NAME :

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YEAR 12 – MATHS EXT.2

REVIEW TOPIC (PAPER 1): MATHEMATICAL INDUCTION

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CSSA 2001

7.

(b) A sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 2$, $u_2 = 12$ and
$u_n = 6u_{n-1} - 8u_{n-2}$ for $n \geq 3$.

(i) Use Mathematical Induction to show that $u_n = 4^n - 2^n$ for $n \geq 1$. **4**

RC 2002

8.

b) A sequence T_n is such that $T_1 = 4$ and $T_2 = 8$ and $T_{n+2} = 6T_{n+1} - 5T_n$

Prove by mathematical induction that $T_n = 5^{n-1} + 3$ 6

SGHS 2002

8.

1. Use mathematical induction to prove that $x^{2n} - y^{2n}$ is divisible by $(x+y)$ for $n \geq 1$ (n an integer) [4]

SBHS 2001

7.

(c) Let $T(m,y)=\frac{{}^mC_0}{y}-\frac{{}^mC_1}{y+1}+\frac{{}^mC_2}{y+2}-\;\ldots\ldots\;+(-1)^m\frac{{}^mC_m}{y+m}$. 7

(i) If it is given that $T(k,x)=\frac{k!}{x(x+1)(x+2)\ldots(x+k)}$ for a particular value of k, show that

$$T(k,x)-T(k,x+1)=T(k+1,x)$$

(ii) Hence prove, using Mathematical Induction or otherwise, that for $n\geq 1$

$$T(n,x)=\frac{{}^nC_0}{x}-\frac{{}^nC_1}{x+1}+\frac{{}^nC_2}{x+2}-\;\ldots\;+(-1)^n\frac{{}^nC_n}{x+n}=\frac{n!}{x(x+1)(x+2)\ldots(x+n)}$$

(NOTE: you may use without proof the result ${}^{m+1}C_r={}^mC_r+{}^mC_{r-1}$)

(iii) Hence prove that

$$\frac{{}^nC_0}{1}-\frac{{}^nC_1}{3}+\frac{{}^nC_2}{5}-\ \ldots\ldots\ +(-1)^n\frac{{}^nC_n}{2n+1}=\frac{2^n n!}{1.3.5\ldots..(2n+1)}$$

CSSA 2000

7.

(b) (i) If $I_n = \int_0^1 \left(x^2 - 1\right)^n \, dx$, $n = 0, 1, 2, \ldots$, show that 8

$I_n = \frac{-2n}{2n+1} I_{n-1}$, $n = 1, 2, 3, \ldots$

(ii) Hence use the method of Mathematical Induction to show that $I_n = \dfrac{(-1)^n\, 2^{2n}\, (n!)^2}{(2n+1)\,!}$

for all positive integers n.

NEAP 2000

8.

(b) (i) Use the principle of mathematical induction to prove that $3^n > n^3$ for all integers $n \geq 4$. 5

(ii) Hence or otherwise show that $\sqrt[3]{3} > \sqrt[n]{n}$ for all integers $n \geq 4$.

JAMES RUSE 2000

5.

(b) Prove by induction that $u_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$ and $u_1 = 1$ and $u_2 = 1$

given the recurrence relation $u_{n+2} = u_n + u_{n+1}$

ST IGNATIUS 2002

8.

b) Prove by Mathematical Induction that: 5

$$\sum_{r=1}^{n} \sin((2r-1)\theta) = \frac{\sin^2 n\theta}{\sin\theta}, \text{ where } n \text{ is a positive integer.}$$

SOLUTIONS

CSSA 2001 Q7

Answer

Let $A(n)$ be the statement : $u_n = 4^n - 2^n$, $n = 1, 2, 3, \ldots$

(i) *Consider* $A(1), A(2)$: $4^1 - 2^1 = 2 = u_1$, $4^2 - 2^2 = 12 = u_2$ $\therefore A(1), A(2)$ are both true.

If $A(n)$ *is true for positive integers* $n \le k$ (k some positive integer, $k \ge 2$) , *then*

$$u_n = 4^n - 2^n, \quad n = 1, 2, 3, \ldots, k \quad **$$

Consider $A(k+1)$, $k \ge 2$: $u_{k+1} = 6u_k - 8u_{k-1}$

$$\therefore u_{k+1} = 6\left(4^k - 2^k\right) - 8\left(4^{k-1} - 2^{k-1}\right) \qquad \textit{if } A(n) \textit{ is true for } n \le k, \textit{ using } **$$
$$= 6.4^k - 6.2^k - 2.4.4^{k-1} + 4.2.2^{k-1}$$
$$= (6-2)4^k - (6-4)2^k$$
$$= 4^{k+1} - 2^{k+1}$$

Hence if $A(n)$ is true for $n \le k$ (k some integer, $k \ge 2$) , then $A(k+1)$ is true. But $A(1)$ and $A(2)$ are true, and hence $A(3)$ is true; then $A(n)$ is true for $n = 1, 2, 3$ and hence $A(4)$ is true and so on. Hence by mathematical induction, $A(n)$ is true for all positive integers $n \ge 1$.

