|x-2|+c}$

HSC '97

(1) (a) Evaluate $\int_0^5 \frac{2}{\sqrt{x+4}}\,dx$ **2**

$= 2\int_0^5 (x+4)^{-\frac{1}{2}}\,dx$ ✓

$= 2\left[2(x+4)^{\frac{1}{2}}\right]_0^5$

$= 4(3-2)$ ✓

$= 4.$

$\boxed{4}$

(c) Find $\int \frac{1}{x^2+2x+3}\,dx$ **2**

$= \int \frac{1}{(x+1)^2+2}\,dx$ ✓

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C$ ✓

$\boxed{\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+c}$

(d) Find $\int \frac{4t-6}{(t+1)(2t^2+3)}\,dt$ $\quad \frac{A}{t+1} + \frac{Bt+C}{2t^2+3}$ **4**

$A(2t^2+3) + (Bt+C)(t+1) = 4t-6.$

$t=-1$: $5A = -10$ ✓

$A = -2.$

$t=0$ $\quad 3A + C = -6.$ ✓

$C = 0$

$t=1$ $\quad A(5) + B(2) = 4-6$

$-10 + 2B = -2.$ ✓

$B = 4.$

$\therefore \int \frac{4t-6}{(t+1)(2t^2+3)}\,dt = \int \frac{4t}{2t^2+3} - \frac{2}{t+1}\,dt.$

$= \log_e|2t^2+3| - 2\log_e|t+1|$ ✓

$= \log_e\left|\frac{2t^2+3}{(t+1)^2}\right| + C.$

$\boxed{\ln\left[\frac{2t^2+3}{(t+1)^2}\right]+c}$

HSC 96

(1) (a) Evaluate $\int_1^3 \frac{4}{(2+x)^2}\,dx$

$= 4\int_1^3 (2+x)^{-2}\,dx$ ✓

$= 4\left[\frac{(2+x)^{-1}}{-1}\right]_1^3$

$= -4\left(\frac{1}{2+x}\right]_1^3$

$= -4\left(\frac{1}{5} - \frac{1}{3}\right)$

$= \frac{8}{15}$ ✓

$\boxed{\frac{8}{15}}$

(c) Find $\int \frac{5t^2+3}{t\left(t^2+1\right)}dt$

$$\frac{A}{t} + \frac{Bt+C}{t^2+1} = \frac{5t^2+3}{t(t^2+1)}$$

$$A(t^2+1) + t(Bt+C) = 5t^2+3.$$ ✓

$t=0$: $A = 3.$

$t=1$ $2A + B + C = 8.$

$B + C = 2.$ ✓

$t=-1$ $2A - (-B+C) = 8.$

$B - C = 2.$

$2B = 4$ ✓

$B = 2.$ ✓

$C = 0.$ ✓

$$\therefore \int \frac{5t^2+3}{t(t^2+1)}\,dt = \int \frac{3}{t} + \frac{2t}{t^2+1}\,dt$$

$$= 3\log_e|t| + \log_e|t^2+1| + C$$ ✓

$\boxed{3\ln|t|+\ln\left(t^2+1\right)+c}$