NAME :

Centre of Excellence in Mathematics
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YEAR 12 – MATHS EXT.2

REVIEW TOPIC (PAPER 1): INTEGRATION OF TRIG FNS

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CSSA 2000 Q2

(b) Find

(i) $\int \left(1+\tan^2 x\right) e^{\tan x} \, dx$

(iii) $\int \cos^3 x \, dx$

(d) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin x}\,dx$, giving your answer in simplest exact form. 3

CSSA 2001 Q2

(ii) Find $\int \frac{x^2+x+1}{x(x^2+1)}\,dx$. 2

(c) (i) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x+\sin x}\,dx$. 2

(ii) Hence use the substitution $u = \frac{\pi}{2} - x$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{x}{1 + \cos x + \sin x}\, dx$. **2**

HEFFERNAN 2002 Q1

(d) Use the substitution $t = \tan\frac{\theta}{2}$ to find $\int \frac{\sin\theta}{1+\cos\theta}\,d\theta$ 4

INDEPENDENT 2001 Q2

(b) Find $\int \frac{x^2}{(1-x^2)^{\frac{3}{2}}}\,dx$ using the substitution $x = \sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. 3

(c) Evaluate $\int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x}\,dx$ using the substitution $t=\tan\frac{x}{2}$. 3

JAMES RUSE 2000 Q1

(i) $\int e^x \sin e^x \, dx$

(iii) $\int x \cos 2x \, dx$

Challenge question:

Use the substitution $u = t - \frac{1}{t}$ to show that $\int \frac{1+t^2}{1+t^4}\,dt = \frac{\sqrt{2}}{2}\tan^{-1}\frac{\sqrt{2}\left(t^2-1\right)}{2t} + c$

SOLUTIONS

CSSA 2000 Q2

(b)(i) $\int (1+\tan^2 x)\ e^{\tan x}\,dx = \int \sec^2 x\ e^{\tan x}\,dx$

$= e^{\tan x} + c$

(iii) $\int \cos^3 x\ dx = \int \cos x(1-\sin^2 x)\ dx$

$= \sin x - \frac{1}{3}\sin^3 x + c$

(d)

$t = \tan\frac{x}{2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$dt = \frac{1}{2}(1+t^2)\,dx$

$\frac{2}{1+t^2}\,dt = dx$

$x = 0 \Rightarrow t = 0$

$x = \frac{\pi}{3} \Rightarrow t = \frac{1}{\sqrt{3}}$

$1-\sin x = 1-\frac{2t}{1+t^2}$

$= \frac{1+t^2-2t}{1+t^2}$

$= \frac{(1-t)^2}{1+t^2}$

$\frac{1}{1-\sin x} = \frac{1+t^2}{(1-t)^2}$

$\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin x}\,dx$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1+t^2}{(1-t)^2}\,\frac{2}{1+t^2}\,dt$

$= 2\int_0^{\frac{1}{\sqrt{3}}} (1-t)^{-2}\ dt$

$= 2\left[(1-t)^{-1}\right]_0^{\frac{1}{\sqrt{3}}}$

$2\left\{\frac{1}{1-\frac{1}{\sqrt{3}}} - 1\right\}$

$= 2\left\{\frac{\sqrt{3}(\sqrt{3}+1)}{3-1} - 1\right\}$

$= 3+\sqrt{3} - 2$

$= 1+\sqrt{3}$

CSSA 2001 Q2

(b)

(ii) $\int \frac{x^2+x+1}{x(x^2+1)}\,dx = \int \frac{(x^2+1)+x}{x(x^2+1)}\,dx = \int \left(\frac{1}{x} + \frac{1}{x^2+1}\right)dx = \ln|x| + \tan^{-1}x + c$

(c)

(i)

$t = \tan\frac{x}{2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\ dx$

$2dt = (1+\tan^2\frac{x}{2})\,dx$

$dx = \frac{2}{1+t^2}\ dt$

$1+\cos x+\sin x = 1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2} = \frac{2+2t}{1+t^2}$

$x = 0 \Rightarrow t = 0$

$x = \frac{\pi}{2} \Rightarrow t = 1$

$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x+\sin x}\,dx = \int_0^1 \frac{1+t^2}{2(1+t)}\cdot\frac{2}{1+t^2}\ dt$

$= \int_0^1 \frac{1}{1+t}\ dt$

$= \left[\ \ln|1+t|\ \right]_0^1$

$= \ln 2$

(ii) Let $I = \int_0^{\frac{\pi}{2}} \frac{x}{1+\cos x + \sin x}\, dx$

$u = \frac{\pi}{2} - x$

$du = -dx$

$x = 0 \Rightarrow u = \frac{\pi}{2}$

$x = \frac{\pi}{2} \Rightarrow u = 0$

$x = \frac{\pi}{2} - u$

$\cos x + \sin x = \sin u + \cos u$

$$I = \int_{\frac{\pi}{2}}^{0} \frac{\frac{\pi}{2} - u}{1+\sin u + \cos u} \cdot -du = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2} - u}{1+\cos u + \sin u}\, du$$

$$\therefore I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos u+\sin u}\, du - \int_0^{\frac{\pi}{2}} \frac{u}{1+\cos u + \sin u}\, du$$

$$I = \frac{\pi}{2}\ln 2 - I$$

$$2I = \frac{\pi}{2}\ln 2$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{x}{1+\cos x + \sin x}\, dx = \frac{\pi}{4}\ln 2$$

HEFFERNAN 2002 Q1

(d) We have $\int \frac{\sin\theta}{1+\cos\theta}\, d\theta$.