(ii) $S_n = \sum_{k=1}^{n} u_k = \sum_{k=1}^{n}\left(4^k - 2^k\right) = \sum_{k=1}^{n} 4^k - \sum_{k=1}^{n} 2^k$

$\sum_{k=1}^{n} 4^k = \dfrac{4\left(4^n - 1\right)}{4-1} = \frac{4}{3}\left(4^n - 1\right)$ (sum of n terms of geometric progression, $a = 4$, $r = 4$)

$\sum_{k=1}^{n} 2^k = \dfrac{2\left(2^n - 1\right)}{2-1} = 2\left(2^n - 1\right)$ (sum of n terms of geometric progression, $a = 2$, $r = 2$)

$\therefore S_n = \frac{4}{3}\left(4^n - 1\right) - 2\left(2^n - 1\right) = \frac{1}{3}\, 2^{2n+2} - \frac{4}{3} - 2^{n+1} + 2 = \frac{1}{3}\, 2^{2n+2} - 2^{n+1} + \frac{2}{3}$

RC 2002 Q8

b)

$n = 1 \Rightarrow T_1 = 5^0 + 3 = 4$, true

$n = 2 \Rightarrow T_2 = 5^1 + 3 = 8$, true 1

Take $n = k$ and $n = k - 1$

$T_k = 5^{k-1} + 3$ and $T_{k-1} = 5^{k-2} + 3$ 1

Assume true and use this to prove true for $n = k + 1$, ie. $T_{k+1} = 5^k + 3$

$T_{k+1} = 6T_k - 5T_{k-1}$

$= 6\left(5^{k-1} + 3\right) - 5\left(5^{k-2} + 3\right)$ 1

$= 6.5^{k-1} + 18 - 5.5^{k-2} - 15$

$= 5^{k-2}(30 - 5) + 3$ 1

$= 5^{k-2}.5^2 + 3$

$= 5^k + 3$ 1

Hence

If true for $n = k - 1$ and $n = k$

then true for $n = k + 1$

However, it was true for $n = 1, n = 2$ 1

$\therefore$ true for $n = 3, n = 4$, etc.

$\therefore$ true for all positive integers n.

SGHS 2002 Q8

Prove by mathematical induction $x^{2n} - y^{2n}$ is divisible by $(x+y)$ for $n \geq 1$.

Step 1: Prove true for $n=1$

$x^2 - y^2 = (x-y)(x+y)$ (½)

∴ true for $n = 1$.

Step 2: Assume true for $n=k$ where k is a +ve integer

(½) i.e. $x^{2k} - y^{2k} = (x+y)M$ where M is positive multiple

Now: Prove true for $n = k+1$

∴ $x^{2(k+1)} - y^{2(k+1)} = x^{2k+2} - y^{2k+2}$

(1) $= x^2 . x^{2k} - y^{2k+2} = x^2 . x^{2k} - x^2 y^{2k} + x^2 y^{2k} - y^2 . y^{2k}$

$= x^2(x^{2k} - y^{2k}) + y^{2k}(x^2 - y^2)$

(½) Now: (from the assumption) $= x^2 . M(x+y) + y^{2k}(x-y)(x+y)$

(1) $= (x+y)\{x^2 M + (x-y)y^{2k}\}$

which is divisible by $(x+y)$.

Step 3 Now the statement is true for $n=k+1$ if true for $n=k$. Statement is true for $n=1$, so is (½) true for $n=1+1=2$ and $n=2+1=3$ and so on for all +ve integer values of n

SBHS 2001 Q7

(c)

(i) $T(k,x) - T(k, x+1)$

$$= \frac{k!}{x(x+1)\cdots(x+k)} - \frac{k!}{(x+1)(x+2)\cdots(x+k)(x+k+1)}$$

$$= \frac{k!\,(x+k+1) - k!\,x}{x(x+1)\cdots(x+k)(x+k+1)}$$

$$= \frac{(k+1)!}{x(x+1)\cdots(x+k+1)} = T(k+1, x).$$

7 (ii) Test $n = 1$:

$$\text{LHS} = T(1,x) = \frac{{}^1C_0}{x} - \frac{{}^1C_1}{x+1} = \frac{1}{x} - \frac{1}{x+1} = \frac{1}{x(x+1)} = \frac{1!}{x(x+1)} = \text{RHS}$$

So the statement is true for $n = 1$

Assume true for some integer $n = k$.

ie $T(k,x) = \dfrac{k!}{x(x+1)(x+2)\cdots(x+k)}$

We need to prove the statement is true for $n = k+1$

ie $T(k+1,x) = \dfrac{(k+1)!}{x(x+1)(x+2)\cdots(x+k)(x+k+1)}$

LHS $= T(k+1,x)$

$= T(k,x) - T(k,x+1)$ | from (i) |

$= \dfrac{(k+1)!}{x(x+1)(x+2)\cdots(x+k)(x+k+1)}$

$=$ RHS

(iii) Substitute $x = 1/2$ into both sides of the result from (ii) and simplify

$$\frac{{}^nC_0}{\frac{1}{2}} - \frac{{}^nC_1}{\frac{1}{2}+1} + \frac{{}^nC_2}{\frac{1}{2}+2} + \dots (-1)^n \frac{{}^nC_n}{\frac{1}{2}+n} = \frac{n!}{\frac{1}{2}\cdot\frac{3}{2}\cdots(\frac{2n+1}{2})}$$

$$= \frac{2^{n+1}\,n!}{1\cdot 3\cdot 5\cdots(2n+1)}$$

$$\frac{{}^nC_0}{1} - \frac{{}^nC_1}{3} + \frac{{}^nC_2}{5} + \dots (-1)^n \frac{{}^nC_n}{2n+1} = \frac{2^n\,n!}{1\cdot 3\cdot 5\cdots(2n+1)}$$

CSSA 2000 Q7

(b)(i) For $n = 1, 2, 3, \ldots$

$$I_n = \int_0^1 (x^2-1)^n \, dx$$

$$= \left[x(x^2-1)^n\right]_0^1 - 2n\int_0^1 x^2 (x^2-1)^{n-1} \, dx$$

$$= 0 - 2n\int_0^1 (x^2-1+1)(x^2-1)^{n-1} \, dx$$

$$= -2n\int_0^1 (x^2-1)^n + (x^2-1)^{n-1} \, dx$$

$$= -2n\left(I_n + I_{n-1}\right)$$

$$(2n+1)\, I_n = -2n\, I_{n-1}$$

$$I_n = \frac{-2n\, I_{n-1}}{2n+1}$$

(ii) $I_0 = \int_0^1 1 \, dx = 1 \Rightarrow I_1 = \frac{-2}{2+1} I_0 = -\frac{2}{3}$

For $n = 1, 2, 3, \ldots$

let $S(n)$ be the statement

$$I_n = \frac{(-1)^n\, 2^{2n}\, (n!)^2}{(2n+1)!}$$

$$\frac{(-1)^1\, 2^2\, (1!)^2}{(2+1)!} = \frac{-4}{3\times 2} = -\frac{2}{3} = I_1$$

$\therefore S(1)$ is true.

(b) (ii) (continued)

If $S(k)$ is true, $I_k = \dfrac{(-1)^k\, 2^{2k}\, (k!)^2}{(2k+1)!}$ **

Consider $S(k+1)$, k some positive integer.

$$I_{k+1} = \frac{-2(k+1)}{2(k+1)+1} I_{k}$$

$$= \frac{-2(k+1)}{2(k+1)+1} \cdot \frac{(-1)^k 2^{2k}(k!)^2}{(2k+1)!}, \quad \text{if } S(k) \text{ is true using } **$$

$$= \frac{(-1)^{k+1}2^{2k+1}(k+1)(k!)^2}{(2k+3)(2k+1)!}$$

$$= \frac{(-1)^{k+1}2^{2k+1}(2k+2)(k+1)(k!)^2}{(2k+3)(2k+2)(2k+1)!}$$

$$= \frac{(-1)^{k+1}2^{2(k+1)}(k+1)^2(k!)^2}{(2k+3)!}$$

$$= \frac{(-1)^{k+1}2^{2(k+1)}\{(k+1)!\}^2}{\{2(k+1)+1\}!}$$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(1)$ is true, hence $S(2)$ is true and then $S(3)$ is true, and so on. By Mathematical Induction, $S(n)$ is true for $n = 1, 2, 3, \ldots$

Hence $I_n = \dfrac{(-1)^n\, 2^{2n}\, (n!)^2}{(2n+1)!}$ for all positive integers n.