Using the substitution $t = \tan\frac{\theta}{2}$

We have $\sin\theta = \frac{2t}{1+t^2}$

$\cos\theta = \frac{1-t^2}{1+t^2}$ **(1 mark)**

$$\text{So, } \frac{\sin\theta}{1+\cos\theta} = \frac{2t}{1+t^2} \div \left(1 + \frac{1-t^2}{1+t^2}\right)$$
$$= \frac{2t}{1+t^2} \div \frac{1+t^2+1-t^2}{1+t^2}$$
$$= \frac{2t}{1+t^2} \div \frac{2}{1+t^2}$$
$$= \frac{2t}{1+t^2} \times \frac{1+t^2}{2}$$
$$= t$$

So, $\int \frac{\sin\theta}{1+\cos\theta}\, d\theta = \int t \cdot \frac{d\theta}{dt}\, dt$ where $\frac{d\theta}{dt} = \frac{2}{1+t^2}$ **(1 mark)**

$$= \int t \cdot \frac{2}{1+t^2}\, dt$$

$$= \int \frac{2t}{1+t^2}\, dt \quad \textbf{(1 mark)}$$

$$= \ln\left(1+t^2\right) + c$$

$$= \ln\left(1 + \tan^2\frac{\theta}{2}\right) + c \quad \textbf{(1 mark)}$$

INDEPENDENT 2001 Q2

(b)

$x = \sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = \cos\theta \, d\theta$

$(1-x^2)^{\frac{3}{2}} = (\cos^2\theta)^{\frac{3}{2}} = \cos^3\theta$

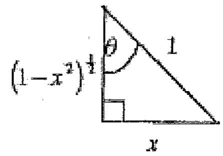
$$\frac{x^2}{(1-x^2)^{\frac{3}{2}}} = \frac{\sin^2\theta}{\cos^3\theta} = \frac{\tan^2\theta}{\cos\theta}$$

$$\int \frac{x^2}{(1-x^2)^{\frac{3}{2}}} \, dx = \int \frac{\tan^2\theta}{\cos\theta} \cos\theta \, d\theta$$

$$\int \tan^2\theta \, d\theta = \int (\sec^2\theta - 1) \, d\theta$$

$$= \tan\theta - \theta + c$$

$$= \frac{x}{(1-x^2)^{\frac{1}{2}}} - \sin^{-1} x + c$$



(c)

$t = \tan\frac{x}{2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2} \, dx \qquad x = 0 \Rightarrow t = 0$

$2\,dt = (1+t^2)\,dx \qquad x = \frac{2\pi}{3} \Rightarrow t = \sqrt{3}$

$$dx = \frac{2}{1+t^2} \, dt$$

$$5 + 4\cos x = 5 + \frac{4(1-t^2)}{1+t^2} = \frac{9+t^2}{1+t^2}$$

$$\int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x} \, dx = \int_0^{\sqrt{3}} \frac{1+t^2}{9+t^2} \frac{2}{1+t^2} \, dt$$

$$= \frac{2}{3}\int_0^{\sqrt{3}} \frac{3}{9+t^2} \, dt$$

$$= \frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^{\sqrt{3}}$$

$$= \frac{2}{3}\left(\frac{\pi}{6} - 0\right)$$

$$= \frac{\pi}{9}$$

JAMES RUSE 2000 Q1

1. (a)(i) $\int e^x \sin e^x \, dx$

$= -\cos(e^x) + c.$

(iii). $\int x \cos 2x \, dx$

$u = x \qquad v = \frac{\sin 2x}{2}$

$u' = 1 \qquad v' = \cos 2x$

$I = x\left(\frac{x \sin 2x}{2}\right) - \frac{1}{2}\int \sin 2x \, dx$

$= \left(\frac{x \sin 2x}{2}\right) + \frac{1}{4}\cos 2x + c.$

Challenge question:

$$\frac{du}{dt} = 1 + \frac{1}{t^2} \Rightarrow du = \frac{t^2+1}{t^2}$$

$$\int \frac{1+t^2}{t^2(\frac{1}{t^2}+t^2)} dt = \int \frac{du}{u^2+2} \quad \text{Note: } u^2 = t^2 + \frac{1}{t^2} - 2$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{\sqrt{2}}{u} + c = \frac{\sqrt{2}}{2} \tan^{-1} \frac{\sqrt{2}}{\frac{t^2-1}{t}} + c = \frac{\sqrt{2}}{2} \tan^{-1} \frac{\sqrt{2}\,t}{t^2-1} + c$$