NEAP 2000 Q8

(b) (i) When $n = 4$, $3^4 = 81$ and $4^3 = 64$. So $3^n > n^3$ is true for $n = 4$. ✓

Assume the statement is true for $n = k$, i.e. assume $3^k > k^3$.

We must prove that the statement is true for $n = k + 1$, i.e. that $3^{k+1} > (k+1)^3$.

Consider $3^{k+1} - (k+1)^3$.

$$\begin{aligned}3^{k+1} - (k+1)^3 &= 3 \times 3^k - k^3 - 3k^2 - 3k - 1\\ &= 3(3^k - k^3) + 2k^3 - 3k^2 - 3k - 1\\ &= 3(3^k - k^3) + (k^3 - 3k^2 + 3k - 1) + (k^3 - 6k)\\ &= 3(3^k - k^3) + (k-1)^3 + k(k^2 - 6) \quad ✓✓\end{aligned}$$

Now $3^k - k^3$ is positive, by assumption,

and $(k-1)^3$ is positive since $k > 3$,

and $k(k^2 - 6)$ is positive since $k > 3$.

Thus $3^{k+1} - (k+1)^3 > 0$, i.e. $3^{k+1} > (k+1)^3$. ✓

Therefore by the principle of mathematical induction, $3^n > n^3$ for all $n > 3$.

(ii) $3^n > n^3$

$(3^n)^{\frac{1}{3n}} > (n^3)^{\frac{1}{3n}}$ (taking the $3n$th root of both sides)

i.e. $3^{\frac{1}{3}} > n^{\frac{1}{n}}$

$\sqrt[3]{3} > \sqrt[n]{n}$ ✓

ST IGNATIUS 2002 Q8

(b) $\sum_{r=1}^{n} \sin(2r-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}$

Let $T_n = \sin(2n-1)\theta$

Let $S_n = \frac{\sin^2 n\theta}{\sin\theta}$

$\sum_{r=1}^{n} \sin(2r-1)\theta = \sin\theta + \sin 3\theta + \sin 5\theta + \ldots + \sin(2n-1)\theta$

when $n = 1$

$\frac{\sin^2 n\theta}{\sin\theta} = \frac{\sin^2\theta}{\sin\theta} = \sin\theta$ which is true.

Let $n = k$ be true.

$\therefore \quad S_k = \frac{\sin^2 k\theta}{\sin\theta}$

Now $S_k + T_{k+1} = \frac{\sin^2 k\theta}{\sin\theta} + \sin[2(k+1)-1]\theta$

$= \frac{\sin^2 k\theta}{\sin\theta} + \sin(2k+1)\theta$

$= \frac{2\sin^2 k\theta + 2\sin(2k+1)\theta\sin\theta}{2\sin\theta}$

$= \frac{1-\cos 2k\theta + 2\sin\theta[\sin 2k\theta\cos\theta + \cos 2k\theta\sin\theta]}{2\sin\theta}$

$= \frac{1-\cos 2k\theta + \sin 2k\theta\sin 2\theta + 2\sin^2\theta\cos 2k\theta}{2\sin\theta}$

$= \frac{1-\cos 2k\theta + \sin 2k\theta\sin 2\theta + 2\times\frac{1}{2}(1-\cos 2\theta)\cos 2k\theta}{2\sin\theta}$

$= \frac{1-\cos 2k\theta + \sin 2k\theta\sin 2\theta + \cos 2k\theta - \cos 2k\theta\cos 2\theta}{2\sin\theta}$

$= \frac{1-\cos(2k\theta+2\theta)}{2\sin\theta}$

$= \frac{1-\cos 2(k\theta+\theta)}{2\sin\theta}$

$S_k + T_{k+1} = \frac{2\sin^2(k\theta+\theta)}{2\sin\theta} = \frac{\sin^2(k+1)\theta}{\sin\theta}$

(b) This result is the required form ie S_k with $(k+1)$ in place of k.

The result is true for $n = k+1$ if it is true for $n=k$ ie if it is true for one integer then it is true for the next consecutive integer.

∴ true when $n=1$, also true when $n=2$

" ✓ ✓ $n=2$, ✓ ✓ ✓ ✓ $n=3$

" ✓ ✓ $n=3$, ✓ ✓ ✓ ✓ $n=4$ etc

Hence $\sum_{r=1}^{n} \sin (2r-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}$ true for all positive integers n.